MODULE 5

Making Informed Decisions

GEOMETRY

TOPIC 1

Does the probability of bad weather affect the probability of your flight being delayed?

Independence and Conditional Probability

1 Compound Sample Spaces

OBJECTIVES

- List the sample space for situations involving probability.
- Construct a probability model for a situation.
- Differentiate between uniform and non-uniform probability models through writing and discussions.
- Develop a rule to determine the total number of outcomes in a sample space without listing each outcome.
- Classify events as independent or dependent.
- Use the Fundamental Counting Principle to calculate the size of sample spaces and explain the process through writing and class discussions.

KEY TERMS

- outcome
- sample space
- event
- probability
- probability model
- uniform probability model
- non-uniform probability model
- complement of an event
- tree diagram
- organized list
- set
- element
- disjoint sets
- intersecting sets
- union of sets
- independent events
- dependent events
- Fundamental Counting Principle

In previous courses you have explored simple and compound probabilities.

How are the possible outcomes of an experiment affected by the types of events and sets in the experiment?

Getting Started

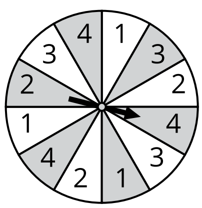
A *sample space* is typically enclosed in braces, { }, with commas between the outcomes.

Take a Spin

An **outcome** is a result of an experiment. The **sample space** is the set of all the possible outcomes of an experiment. An **event** is an outcome or set of outcomes in a sample space.

The **probability** of an event is the ratio of the number of desired outcomes to the total number of possible outcomes. The probability of event A is

$$P(A) = \frac{\text{number of outcomes in } A}{\text{total possible outcomes}}$$



1. A board game includes the spinner shown that players must use to advance a game piece around the board.

   a. What is the sample space when a player spins the spinner shown one time?

   b. What is the probability of spinning the number 3, $P(3)$?

   c. What is the probability of spinning a number greater than 1?

In a probability model, the sum of the probabilities must equal 1.

It is often helpful to construct a model when analyzing situations involving probability. A **probability model** lists the possible outcomes and the probability for each outcome.

In a **uniform probability model**, the probabilities for each outcome are equal. When the probabilities of the outcomes are not all equal, the model is a **non-uniform probability model**.

2. Consider the spinner.

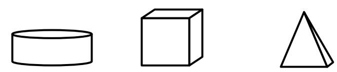
   a. Complete the probability model for spinning the spinner.

Outcomes	1	2	3	4
Probability				

b. Is this a *uniform probability model* or a *non-uniform probability model*? Explain how you know.

3. What is the probability of a spin not resulting in a 3?

In this game, players can earn different types of tokens as they move around the board. When a player lands on certain spaces, the player can randomly choose a token from a box. The token is then replaced before the next player's turn.



6 tokens 4 tokens 2 tokens

4. Isaiah and Camilla each determined the sample space for choosing a token.

Isaiah: Cylinder, Cube, Pyramid

Camilla: Cylinder, Cylinder, Cylinder, Cylinder, Cylinder, Cylinder
Cube, Cube, Cube, Cube
Pyramid, Pyramid

Who determined the sample space correctly? Explain your reasoning.

The **complement of an event** is an event that contains all the outcomes in the sample space that are not outcomes in the event. If E is an event, then the complement of E is often denoted as $\overline{E}$ or E^c. An event and its complement are called complementary events.

5. Consider the choice of tokens.

   a. What is the probability of choosing a pyramid, *P*(pyramid)? A cube, *P*(cube)?

   b. Construct the probability model for choosing one of the tokens.

Outcomes	cylinder	cube	pyramid
Probability			

   c. Is this a uniform probability model or a non-uniform probability model? Explain how you know.

   d. What is the probability of choosing a token that is not a cylinder?

ACTIVITY 1.1

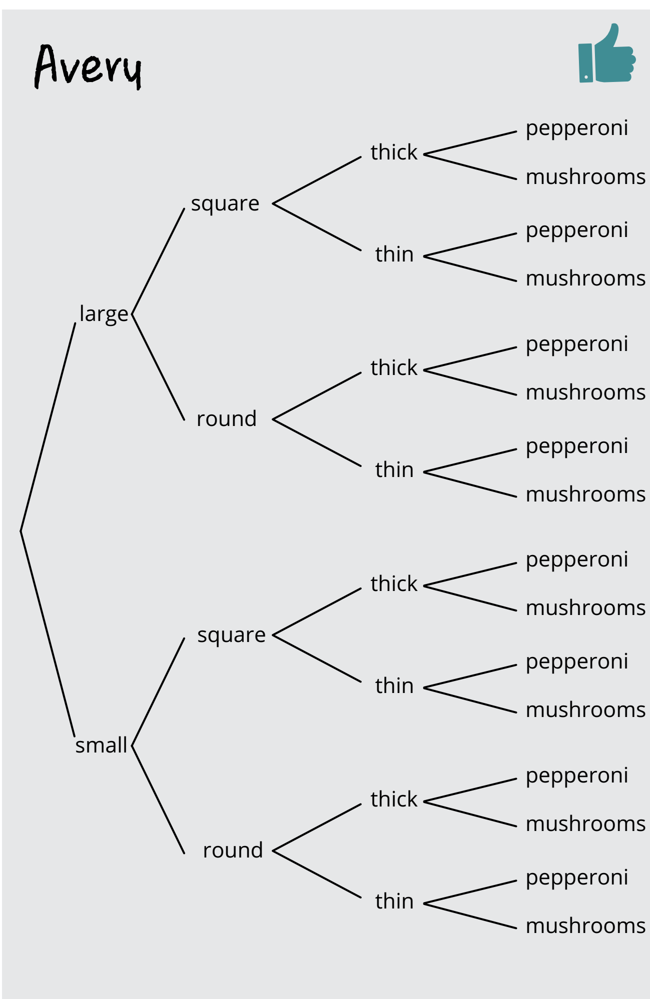
Sample Space for Pizza Special

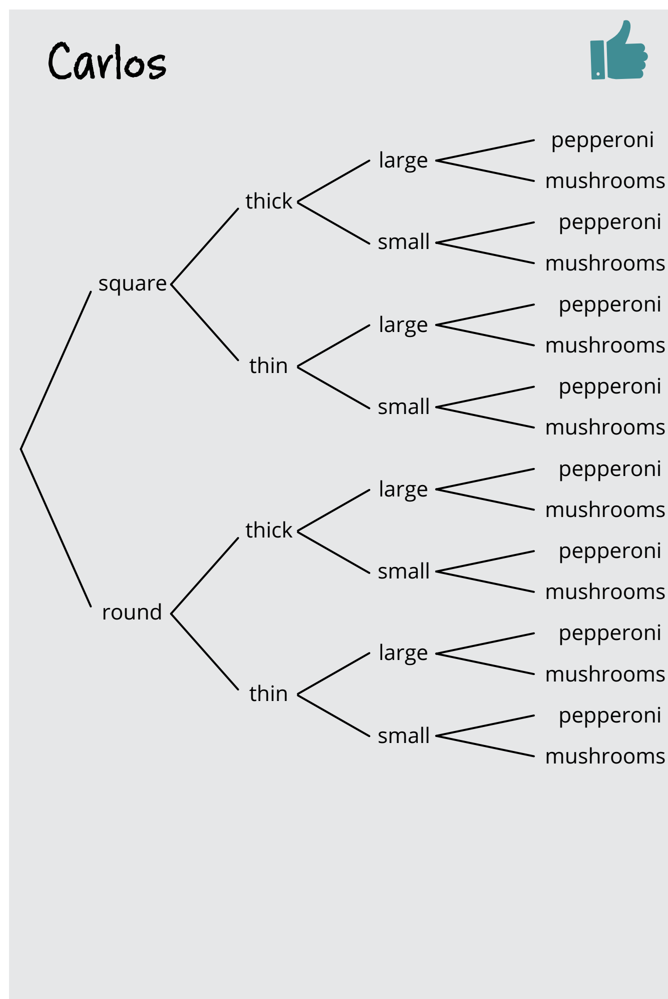
A pizzeria advertises special deals in the newspaper.

> **Today's Special at the Pizzeria**
> Large one-topping pizza $9.00
> Small one-topping pizza $6.50
> Choose either a square or a round pizza with thick or thin crust.
> Available toppings: pepperoni or mushrooms
> Enjoy a fresh-baked pizza!!!

A **tree diagram** is a visual model for determining the sample space of multiple events.

Avery and Carlos sketched tree diagrams to show the possible pizza specials at the pizzeria.





Both tree diagrams show the same information, but each tree diagram is organized differently.

1. How many outcomes are included in the sample space of each tree diagram? Explain how you determined your answer.

2. What does each outcome of the sample space represent?

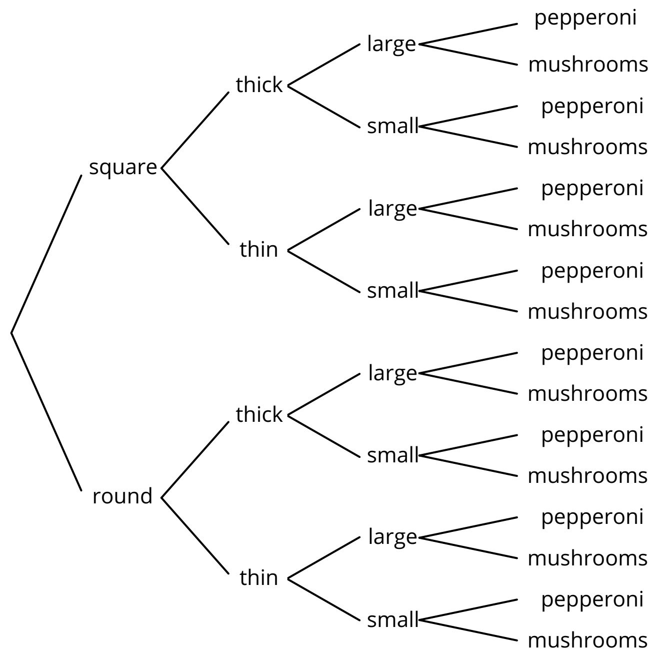
3. Compare the tree diagrams created by Avery and Carlos.

   a. How are the tree diagrams similar and different?

   b. Does the arrangement of the tree diagram affect the total number of possible outcomes? Explain why or why not.

An **organized list** is a visual model for determining the sample space of events.

4. Use the tree diagram to write an organized list that displays all the possible pizza specials at the pizzeria.



Think About . . .

There are descriptions that begin with the same letter. When you are abbreviating the names, what makes sense? Be sure to include a key so others know what you mean.

5. Analyze the sample space to answer each question.

   a. How many possible round pizza specials can you order?

   b. How many possible thick and round pizza specials can you order?

   c. How many of the possible pizza choices are square but do not have mushrooms?

ACTIVITY 1.2 Sample Space for Student Council Election

Jasmine, Gabriel, Kayla, Mei, and Parker are running for student council offices. The student with the greatest number of votes is elected president, and the student receiving the next greatest number of votes is elected vice president.

1. Create a tree diagram to represent the possible outcomes for the election.

2. What does each level in your diagram represent? Does order matter? Explain your reasoning.

3. Write the sample space as an organized list.

4. Analyze the sample space to answer each question.

   a. How many outcomes result in Gabriel being elected president or vice president?

   b. How many outcomes result in Parker not being elected president?

   c. How many outcomes result in Jasmine or Kayla as president and Parker or Gabriel as vice president?

5. Logan used the first initial of each student's name as an abbreviation in his sample space. He included both JM and MJ as different outcomes. Diego says these two outcomes mean the same thing and only one should be included in the sample space. Who's correct? Explain your reasoning.

6. Can you rearrange the tree diagram to produce a different number of total possible outcomes for the election? Explain why or why not.

7. Consider your sample spaces from *Pizza Special* and *Student Council Election*.

   a. Does choosing a topping for a pizza at the pizzeria affect any of the other choices you have? Does choosing a size affect any of the other choices? Explain your reasoning.

   b. Does electing the student council president affect the choices for vice president? Explain your reasoning.

ACTIVITY 1.3

Categorizing Scenarios Involving Events

Two or more sets are **disjoint sets** when they do not have any common elements.

A **set** is a collection of items. If *x* is a member of set *B*, then *x* is an **element** of set *B*.

Consider the examples of disjoint sets, intersecting sets, and the union of sets.

Two or more sets are **intersecting sets** when they do have common elements.

A **union of sets** is a set formed by combining all the members of the sets. A member is listed only once.

WORKED EXAMPLE

Disjoint Sets

Let *N* represent the set of 9th grade students. Let *T* represent the set of 10th grade students.

The set of 9th grade students and the set of 10th grade students are disjoint because the two sets do not have any common elements. Any student can be in one grade only.

Intersecting Sets

Let *V* represent the set of students who are on the volleyball team. Let *M* represent the set of students who are in the math club. Sarah is on the volleyball team and belongs to the math club.

The set of students who are on the volleyball team and the set of students who are in the math club are intersecting because we know they have at least one common element, Sarah.

Union of Sets

Let *B* represent the set of students in the 11th grade band. Let *C* represent the set of students in the 11th grade chorus. The union of these two sets would be all the students in the 11th grade band or the 11th grade chorus. A student in both would be listed only once.

1. Identify the sets in each scenario in the previous two activities. Then, determine whether the sets are disjoint or intersecting. Explain your reasoning.

   a. *Pizza Special*

   b. *Student Council Election*

2. Determine whether each set is disjoint or intersecting. Explain your reasoning.

   a. Let E represent the set of even integers. Let O represent the set of odd integers.

   b. Let T represent the set of multiples of 2. Let F represent the set of multiples of 5.

Independent events are events for which the occurrence of one event has no impact on the occurrence of the other event.

Dependent events are events for which the occurrence of one event has an impact on the occurrence of subsequent events.

3. Consider the letters in the words *baker* and *crate*.

   a. What set would be formed by taking the intersection of the letters in the two words?

   b. What set would be formed by taking the union of the letters in the two words?

Consider the examples of independent events and dependent events.

WORKED EXAMPLE

A jar contains 1 blue marble, 1 green marble, and 2 yellow marbles.

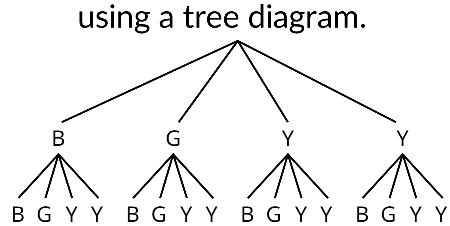
Independent Events

You randomly choose a yellow marble, replace the marble in the jar, and then randomly choose a yellow marble again.

The event of choosing a yellow marble first does not affect the event of choosing a yellow marble second because the yellow marble chosen first is replaced in the jar.

The events of randomly choosing a yellow marble first and randomly choosing a yellow marble second are independent events because the first yellow marble was replaced in the jar.

You can visualize this using a tree diagram.



WORKED EXAMPLE

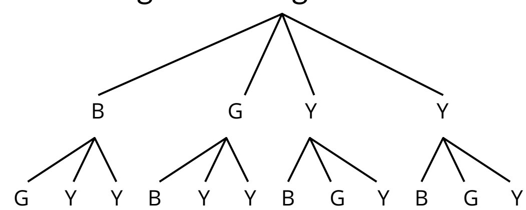
Dependent Events

A jar contains 1 blue marble, 1 green marble, and 2 yellow marbles. You randomly choose a yellow marble without replacing the marble in the jar, and then randomly choose a yellow marble again.

The event of choosing a yellow marble first does affect the event of choosing a yellow marble second because the yellow marble chosen first is not replaced in the jar.

The events of randomly choosing a yellow marble first and randomly choosing a yellow marble second are dependent events because the first yellow marble was not replaced in the jar.

You can see this visually using a tree diagram.



4. Explain why Emily's statement is incorrect.

Emily

I think that choosing the first marble and choosing the second marble are dependent events.

5. Identify one outcome from *Pizza Special* and *Student Council Election*. Then, state whether the events that result in that outcome are independent or dependent. Explain your reasoning.

   a. *Pizza Special*

   b. *Student Council Election*

ACTIVITY **1.4**

The Fundamental Counting Principle

Emily's aunt owns an ice cream shop. She offers her customers a choice of chocolate, vanilla, or strawberry ice cream in a cone or cup and with or without sprinkles. Her aunt was trying to determine the number of ways that a customer could order ice cream. Emily said, "There are 12 ways!" Emily used a mathematical principle called the *Fundamental Counting Principle*. The principle is used to determine the number of outcomes in the sample space.

The **Fundamental Counting Principle** states: "If an action A can occur in m ways and for each of these m ways an action B can occur in n ways, then actions A and B can occur in $m \cdot n$ ways."

Think About . . .
What is the difference between an event and an action?

The Fundamental Counting Principle can be generalized to more than two actions that happen in succession. If for each of the m and n ways A and B occur there is also an action C that can occur in s ways, then Actions A, B, and C can occur in $m \cdot n \cdot s$ ways.

1. Sebastian has an all-day pass for a local amusement park. His favorite rides are the Bungee-Buggy, Head Rush roller coaster, Beep Beep go-karts, and Tsunami Slide water roller coaster. He never rides any other rides, and he can ride each of his favorite rides as many times as he wants.

   a. Describe the type of event for selecting ride order. Explain your reasoning.

A situation can involve independent events from disjoint sets, independent events from the same set with repetitions allowed, or dependent events from the same set without repetitions.

   b. Use the Fundamental Counting Principle to determine the number of possible ride orders for Sebastian's next two rides. Explain your reasoning.

c. How many ride order possibilities are there for Sebastian's next five rides?

2. Valentina stayed home from school Wednesday because she was ill. She watched a television program from 12:00 p.m. until 12:30 p.m. and another program from 12:30 p.m. until 1:00 p.m. From 12:00 p.m. until 12:30 p.m., her program choices were the news, cartoons, or a talk show. From 12:30 p.m. until 1:00 p.m., her program choices were a comedy, a soap opera, a game show, or a cooking show.

   a. Describe the type of event for Valentina's television program selections. Explain your reasoning.

   b. How many program selections can Valentina watch from 12:00 p.m. until 1:00 p.m?

3. A student's daily schedule includes math, English, science, social studies, music, art, and physical education. Students are enrolled in each class for one period per day.

   a. Describe the type of event for creating a student's daily schedule. Explain your reasoning.

   b. Determine how many different orders the classes can be arranged to fill a seven-period daily schedule.

   c. Lunch period is directly after fourth period. How many different class schedule arrangements are possible before lunch period? Explain your reasoning.

4. The cell phone PIN to access voicemail is a 4-digit number. Each digit can be a number from 0 to 9, including 0 and 9.

   a. How many 4-digit numbers are possible? Repetition of numbers is allowed. Explain your calculation.

   b. When repeating digits is not permitted, how many different 4-digit PINs are possible?

5. A typical license plate number for a car consists of three letters followed by four numbers ranging from 0 through 9, including 0 and 9.

   a. How many different license plate numbers are possible when letters and numbers can be repeated? Explain your calculation.

   b. How many different 3-letter, 4-digit license plate numbers are possible when letters and digits cannot be repeated?

Talk the Talk

Counting Without Counting

The Fundamental Counting Principle is used to determine the number of outcomes in a sample space. The calculations vary depending upon the situation.

1. For each given type of event and type of set, write a scenario and sample space calculation that represents the type of event and type of set.

Type of Event (independent or dependent)	Type of Set (disjoint or intersecting)	Scenario	Sample Space Calculation
independent	three disjoint sets		

Type of Event (independent or dependent)	Type of Set (disjoint or intersecting)	Scenario	Sample Space Calculation
dependent	intersecting sets		
dependent	two disjoint sets		

2. Nahimana and Angelina each describe a scenario with a sample space of $4 \cdot 3 \cdot 2 = 24$.

Nahimana	Angelina
The scenario has independent events and disjoint sets.	The scenario has dependent events and intersecting sets.

a. Based on Nahimana's description, write an example scenario involving 3 disjoint sets and 24 outcomes in the sample space.

b. Based on Angelina's description, write an example scenario involving 3 intersecting sets and 24 outcomes in the sample space.

3. Describe how to calculate the number of outcomes in the sample space of situations involving either independent or dependent events.

4. List advantages and disadvantages of using a tree diagram or an organized list to represent a sample space.

Lesson 1 Assignment

Write

1. A ________________ is a collection or group of items and each item within it is called an ________________.

2. The ________________ is a set formed by combining all the members of the sets, such that all the members are only listed once.

3. Sets that do not have common elements are called ________________.

4. Sets that do have common elements are called ________________.

5. ________________ and ________________ are two types of visual models that display sample space.

6. Events for which the occurrence of one event has no impact on the occurrence of the other event are ________________.

7. Events for which the occurrence of one event has an impact on the following events are ________________.

8. The ________________ states that if an action A can occur in m ways, and for each of these m ways, an action B can occur in n ways, then Actions A and B can occur in $m \cdot n$ ways.

Remember

The Fundamental Counting Principle states: "If an action A can occur in m ways, and for each of these m ways, an action B can occur in n ways, then actions A and B can occur in $m \cdot n$ ways." The values for m and n are determined by whether the events are independent or dependent.

Lesson 1 Assignment

Practice

1. Suppose you roll a number cube once.

    a. Identify the sample space.

    b. What is the probability of rolling a 5, $P(5)$?

    c. What is the probability of rolling an even number, $P(\text{even})$?

    d. What is the probability of rolling a number greater than 2, $P(\text{greater than } 2)$?

    e. Construct the probability model for rolling a number cube.

Outcomes						
Probability						

Lesson 1 Assignment

f. Is the probability model from part (e) a uniform or non-uniform probability model? Explain your reasoning.

g. What is the probability of rolling a number that is not a multiple of 3, $P(\text{not a multiple of } 3)$?

2. For each scenario,
   - Determine the actions.
   - Determine the outcomes of each action.
   - Determine whether the outcomes of each action belong to disjoint sets or intersecting sets. Explain your reasoning.
   - Sketch a tree diagram or write an organized list to represent the sample space.
   - Determine whether the events in each outcome of the sample space are independent or dependent.
   - Determine the size of the sample space using the Fundamental Counting Principle. Show your calculation.

Deck A

Lose a Turn	Take 2 Tokens	Take an Extra Turn

Deck B

Go Back 2 Spaces	Go Back 1 Space	Go Ahead 2 Spaces	Go Ahead 1 Space

Lesson 1 Assignment

a. While playing a board game, a player randomly chooses one card from each of the two decks and then replaces the cards in the decks.

b. Adriana randomly chooses a card from a deck of six cards without replacing it, and she then chooses another card. The cards are numbered 1 through 6.

Lesson 1 Assignment

Prepare

Determine each product.

1. $\frac{4}{5} \cdot \frac{1}{4}$

2. $\frac{1}{6} \cdot \frac{3}{5}$

3. $\frac{5}{8} \cdot \frac{3}{7}$

4. $\frac{1}{4} \cdot \frac{2}{3} \cdot \frac{1}{2}$

2 Compound Probability with *And*

OBJECTIVES

- Determine the probability of two or more independent events by analyzing texts and justifying reasoning through writing.
- Determine the probability of two or more dependent events.

KEY TERMS

- compound event
- compound probability rule involving *and*
- conditional probability

You have used tree diagrams and organized lists to determine the sample space and calculate the probability of events.

How can you calculate the probability of compound events that are combined by the word *and* using a mathematical rule?

Getting Started

Frundaes!

Stan's Frozen Yogurt Shop offers frundaes, which are frozen yogurt sundaes. The shop advertises different frundae options for customers.

BUILD YOUR OWN FRUNDAE

Choose one yogurt flavor and one topping.

Frozen Yogurt Flavors	Toppings
vanilla	nuts
chocolate	sprinkles
strawberry	granola
peach	

1. Consider the number of possible frundae options.

   a. Write an organized list for the sample space that represents different frundae options consisting of one yogurt flavor and one topping.

b. How many different frundae options are possible by first selecting one frozen yogurt flavor and then one topping? Explain how you determined your answer.

2. Parker and Avery each order a single flavor cone with no toppings.

   a. Write an organized list for the sample space that represents the different single flavor cones that Parker and Avery could buy.

   b. How many different single flavor cones purchase options are possible for Parker and and Avery? Explain how you determined your answer.

3. A Triple-Decker Froyanza is a frozen yogurt cone with three servings of yogurt. Luna orders a Triple-Decker Froyanza with one serving of chocolate, one serving of vanilla, and one serving of strawberry frozen yogurt.

   a. Write an organized list for the sample space that represents the order in which the server could stack the three servings of frozen yogurt on Luna's cone.

   b. How many different stacking orders are possible for the frozen yogurt flavors of Luna's Triple-Decker Froyanza? Explain how you determined your answer.

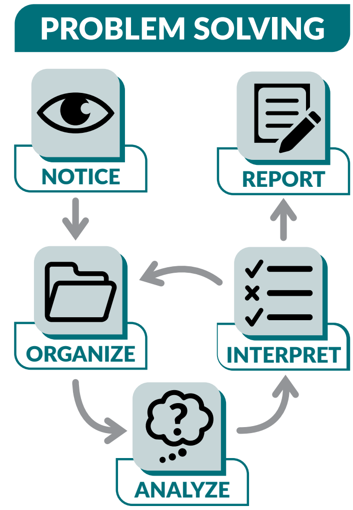
4. Compare and contrast the sample spaces from Questions 1, 2, and 3.

   a. How are the sample spaces alike?

   b. How is Question 1 and its sample space different from Question 2 and its sample space?

   c. How is Question 3 and its sample space different from Questions 1 and 2 and their sample spaces?

ACTIVITY
2.1 Exploring Compound Events with *And*



A **compound event** is an event that consists of two or more events.

1. Consider the different ordering situations of frozen yogurt from the Getting Started.

   a. Which situations represent compound events?
   Explain your reasoning.

   b. Describe the events that make up the compound event in ordering a frundae. Then, list the outcomes for each event.

   c. Are the events that make up the compound event in ordering a frundae independent or dependent events? Explain your reasoning.

Frozen Yogurt Flavors
vanilla
chocolate
strawberry
peach

Toppings
nuts
sprinkles
granola

Suppose you decide to build your next frundae by randomly selecting a yogurt flavor and topping.

2. What is the probability of choosing a frundae that does not have chocolate yogurt and does have a granola topping?

   a. Let A represent the event of choosing a flavor that is not chocolate. What is $P(A)$, the probability of choosing a flavor that is not chocolate?

Ask Yourself . . .
How can you use the sample spaces you created to determine each probability?

b. Let B represent the event of choosing granola as a topping. What is $P(B)$, the probability of choosing granola as a topping?

$P(A \text{ and } B)$ can be written as $P(A \cap B)$, where $A \cap B$ represents an intersection of Events A and B. The intersection of Events A and B is the set of all of the outcomes where both Event A and Event B occur.

c. What is $P(A \text{ and } B)$, the probability of choosing a flavor other than chocolate that has granola as a topping? Explain how you determined your answer.

3. What is the probability of choosing a frundae with chocolate flavored yogurt and a topping that is not granola?

   a. Let A represent the event of choosing a frundae with chocolate flavored yogurt. What is $P(A)$, the probability of choosing a frundae with chocolate flavored yogurt?

   b. Let B represent the event of choosing a topping that is not granola. What is $P(B)$, the probability of choosing a topping that is not granola?

   c. What is $P(A \text{ and } B)$, the probability of choosing a frundae with chocolate flavored yogurt and a topping that is not granola? Explain how you determined your answer.

4. Compare the probabilities you determined in Questions 2 and 3. What mathematical relationship exists between $P(A)$, $P(B)$, and $P(A \text{ and } B)$?

5. Why do you think that the probability of both events occurring is less than the probability of either event occurring by itself?

6. Why do you think multiplication is performed in compound probability problems using the word *and*?

ACTIVITY 2.2

Calculating Compound Probabilities with *And*

You just calculated probabilities for a compound event comprised of two events. Now, let's consider a compound event comprised of more than two events.

1. Suppose you flip 3 coins and record the result of each flip.

   a. Are the events independent events? Explain your reasoning.

   b. What is the probability that all 3 coins land heads up?

   c. What is the probability that the first 2 coins land heads up and the third coin lands tails up?

   d. What is the probability of the second coin landing tails up? Explain how you determined your answer.

   e. What is the probability of flipping heads on the first coin and tails on the third coin? Explain how you determined your answer.

2. Suppose you flip 50 coins. What is the probability of flipping heads on the first and second coins? Diego and Sarah each answered this question differently.

> **Diego**
>
> To calculate the number of total possible outcomes, I multiply 2 by itself 50 times, which is 2^{50}, or 1,125,899,906,842,624. There is 1 desired outcome for each of the first 2 coins, or $1 \cdot 1$. For the other 48 coins, there are 2 desired outcomes, because they can be heads or tails. This is 2^{48}. So, the number of desired outcomes is $1 \cdot 1 \cdot 2^{48}$, or 281,474,976,710,656.
>
> The probability of flipping heads on the first and second coins, then, is
>
> $$\frac{281{,}474{,}976{,}710{,}656}{1{,}125{,}899{,}906{,}842{,}624} = \frac{2^{48}}{2^{50}} = \frac{1}{2^2} = \frac{1}{4}.$$

> **Sarah**
>
> I only need to do calculations with the probabilities for the first 2 coins because the probabilities for the other 48 coins are each $\frac{2}{2}$, or 1.
>
> The probability of flipping heads on the first two of 50 coins is $\frac{1}{2} \cdot \frac{1}{2}$, or $\frac{1}{4}$.

Which strategy is more efficient? Why?

Some calculators display a maximum of 10 digits. Very large numbers are often displayed using scientific notation.

The **compound probability rule involving *and*** states: "If Event *A* and Event *B* are independent events, then the probability that Event *A* happens and Event *B* happens is the product of the probability that Event *A* happens and the probability that Event *B* happens, given that Event *A* has happened."

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Think About . . .
If Events *A* and *B* are independent, does the occurrence of Event *A* affect the probability of the occurrence of Event *B*?

ACTIVITY 2.3

Dependent Events with *And*

You can use small numbers called subscripts to indicate the different red or green socks. For example, R_1 and R_2 can represent the two red socks.

You have 2 red, 1 blue, and 3 green socks in a drawer.

Suppose you reach into the drawer without looking and choose a sock, replace it, and then choose another sock. You choose a total of 2 socks.

1. Use this information to answer each question.

   a. Are the events "choosing the first sock" and "choosing the second sock" dependent or independent events? Explain your reasoning.

   b. Use a tree diagram or organized list to represent the sample space for this situation.

   c. How can you use the Fundamental Counting Principle to determine the total number of possible outcomes? Explain your reasoning.

d. Calculate the probability of choosing:

- A blue sock and then a red sock.
- A red sock and then a sock that is not blue.
- Two socks with the first sock being green.

Ask Yourself . . .
How can a tree diagram or organized list be used to check your answers?

Suppose you reach into the drawer without looking and choose a sock, do not replace it, and then choose another sock. You choose a total of 2 socks. Remember, there are 2 red, 1 blue, and 3 green socks in a drawer.

2. Use this information to answer the questions.

   a. Are the actions "choosing the first sock" and "choosing the second sock" dependent or independent events? Explain your reasoning.

   b. Use a tree diagram or organized list to represent the sample space for this situation.

c. How can you use the Fundamental Counting Principle to determine the total number of possible outcomes? Explain your reasoning.

d. Calculate the probability of choosing:

- A blue sock and then a red sock.

- A red sock and then a sock that is not blue.

- Two socks with the first sock being green.

3. What's different about the probability calculations in Question 1, part (d) and Question 2, part (d)?

If Event A and Event B are dependent, then the probability that Event A happens and Event B happens is the product of the probability that Event A happens and the probability that Event B happens, given that Event A has already occurred.

If Events A and B are dependent, then the probability of Event B occurring is affected by the occurrence of the first event. Therefore, the notation for the compound probability rule involving *and* now states that $P(A \text{ and } B) = P(A) \cdot P(B|A)$.

4. Two red socks, 1 blue sock, and 3 green socks are in a drawer. One sock is randomly chosen without replacing it in the drawer, then a second sock is randomly chosen. What is the probability that the second sock is red?

   Gabriel explained his solution method.

> Gabriel
>
> We want the second sock to be red, but the first sock can be red or not red. The desired outcomes are "red and red" or "not red and red."

Calculate the probability of choosing a red sock second.

Conditional probability is the probability of an event which assumes the occurrence of some other event. $P(B|A)$ means "the probability of the occurrence of Event B given the occurrence of Event A."

Remember . . .
There are two cases that you should consider, depending on whether the first sock was red or not.

5. Two red socks, 1 blue sock, and 3 green socks are in a drawer. One sock is randomly chosen without replacing it in the drawer, and then a second sock is randomly chosen. Use Gabriel's method to answer each question.

   a. What is the probability that the second sock is blue?
      Explain your reasoning.

b. What is the probability that the second sock is green? Explain your reasoning.

Talk the Talk

That Depends

Consider the situations that you analyzed in this lesson.

1. Compare the methods you used to determine the compound probability of two independent events both occurring and the compound probability of two dependent events both occurring. Describe the similarities and differences between the methods.

2. What rules could you write to determine the compound probability of three or more independent events? Three or more dependent events? Include examples to support your conclusions.

Lesson 2 Assignment

Write

Describe the difference of the compound probability rule involving *and* for independent and dependent events in your own words.

Remember

A compound event is an event that consists of two or more events. If the events are combined by the word *and*, you can multiply the probability of each event to determine the compound probability.

Practice

1. Suppose a player chooses cards from the two decks shown. The subsets of cards are labeled C1 to C7 (see figure).

   a. A player chooses one card from Deck A and one card from Deck B. What is the probability that the player will choose cards C1 and C4?

Deck A

Lose a Turn	Take 2 Tokens	Take an Extra Turn
5 cards	4 cards	3 cards
C1	C2	C3

   b. A player chooses one card from Deck A and replaces it. Then the next player chooses one card from Deck A. What is the probability that both players will choose a C2 card?

Deck B

Go Back 2 Spaces	Go Back 1 Space	Go Ahead 2 Spaces	Go Ahead 1 Space
5 cards	4 cards	3 cards	2 cards
C4	C5	C6	C7

   c. A player chooses two cards at the same time from Deck B. What is the probability that the player will choose two C5 cards?

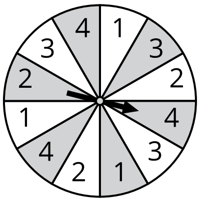
Lesson 2 Assignment

d. A player chooses one card from Deck A and one card from Deck B. What is the probability of not choosing a C1 card from Deck A and the probability of not choosing a C7 card from Deck B?

e. A player chooses one card from Deck A and then, without replacing it, chooses another card from Deck A. What is the probability that the first card will be a C2 and the second card will not be a C2?

2. The board game includes both the spinner and the set of tokens shown in the figure.

a. A player spins the spinner once and then randomly chooses a token. What is the probability that the spinner will land on a 4 and the player will choose a cube token?



b. A player spins the spinner twice. What is the probability that the second spin will land on a 3?

c. A player chooses a token from the set, replaces it, and then chooses another token from the set. What is the probability that the first token chosen will be a cube and the second will be a disk?

d. A player chooses two tokens from the set at the same time.
What is the probability that both will be pyramids?

e. A player spins the spinner once and then randomly chooses a token. What is the probability that the spinner will not land on a 3 and the player will choose a disk token?

f. A player randomly chooses three tokens at once from the set.
What is the probability that the first two tokens are cubes?

Lesson 2 Assignment

Prepare

Determine the value of each expression.

1. $\frac{1}{2} + \frac{1}{3} - \frac{1}{4}$

2. $\frac{1}{6} + \frac{3}{4} - \frac{2}{5}$

3. $\frac{2}{3} + \frac{1}{8} - \frac{1}{2}$

3 Compound Probability with *Or*

OBJECTIVES

- Determine the probability of one or another independent event.
- Determine the probability of one or another dependent event and communicate reasoning through writing.

KEY TERM

- Addition Rule for probability

You have determined the probability of compound events related by the word *and*.

How do the methods for determining the probability of compound events related by the word *or* compare?

Getting Started

Making Heads or Tails of It

Suppose you flip two coins. What is the probability of the first coin landing heads up or the second coin landing tails up?

1. Describe the events in this probability situation and then determine whether the events are independent or dependent.

Ask Yourself . . .
What does it mean to be "heads up OR tails up"?

2. Complete the table to construct the sample space for this situation.

		Second Coin	
		H	T
First Coin	H		
	T		

   a. Draw a circle around the outcomes that match the first event.

   b. Draw a rectangle around the outcomes that match the second event.

3. How can you describe the outcome that is circled and has a rectangle around it in the sample space?

4. Sebastian and Mason described the probability of a heads up result for the first coin or a tails up result for the second coin.

Sebastian

I circled 2 outcomes and drew a rectangle around 2 outcomes, and there are 4 possible outcomes. So, the probability of flipping a heads on the first coin or a tails on the second is

$\frac{2+2}{4} = \frac{4}{4} = 1.$

Mason

I marked 4 outcomes, and there are 4 possible outcomes. But, I marked 1 of the outcomes twice. So, I have to subtract one of the outcomes and the probability of a heads up result for the first coin or a tails up result for the second coin is $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} - \frac{1}{4} = \frac{3}{4}.$

Explain in your own words why Mason is correct and Sebastian is incorrect.

ACTIVITY 3.1

Exploring Compound Events with *Or*

In a compound event that is related by the word *or*, there can be possible outcomes that are in the sample space for each event. These outcomes should be counted only once when determining the compound probability.

1. Use the sample space and what you know about probability to answer each question.

   a. What is the probability of a heads up result for flipping the first coin, $P(A)$?

   b. What is the probability of a tails up result for flipping the second coin, $P(B)$?

   c. What is the probability of a heads up result for the first coin flip AND a tails up result for the second coin flip, $P(A \text{ and } B)$?

   d. What is the probability of a heads up result for the first coin flip OR a tails up result for the second coin flip $P(A \text{ or } B)$?

Ask Yourself . . .
How can I combine the probabilities that I determined in parts a and b to calculate $P(A \text{ or } B)$, knowing that $P(A \text{ and } B)$ is included in both parts a and b?

$P(A \text{ or } B)$ is also known as $P(A \cup B)$, where $A \cup B$ represents a union of Events A and B. The union of Events A and B is the set all of the outcomes where Event A or Event B occurs.

2. Create a formula you could use to relate the probability of each event by itself, $P(A)$, $P(B)$, and the probability of the first event OR the second event $P(A \text{ or } B)$. Explain why your formula works.

ACTIVITY 3.2 Calculating Compound Probabilities with *Or*

A new holiday—Probability Day—is going to be celebrated at your school. It may be celebrated on any of the first 3 days of any month. The problem now is to choose which day it will fall on. Of course, the day will be selected at random. First the month will be selected and then the day.

1. How many dates for Probability Day are in the sample space? Explain how you determined the answer.

2. Construct a model to represent the sample space for this situation.

3. Determine the probability that each day is randomly chosen. Explain your reasoning.

   a. January 3rd

b. May 2nd

c. Any specific day and month

4. Study the sample space. Let $P(A)$ represent the probability of randomly choosing a summer month, and let $P(B)$ represent the probability of randomly choosing the first day of the month.

The summer months are June, July, and August.

a. Calculate $P(A)$.

b. Calculate $P(B)$.

c. Calculate $P(A \text{ and } B)$.

d. Calculate $P(A \text{ or } B)$. Use the organized list you created to explain your answer.

5. Use your answers from Question 4. Describe how you can calculate the probability of randomly choosing a summer month or the first day of a month without using a tree diagram or organized list.

The **Addition Rule for probability** states: "The probability that Event *A* occurs or Event *B* occurs is the probability that Event *A* occurs plus the probability that Event *B* occurs minus the probability that both *A* and *B* occur."

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

6. What is the probability of randomly choosing a summer month or a winter month?

The winter months are December, January, and February.

Remember . . .
Disjoint sets are sets with no outcomes in common.

7. Lucas says that because the summer month outcomes and the winter month outcomes are disjoint sets, the probability that one event or the other occurs is just the sum of the probabilities. It's not necessary to subtract the probability of both events. Is Lucas correct? Explain why or why not.

ACTIVITY 3.3

Dependent Events with *Or*

You have seen that you can use the Addition Rule for probability to determine the probability that one or another of two independent events occurs. Can you also use this rule to determine the probability of one or another of two dependent events occurring? Let's find out!

Suppose six cards are taken from a set of cards and that 3 of the cards have a yellow dot on the back, 1 of the cards has a purple dot on the back, and 2 of the cards have a blue dot on the back. You are asked to randomly select two cards, one at a time.

1. Will the actions "choosing the first card" and "choosing the second card" create independent events or dependent events?

2. Does the action "choosing the first card" affect the action of "choosing the second card"? When yes, how? Explain your reasoning.

3. Write an organized list to represent the sample space for this situation.

4. Use the sample space to determine the probability of randomly choosing a yellow card first or a blue card second.

   a. Describe the events in this probability situation.

   b. Draw a circle around the outcomes that match the first event and a rectangle around the outcomes that match the second event. How can you describe the outcomes that are circled and have a rectangle around them in the sample space?

   c. What is the probability of randomly choosing a yellow card first, $P(A)$?

Remember . . .
When only the second event is specified, you can consider it as an independent event and determine its probability without considering the first event.

   d. What is the probability of randomly choosing a blue card second, $P(B)$?

e. What is the probability of randomly choosing a yellow card first AND a blue card second, $P(A \text{ and } B)$?

f. What is the probability of randomly choosing a yellow card first OR a blue card second, $P(A \text{ or } B)$?

5. Camilla wrote an explanation for randomly choosing a blue card second. Explain why Camilla's reasoning is incorrect.

Camilla

Because the actions are dependent in this problem, the probability of choosing a blue card second is $\frac{2}{5}$. There are 5 cards left after the first pick, and 2 of the cards left are blue.

6. Isaiah wrote an explanation for randomly choosing a yellow card first and a blue card second. Explain why Isaiah's reasoning is incorrect.

> Isaiah
>
> In this problem, the probability of choosing a yellow card first is $\frac{1}{2}$, and the probability of choosing a blue card second is $\frac{1}{3}$. So, the probability of choosing both a yellow card first and a blue card second is $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$.

7. Calculate the probability of randomly choosing a yellow card first or a blue card second, $P(A \text{ or } B)$, using the Addition Rule for probability. Compare this to your answer in Question 4, part (f). What do you notice?

8. Determine the probability of choosing a yellow card first or a yellow card second using the Addition Rule for probability. Verify your answer using the sample space in Question 3.

9. Does the Addition Rule for probability apply to independent events as well as dependent events? Explain your reasoning.

Remember . . .
Look back at your process. Is there a more efficient strategy?

Talk the Talk

Don't Compound the Problem

In this lesson and the previous lesson, you investigated the probabilities of different kinds of compound events.

1. Complete the graphic organizer to record examples of the types of compound events you have studied. For each type of compound event, describe the methods you used to determine the probability.

Independent Events *P*(*A* and *B*)

Scenario:

Question:

Method:

Independent Events *P*(*A* or *B*)

Scenario:

Question:

Method:

Compound Events

Scenario:

Question:

Method:

Dependent Events *P*(*A* and *B*)

Scenario:

Question:

Method:

Dependent Events *P*(*A* or *B*)

Lesson 3 Assignment

Write

1. In symbols, what is the Addition Rule for probability?

2. When should you use the Addition Rule for probability?

Remember

When two compound events are combined by the word *or*, you can add the probabilities of each event occurring separately and subtract the probability of both events occurring.

Practice

1. Two decks of cards are used for a game.

   a. A player chooses one card from Deck A and one card from Deck B. What is the probability that the player will choose a C2 card from the first deck or a C6 card from the second deck?

   b. A player chooses one card from Deck A and one card from Deck B. What is the probability that the player will choose a C3 card from the first deck or a C5 card from the second deck?

Deck A

Lose a Turn	Take 2 Tokens	Take an Extra Turn
5 cards	4 cards	3 cards
C1	C2	C3

Deck B

Go Back 2 Spaces	Go Back 1 Space	Go Ahead 2 Spaces	Go Ahead 1 Space
5 cards	4 cards	3 cards	2 cards
C4	C5	C6	C7

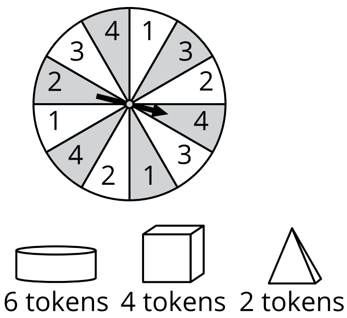
Lesson 3 Assignment

c. A player chooses two cards from Deck A. What is the probability that the player will choose a C1 card first or a C2 card second?

d. A player chooses two cards from Deck B. What is the probability that the player will choose a C5 card first or a C4 card second?

2. Consider the spinner and the set of tokens shown in the figure.

a. A player spins the spinner one time and then randomly chooses a token. What is the probability that the spinner will land on a 2 or the player will choose a pyramid?



b. A player spins the spinner two times. What is the probability that the spinner will land on a number greater than 1 the first time or on a number greater than 2 the second time?

c. A player spins the spinner one time and then randomly chooses a token. What is the probability that the spinner will not land on a 2 or the player will not choose a disk?

Lesson 3 Assignment

d. A player spins the spinner two times. What is the probability that the spinner will land on a 1 the first time or on a 4 the second time?

e. A player spins the spinner one time and then randomly chooses a token. What is the probability that the spinner will land on a 2 or the player will choose a cube?

Prepare

The local diner serves five different kinds of pie: apple, blueberry, lemon, rhubarb, and peach.
They serve four flavors of ice cream: vanilla, strawberry, chocolate, and tangerine.

1. What is the probability of a customer ordering peach pie and ordering vanilla ice cream?

2. What is the probability of a customer ordering peach pie or ordering vanilla ice cream?

4 Calculating Compound Probability

OBJECTIVES

- Calculate compound probabilities with and without replacement.
- Interpret probabilities in real-world situations and communicate findings in writing.

You have calculated compound probabilities for compound events combined by the word *and* and the word *or.*

How can you determine when compound events are independent or dependent in order to correctly calculate compound probabilities?

Getting Started

It's All in the Cards

You will often see the phrases "with replacement" and "without replacement" in probability problems. These phrases refer to whether or not the total sample space changes from the first to the last event.

For example, when you draw a marble from a bag, record the color, and then replace the marble, the sample space for the next event remains the same. When you do not replace it, the sample space decreases.

A standard deck of playing cards contains 52 cards. There are 4 suits: spades, clubs, hearts, and diamonds. Each suit contains 13 cards: an ace, the numbers 2–10, and three face cards, which are the Jack, Queen, and King.

A pair is 2 cards with the same number or face. Two aces is an example of a pair. A three of a kind is 3 cards with the same number or face. Three Jacks is an example of a three of a kind.

1. Calculate each of the following probabilities for randomly drawing 2 cards from a standard deck of playing cards, one at a time, *with* replacement. Explain your reasoning.

   a. An ace first and a ten second

b. An ace first or a ten second

2. Calculate each probability for drawing 2 cards, one at a time, *without* replacement.

   a. An ace first and a ten second

   b. An ace first or a ten second

ACTIVITY 4.1

More Than Two Compound Events

You can use the same methods to calculate the probabilities for more than two compound events.

1. Calculate the probability of drawing three of a kind when drawing 3 cards, one at a time, *with* replacement.

2. Calculate the probability of drawing three of a kind when drawing 3 cards, one at a time, *without* replacement.

3. What kinds of events generally correspond to situations "with replacement"? "Without replacement"? Why?

ACTIVITY 4.2

Compound Events Without Repetition

A homeowner's association has 25 members. The association needs to establish a three-member committee.

You can assume that, in this situation, every member of the homeowner's association has a different first name.

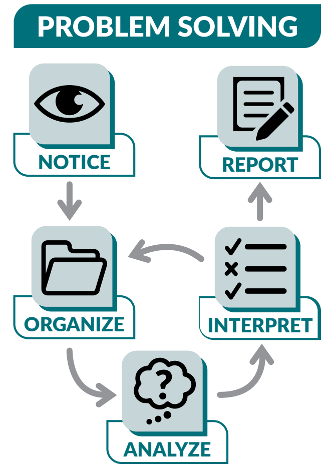
1. Calculate the probability that Carlos, Parker, and Avery are randomly selected to serve on the committee.

2. Calculate the probability that Carlos is randomly selected as president, Parker as vice president, and Avery as treasurer.

3. What is the difference between the situations in Questions 1 and 2?

ACTIVITY 4.3

And or Or



The menu at a local restaurant offers the items shown. A dinner consists of one selection from each category.

Appetizers: Shrimp, Veggies, Avocado Dip, Soup, Stuffed Mushrooms
Salads: Garden, Caesar, Pasta, House
Entrées: Steak, Pizza, Ravioli, Meatloaf, Chicken, Flounder, Spaghetti, Pork, Ham, Shrimp
Desserts: Ice Cream, Cookies, Fruit, Chocolate Cake, Pie, Cheese Cake, Sorbet

1. How many different dinners exist?

2. What is the probability that a patron selects chicken?

3. What is the probability that a patron selects meatloaf and chocolate cake?

4. What is the probability that one patron selects steak or another patron selects pie?

5. What is the probability that a patron selects meatloaf? Any entrée except meatloaf?

6. What is the probability that a patron selects flounder, Caesar salad, and fruit?

7. What is the probability that a patron selects pizza, cookies, and any salad except a garden salad?

8. What is the probability that a patron selects soup, chicken, any salad, and any dessert?

Talk the Talk

Do You Get It?

A pack of 200 trivia cards is divided evenly into five categories: sports, music, movies, books, and occupations.

1. Suppose you randomly choose two cards with replacement. What is the probability of choosing two cards in the sports category?

2. Suppose you choose three cards without replacement. What is the probability of drawing three cards in the sports category?

Lesson 4 Assignment

Write

Describe in your own words the effects of the phrases "with replacement" and "without replacement" on sample space in a compound probability problem.

Remember

Situations "with replacement" generally involve independent events. In these types of situations, the first event does not affect subsequent events. Situations "without replacement" generally involve dependent events. In these types of situations, the first event affects each subsequent event.

Practice

1. A game includes a deck of cards with an animal picture on each card. The table shows the numbers of each type of card. Suppose each time a card is chosen, the card is replaced before another card is chosen.

Number of Cards	Animal on Card
8	lion
6	giraffe
10	monkey
12	elephant
4	panda

   a. A child draws out two cards. What is the probability that the first card will have a monkey on it and the second card will have an elephant on it?

   b. A child draws out two cards. What is the probability that the first card will have a lion on it or the second card will have a giraffe on it?

   c. A child draws out two cards. What is the probability that the second card will have a panda on it?

Lesson 4 Assignment

d. A child draws out three cards. What is the probability that the first card will have a lion on it and the third card will have a monkey on it?

e. A child draws out five cards. What is the probability that they will each have a different animal on them?

2. Consider the same game and deck of cards from Question 1. This time, when a card is chosen, it is not replaced in the deck.

   a. A child draws out two cards. What is the probability that the first card will have an elephant on it and the second card will have a lion on it?

   b. A child draws out two cards. What is the probability that the first card will have a monkey on it or the second card will have a panda on it?

   c. A child draws out three cards. What is the probability that the second card will have a lion on it?

Lesson 4 Assignment

d. A child draws out two cards. What is the probability that the first card will have a panda on it or the second card will have a giraffe on it?

e. A child draws out three cards. What is the probability that the second and third cards will display elephants?

f. A child draws out two cards. What is the probability that the first card will have a lion on it or the second card will have a monkey on it?

Prepare

A quiz in a magazine contains five true-or-false questions. The questions are written in a language you do not recognize.

1. What is the probability of guessing the correct answers to all five questions?

2. What is the probability of guessing only one question correctly?

Name: ______________________

Independence and Conditional Probability

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me,** 2 represents **I am building proficiency of the skill,** and 3 represents **I have demonstrated proficiency of the skill**.

I can demonstrate an understanding of the standards in the *Independence and Conditional Probability* topic by:

TOPIC 1: *Independence and Conditional Probability*	Beginning of Topic	Middle of Topic	End of Topic
defining *event*, *sample space*, *union*, *intersection*, and *complement*.	◯	◯	◯
understanding events as subsets of a sample space based on the union, intersection, or the complement of other events.	◯	◯	◯
calculating the probability of an event.	◯	◯	◯
defining and identifying independent events.	◯	◯	◯
explaining that for two independent events, the probability of the events occurring together is the product of the probability of each event.	◯	◯	◯
predicting whether two events are independent by determining whether $P(A \text{ and } B) = P(A) \cdot P(B)$.	◯	◯	◯
applying the compound probability rule involving *and* for the intersection of two independent or dependent events.	◯	◯	◯

continued on the next page

TOPIC 1 SELF-REFLECTION *continued*

TOPIC 1: *Independence and Conditional Probability*	Beginning of Topic	Middle of Topic	End of Topic
interpreting a probability based on a given situation.			
applying the Addition Rule for probability to determine the probability of the union of two events.			

At the end of the topic, take some time to reflect on your learning and answer the following questions.

1. Describe a new strategy you learned in the *Independence and Conditional Probability* topic.

2. What mathematical understandings from the topic do you feel you are making the most progress with?

3. Are there any concepts within this topic that you find challenging to understand? How do you plan to improve your understanding of these concepts?

TOPIC 1 SUMMARY

Independence and Conditional Probability Summary

LESSON 1 Compound Sample Spaces

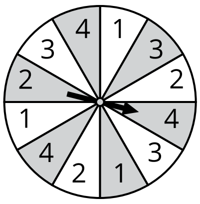
An **outcome** is a result of an experiment. The **sample space** is the set of all the possible outcomes of an experiment. An **event** is an outcome or set of outcomes in a sample space.

The **probability** of an event is the ratio of the number of desired outcomes to the total number of possible outcomes. The probability of event *A* is $P(A) = \frac{\text{desired outcomes}}{\text{possible outcomes}}$.

It is often helpful to construct a model when analyzing situations involving probability. A **probability model** lists the possible outcomes and the probability for each outcome.

In a probability model, the sum of the probabilities must equal 1. In a **uniform probability model**, the probabilities for each outcome are equal. When the probabilities of the outcomes are not all equal, the model is a **non-uniform probability model**.

For example, consider the spinner shown. For the experiment of spinning the spinner, the numbers on the sections of the spinner are the outcomes. The sample space of spinning the spinner is {1, 2, 3, 4}. The probability of spinning the number 1 is $P(1) = \frac{3}{12} = \frac{1}{4}$. The uniform probability model shows the probability of each outcome.



Outcomes	1	2	3	4
Probability	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

The **complement of an event** is an event that contains all the outcomes in the sample space that are not outcomes in the event. If *E* is an event, then the complement of *E* is often denoted as $\overline{E}$ or E^c.

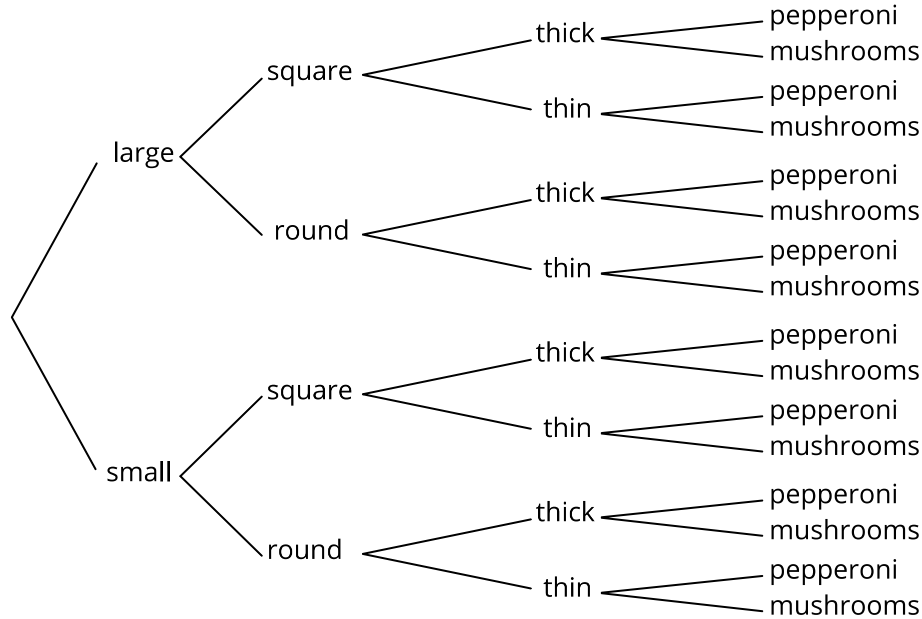
KEY TERMS

- outcome
- sample space
- event [evento]
- probability [probabilidad]
- probability model [modelo de probabilidad]
- uniform probability model [modelo de probabilidad uniforme]
- non-uniform probability model [modelo de probabilidad no uniform]
- complement of an event [complemento de un evento]
- tree diagram
- organized list [lista organizada] set
- element [elemento]
- disjoint sets
- intersecting sets
- union of sets
- independent events [eventos independientes]
- dependent events [eventos dependientes]
- Fundamental Counting Principle [Principio fundamental de conteo]
- compound event [evento compuesto]
- compound probability rule involving *and* [regla de probabilidad compuesta con “y”]
- conditional probability [probabilidad condicional]
- Addition Rule for Probability

For example, the probability of rolling a 1 or a 2 on a six-side number cube is $P(1 \text{ or } 2) = \frac{1}{3}$. The complement of rolling a 1 or a 2 on a six-sided number cube is rolling a 3, 4, 5, or 6, so $P(1 \text{ or } 2^C) = \frac{2}{3}$.

A **tree diagram** is a visual model for determining the sample space of multiple events.

For example, the sample space for all of the possible types of pizza specials that can be made at a pizzeria is be modeled by the tree diagram shown.



An **organized list** is a visual model for determining the sample space of events. You can abbreviate the names of the outcomes as long as a key is provided.

For example, an organized list for the pizza tree diagram could use the following abbreviation key: lg-large, sm-small, sq-square, ro-round, thk-thick, thn-thin, pepp-pepperoni, mush-mushroom. The resulting list would be as follows:

Lg-sq-thk-pepp	Lg-ro-thk-pepp	Sm-sq-thk-pepp	Sm-ro-thk-pepp
Lg-sq-thk-mush	Lg-ro-thk-mush	Sm-sq-thk-mush	Sm-ro-thk-mush
Lg-sq-thn-pepp	Lg-ro-thn-pepp	Sm-sq-thn-pepp	Sm-ro-thn-pepp
Lg-sq-thn-mush	Lg-ro-thn-mush	Sm-sq-thn-mush	Sm-ro-thn-mush

A **set** is a collection of items. If x is a member of set B, then x is an **element** of set B. Two or more sets are **disjoint sets** if they do not have any common elements. Two or more sets are **intersecting sets** if they do have common elements. The **union of sets** is a set formed by combining all the members of the sets. A member may be listed only once.

For example, consider the following sets.

Disjoint Sets:

Let *N* represent the set of 9th grade students. Let *T* represent the set of 10th grade students. The set of 9th grade students and the set of 10th grade students are disjoint because the two sets do not have any common elements. Any student can be in one grade only.

Intersecting Sets:

Let *V* represent the set of students who are on the volleyball team. Let *M* represent the set of students who belong to the math club. Harper is on the volleyball team and belongs to the math club. The set of students who are on the volleyball team and the set of students who are in the math club are intersecting because you know they have at least one common element, Harper.

The Union of Sets:

Let *B* represent the set of students in the 11th grade band. Let *C* represent the set of students in the 11th grade chorus. The union of these two sets would be all the students in the 11th grade band or the 11th grade chorus. A student in both would only be listed once.

Independent events are events for which the occurrence of one event has no impact on the occurrence of the other event.

Dependent events are events for which the occurrence of one event has an impact on the occurrence of subsequent events.

For example, consider a jar which contains 1 blue marble, 1 green marble, and 2 yellow marbles.

Independent Events:

You randomly choose a yellow marble, replace the marble in the jar, and then randomly choose a yellow marble again. The event of choosing a yellow marble first does not affect the event of choosing a yellow marble second because the yellow marble chosen first is replaced in the jar. The events of randomly choosing a yellow marble first and randomly choosing a yellow marble second are independent events.

Dependent Events:

You randomly choose a yellow marble without replacing the marble in the jar, and then randomly choose a yellow marble again. The event of choosing a yellow marble first does affect the event of choosing a yellow marble second because the yellow marble chosen first is not replaced in the jar. The events of randomly choosing a yellow marble first and randomly choosing a yellow marble second are dependent events.

The *Fundamental Counting Principle* is used to determine the number of outcomes in the sample space. The **Fundamental Counting Principle** states that if an action A can occur in m ways, and for each of these m ways an action B can occur in n ways, then actions A and B can occur in $m \cdot n$ ways. The Fundamental Counting Principle can be generalized to more than two actions that happen in succession. If for each of the m and n ways A and B occur, there is also an action C that can occur in s ways, then Actions A, B, and C can occur in $m \cdot n \cdot s$ ways.

For example, if you were to visit an ice cream store that sells either chocolate, vanilla, or strawberry ice cream in a cone or cup with or without sprinkles, you would have $3 \cdot 2 \cdot 2 = 12$ possible outcomes.

LESSON 2 Compound Probability with *And*

A **compound event** is an event that consists of two or more events.

The **compound probability rule involving *and*** states: "If Event A and Event B are independent events, then the probability that Event A happens and Event B happens is the product of the probability that Event A happens and the probability that Event B happens, given that Event A has happened."

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

If Event A and Event B are dependent, then the probability that Event A happens and Event B happens is the product of the probability that Event A happens and the probability that Event B happens, given that Event A has already happened. If two or more of the sets are disjoint, then their probability will be zero, as the two sets will have no elements in common.

For example, suppose you have 2 red, 1 blue, and 3 green socks in a drawer. You reach into the drawer without looking and choose a sock, replace it, and then choose another sock. You choose a total of 2 socks.

These are independent events, so use the compound probability rule to calculate. $P(\text{green and red}) = P(\text{green}) \cdot P(\text{red}) = \frac{3}{6} \cdot \frac{2}{6} = \frac{6}{36} = \frac{1}{6}$

Now suppose you reach into the drawer without looking and choose a sock, do not replace it, and then choose another sock. You choose a total of 2 socks.

These are dependent events, so $P(A \text{ and } B) = P(A) \cdot P(B, \text{ given } A)$. Therefore, $P(\text{green and red}) = \frac{3}{6} \cdot \frac{2}{5} = \frac{6}{30} = \frac{1}{5}$.

A **conditional probability** is the probability of event B, given that event A has already occurred. Consider a number cube that is rolled. The probability of rolling a 4 or less on the second roll of the number cube (B), given that a 5 is rolled first (A), is an example of a conditional probability.

The notation for conditional probability is $P(B|A)$, which reads, "the probability of event B, given event A."

LESSON 3 Compound Probability with *Or*

In a compound event that is related by the word *or*, there can be possible outcomes that are in the sample space for each event. These outcomes should only be counted once when determining the compound probability.

The **Addition Rule for probability** states: "The probability that Event A occurs or Event B occurs is the probability that Event A occurs plus the probability that Event B occurs minus the probability that both A and B occur."

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Keep in mind that if two events are disjoint, then their probability will be $P(A \text{ or } B) = P(A) + P(B)$.

For example, consider determining the probability of drawing a jack or a club from a deck of cards. Let A = a jack and B = a club.

$$P(\text{jack or club}) = P(\text{jack}) + P(\text{club}) - P(\text{jack and club})$$
$$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

LESSON 4 Calculating Compound Probability

You can use the same methods to calculate the probabilities for more than two compound events.

You will often see the phrases "with replacement" and "without replacement" in probability problems. These phrases refer to whether or not the total sample space changes from the first to the last event. For example, if you draw a marble from a bag, record the color and then replace it, the sample space for the next event remains the same. If you do not replace it, then the sample space decreases.

For example, the probability for drawing 3 of a kind of a given number or face card from a deck of cards with replacement is

$P(\text{3 of a kind}) = \frac{4}{52} \cdot \frac{4}{52} \cdot \frac{4}{52} = \frac{1}{2197}$.

The probability of drawing 3 of a kind without replacement is

$P(\text{3 of a kind}) = \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50} = \frac{1}{5525}$.

GEOMETRY

TOPIC 2

In darts, players score points by throwing darts inside the scoring area. A dart that lands outside the scoring area scores no points.

Computing Probabilities

1 Compound Probability for Data Displayed in Two-Way Tables

OBJECTIVES

- Determine probabilities of compound events for data displayed in two-way tables and discuss with a group or partner.
- Determine relative frequencies of events.

KEY TERMS

- two-way table
- frequency table
- two-way frequency table
- categorical data
- relative frequency
- two-way relative frequency table

You have previously learned how to calculate probability as the ratio of desired outcomes divided by total outcomes. You have also calculated the probability for compound events that are independent and dependent.

How can you determine both simple and compound probabilities from data relayed by a two-way table?

Getting Started

Some Sum

When playing board games, it is common to roll two number cubes and calculate their sum.

1. Determine the number of possible outcomes when two number cubes are rolled once. Explain your conclusion.

2. Write the sample space for rolling two number cubes one time.

A **two-way table** shows the relationship between two data sets; one data set is organized in rows, and the other data set is organized in columns.

3. Complete the two-way table to represent all the possible sums that result from rolling two number cubes one time.

2nd Number Cube

1st Number Cube	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

Review the definition of **represent** in the Academic Glossary.

4. What are the possible sums? Explain your reasoning.

5. Use the two-way table to answer each question.

A **frequency table** shows the frequency of an item, number, or event appearing in a sample space.

a. Complete the frequency table shown.

Outcome (Sum of the Number Cubes)	Frequency

b. What is the sum of the frequencies? Why?

c. Which sum appears most often in the frequency table? Why do you think this sum appears the most?

d. Using the frequency table, can you determine the probability of rolling a sum of 7? Why or why not?

e. What is the probability of rolling an odd sum? Explain your reasoning.

f. What is the probability of rolling a sum less than 7? Explain your reasoning.

ACTIVITY 1.1

Converse of the Compound Probability Rule

You have learned that when two events A and B are independent, the probability of both events occurring, $P(A \text{ and } B)$, is $P(A) \cdot P(B)$.

The converse is also true: If the probability of two events A and B, $P(A \text{ and } B)$, occurring together is $P(A) \cdot P(B)$, then the two events are independent.

1. Use the converse of the compound probability rule involving *and* the sample space you created in the previous activity to determine whether the events are independent. Explain your reasoning.

    a. A result of 3 on the first number cube and a result of 3 on the second number cube.

    b. A result of two 3s and a sum of 6.

c. A result of 6 on the first number cube and a sum less than 7.

d. A result of any number on the first number cube and a sum less than 7.

ACTIVITY 1.2

Analyzing Two-Way Tables

A **two-way frequency table** shows the number of data points and their frequencies for two variables. One variable is divided into rows, and the other is divided into columns.

A recent study estimates that between 70% to 90% of the world's population is right-handed. Another study suggests that almost 90% of athletes are right-handed. Yet another study shows that left-handed people have a higher percentage of participation in individual sports like wrestling or golf.

Favored hand and sports participation are examples of data sets that can be grouped into categories. These data are **categorical data**. *Categorical data* are data for which each piece of data fits into exactly one of several different groups or categories.

Mr. Hernandez's math class decides to conduct a survey of 63 people to determine which hand is favored and whether the hand favored affects whether a person participates in certain types of sports or no sports at all. The results are shown in the two-way table.

Sports Participation

Favored Hand		Individual	Team	Does Not Play	Total
	Left	3	13	8	
	Right	6	23	4	
	Mixed	1	3	2	
	Total				

1. Name the two variables displayed in the table.

2. Calculate the total for each row and each column in the table.

Review the definition of **describe** in the Academic Glossary.

3. Use the two-way frequency table to answer each question. Describe how to use the rows and columns to determine each answer.

   a. How many of those surveyed are left-handed and do not play sports?

   b. How many of those surveyed are right-handed or play team sports?

   c. How many of those surveyed play any kind of sport?

4. Camilla claims that the figures in the studies are correct, because there were more right-handed people who participated in the survey than left-handed and mixed-handed people. Do you agree with Camilla? Explain your reasoning.

Because there are three types of handed people who participated in the survey, Mr. Hernandez's students cannot claim that the studies' figures are correct by simply looking at the frequencies. Instead, they must determine the *relative frequencies*. A **relative frequency** is the ratio of occurrences within a category to the total number of occurrences. To determine the ratio for each category, determine the part to the whole for each category.

A **two-way relative frequency table** displays the relative frequencies for two categories of data.

5. Use the survey results to calculate the relative frequency of each entry. Record the results in the two-way relative frequency table. Write each result as a fraction and as a percent rounded to the nearest tenth.

	Individual	Team	Does Not Play	Total
Left	$\frac{3}{63} \approx 4.8\%$			
Right				
Mixed				
Total				

Ask Yourself . . .
Do your percents add up to 100%? If not, why not?

6. Suppose a student is randomly selected from the survey. Determine each probability.

a. The student is left-handed.

b. The student is right-handed.

c. The student participates in some kind of sport.

7. Suppose a student is randomly selected from the survey. Determine each probability. Explain how you calculated your answer.

   a. The student is left-handed and participates in individual sports.

   b. The student is right-handed and participates in team sports.

   c. The student is left-handed and participates in individual sports or right-handed and participates in team sports.

8. Suppose a student is randomly selected from the survey. Determine each probability. Explain how you calculated your answer.

   a. The student is left-handed or participates in team sports.

Ask Yourself . . .
How can you use the probability rules you have learned to answer each question?

   b. The student does not play sports or is mixed-handed.

9. How do the results of the survey compare with the estimate that between 70% and 90% of the world's population is right-handed?

10. How do the results of the survey compare with the estimate that almost 90% of athletes are right-handed?

    Three students attempted to answer this question. Their work and reasoning is shown.

Emily

Out of the 63 people surveyed, 6 + 23, or 29, are both right-handed AND play a sport.

$\frac{29}{63} \approx 46\%$

This is much less than the 90% estimate.

Logan

Out of the 33 people in the survey who are right-handed, 6 + 23, or 29, play a sport.

$\frac{29}{33} \approx 87.9\%$

This is close to the 90% estimate.

Valentina

Out of the 10 + 39, or 49, people in the survey who play a sport, 6 + 23, or 29, are right-handed.

$\frac{29}{49} \approx 59.2\%$

This is much less than the 90% estimate.

Explain why Valentina's reasoning is correct, and describe what Emily and Logan did to get incorrect answers.

ACTIVITY 1.3

Determining Compound Probabilities from Relative Frequencies

You may prefer to use percents when comparing probabilities. Writing probability in fractional form is helpful because you can see the desired outcomes and the total outcomes.

A survey was conducted in one class about students' favorite after-school activities. The frequencies of 9th graders' and 10th graders' responses are shown in the table.

What is your favorite thing to do when you are not in school?

	9th Grade		10th Grade		Total	
Activity	Freq.	Rel. Freq.	Freq.	Rel. Freq.	Freq.	Rel. Freq.
Listen to Music	5		3		8	
Watch TV	5		4		9	
Participate in Sports	1		2		3	
Play Video Games	3		0		3	
Surf the Internet	2		2		4	
Shop	0		1		1	
Read	0		1		1	
Other	1		0		1	
Total	17		13		30	

1. Calculate the relative frequency of each entry. Record the results in the two-way relative frequency table. Write each result as a fraction and as a percent rounded to the nearest tenth.

2. Suppose a student is randomly selected from the class. Determine each probability. Explain how you calculated your answer.

   a. The student is a 9th grader.

   b. The student watches TV as their favorite thing to do when not in school.

   c. The student is a 9th grader who plays video games.

   d. The student is a 10th grader who listens to music.

   e. The student is a 10th grader and watches TV for their favorite pastime.

   f. The student is a 10th grader or watches TV for their favorite pastime.

3. Is being a 9th grader independent of listening to music as a favorite after-school activity? Explain your reasoning.

Talk the Talk

Statistically Speaking

1. Write the term that best completes each statement.

   a. A two-way table is a table that shows the relationship between two data sets, one organized in ________ and one organized in ________.

   b. A ________ table is a table that shows how often each item, number, or event appears in a sample space.

   c. A ____________________ shows the number of data points and their frequencies for two variables.

   d. Data that can be grouped into categories, such as eye color and gender, are called ________ data.

   e. A relative frequency is the ratio of occurrences within a category to the __________ of occurrences.

   f. A two-way relative frequency table displays the ________________ for two categories of data.

2. Of the students in Mei's homeroom, 11 students have brown hair, 7 have black hair, 5 have auburn hair, 4 have blonde hair, and 3 have red hair. Calculate each relative frequency. Round to the nearest thousandth, when necessary.

   a. black hair

   b. auburn or red hair

c. not brown hair

d. brown and black hair

The two-way frequency table shows the number of students from each grade who plan to attend this year's homecoming football game.

		Grade				
		Freshmen	Sophomores	Juniors	Seniors	Total
Homecoming Game Attendance	Attending Homecoming Game	31	28	25	32	116
	Not Attending Homecoming Game	17	24	11	6	58
	Total	48	52	36	38	174

3. Calculate the relative frequency of the entries in the two-way table. Round each relative frequency to the nearest tenth of a percent, when necessary.

   a. A freshman going to the homecoming game.

   b. A sophomore not going to the homecoming game.

   c. Students from all grades going to the homecoming game.

   d. Students who are not juniors and are not attending the homecoming game.

Lesson 1 Assignment

Write

1. A ________ is a table that shows the relationship between two data sets, one organized in rows and one organized in columns.

2. A ________ is a table that shows the frequency of an item, number, or event appearing in a sample space.

3. A ________ shows the number of data points and their frequencies for two variables.

4. A ________ is the ratio of occurrences within a category to the total number of occurrences.

5. A table that displays the relative frequencies for two categories of data is called a ________.

Remember

Two-way tables can be used to determine the probabilities of compound events. The converse of the compound probability rule involving *and* states: "If the probability of two events A and B occurring together is $P(A) \cdot P(B)$, then the two events are independent."

Practice

Angelina rolls two number cubes.

1. Complete a two-way table to represent all the possible products of the numbers rolled on two number cubes.

	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

Lesson 1 Assignment

2. Create a frequency table with the product of the numbers rolled on the two number cubes and their frequency.

Product																		
Frequency																		

3. Use your two-way table and frequency table to answer each question.

   a. What is the probability of rolling an odd product?

   b. What is the probability of rolling a product less than 10?

4. Use the converse of the compound probability rule involving *and* to determine whether the events are independent. Explain your reasoning.

   a. Rolling a 2 on the first number cube and rolling a 6 on the second number cube.

Lesson 1 Assignment

b. Rolling a 3 on the first number cube and a product equal to 20.

A survey was taken of 24 households on Oak Street to compare the number of cars registered to the household and the number of people who live in the house. The responses are shown in the table.

5. Complete the table. Write each result as a fraction and a percent rounded to the nearest tenth.

		Number of People in Household								Total	Rel.Freq.
		2		3		4		5			
		Freq.	Rel. Freq.	Freq.	Rel. Freq.	Freq.	Rel. Freq.	Freq.	Rel. Freq.		
Number of Cars	1	2		1		1		0			
	2	4		3		5		3			
	3	0		1		3		1			
Total											

Lesson 1 Assignment

6. Suppose a house on Oak Street is randomly chosen. Determine each probability.

    a. The house has two cars.

    b. The house has 3 people in the household.

    c. The house has 4 people in the household and 3 cars.

    d. The house has 5 people in the household and 2 cars.

Prepare

A teacher administers a random survey to determine the probabilities of a middle school student with siblings and a middle school student without siblings having their own smartphone. She organizes the data in a two-way frequency table.

1. What could be the variables in her table?

2. What could be the groups within each variable in her table?

2 Conditional Probability

OBJECTIVES

- Use conditional probability to determine the probability of an event given that another event has occurred, and explain the reasoning in writing.
- Use conditional probability to determine whether or not events are independent.

You have calculated probabilities when there was replacement from trial to trial and when there was no replacement.

How does the phrase "without replacement" relate to a conditional probability?

Getting Started

Rollin', Rollin', Rollin'

The two-way table represents a situation in which two number cubes are rolled, one at a time. The sums of the two rolls are shown.

First Roll \ Second Roll	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Suppose Event *A* is rolling a 5 on the first roll, and Event *B* is rolling a 4 or less on the second roll.

1. Determine *P*(*A* and *B*).

   a. What is the probability of rolling a 5 on the first roll, *P*(*A*)? On the two-way table, shade the desired outcomes, and draw a border around the possible outcomes. Explain your reasoning.

   b. What is the probability of rolling a 4 or less on the second roll, *P*(*B*)? On the two-way table, shade the desired outcomes, and draw a border around the possible outcomes. Explain your reasoning.

c. What is the probability of rolling a 5 first AND a 4 or less second, $P(A \text{ and } B)$? Explain your reasoning.

ACTIVITY 2.1 Determining Conditional Probability

The notation for conditional probability is $P(B|A)$, which reads, "the probability of Event B, given Event A."

Recall that a conditional probability is the probability of Event B, given that Event A has already occurred. The probability of rolling a 4 or less on the second roll of the number cube (B), given that a 5 is rolled first (A), is an example of a conditional probability.

1. What is the probability of rolling a 4 or less on the second roll, given that a 5 is rolled first, $P(B|A)$? Shade the desired outcomes, and draw a border around the possible outcomes. Explain your reasoning.

Second Roll

First Roll	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Ask Yourself . . .
When you know that a 5 is rolled first, does that change the possible outcomes?

2. Compare $P(B|A)$ with $P(B)$.

   a. What do you notice?

   b. Do you think Events A and B are independent or dependent events? Explain your reasoning.

3. Suppose Event A is rolling a 5 on the first roll, and Event B is rolling a sum greater than or equal to 8.

   a. What is the probability of rolling a 5 on the first roll, $P(A)$? Shade the desired outcomes, and draw a border around the possible outcomes. Explain your reasoning.

Second Roll

First Roll	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

b. What is the probability of rolling a sum greater than or equal to 8, $P(B)$? In the table in part (a), shade the desired outcomes, and draw a border around the possible outcomes. Explain your reasoning.

c. What is the probability of rolling a 5 first AND a sum greater than or equal to 8, $P(A \text{ and } B)$? Explain your reasoning.

d. What is the probability of rolling a sum greater than or equal to 8, given that a 5 is rolled first, $P(B|A)$? Shade the desired outcomes, and draw a border around the possible outcomes. Explain your reasoning.

Second Roll

First Roll	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

e. Compare $P(B|A)$ with $P(B)$. What do you notice?

f. Do you think events A and B are independent or dependent events? Explain your reasoning.

When two events are independent, $P(B|A) = P(B)$. When two events are dependent, $P(B|A) \neq P(B)$. Thus, two events are independent if and only if $P(B|A) = P(B)$.

ACTIVITY 2.2 The Conditional Probability Formula

Now you can derive the formula for the conditional probability of two independent events, $P(B|A)$.

1. Let A represent the event of rolling a 5 on the first roll. Let B represent the event of rolling a number less than or equal to 4 on the second roll.

Second Roll

First Roll	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

a. Complete the table.

	$P(A)$	$P(B)$	$P(A \text{ and } B)$	$P(B|A)$
Desired Outcomes				
Total Outcomes				
Probability				

Ask Yourself . . .
What observations can you make?

b. Use the table to verify that Event A and Event B are independent.

c. What quantity in the table is equivalent to the number of desired outcomes for $P(B|A)$?

d. What quantity in the table is equivalent to the number of total outcomes for $P(B|A)$?

WORKED EXAMPLE

The conditional probability, $P(B|A)$, for independent events can be represented as $\frac{\text{desired outcomes}}{\text{total outcomes}}$.

$$
\begin{aligned}
P(B|A) &= \frac{\text{desired outcomes}}{\text{total outcomes}} \\
&= \frac{A \text{ and } B}{A} \\
&= \frac{A \text{ and } B}{A} \cdot \frac{\frac{1}{\text{total}}}{\frac{1}{\text{total}}} \\
&= \frac{\frac{A \text{ and } B}{\text{total}}}{\frac{A}{\text{total}}} \\
&= \frac{P(A \text{ and } B)}{P(A)}
\end{aligned}
$$

2. Adriana says the probability of rolling a number less than or equal to 4 on the second roll, given a 5 on the first roll, $P(B|A)$, is equal to the probability of rolling a number less than or equal to 4 on the second roll, $P(B)$, because Events A and B are independent. Is Adriana correct? Explain your reasoning.

3. How is the formula in the Worked Example related to Adriana's statement in Question 2?

ACTIVITY 2.3

Conditional Probability for Dependent Events

Let's see whether the formula works for actions that affect the outcomes of other actions.

Suppose you have 2 red socks, 1 blue sock, and 1 yellow sock in a drawer. You randomly choose a sock without replacing it and then randomly choose a second sock.

1. Write an organized list to represent all of the possible outcomes.

2. What is the probability of randomly choosing a red sock first, $P(A)$?

3. What is the probability of randomly choosing a red sock second, given that a red sock is randomly chosen first, $P(B|A)$?

4. What is the probability of randomly choosing a red sock first and a red sock second, $P(A \text{ and } B)$?

5. Are Events A and B independent or dependent events? Explain your reasoning.

6. Is the probability of randomly choosing a red sock second, given that a red sock is randomly chosen first, $P(B|A)$, equal to the probability of $\frac{P(A \text{ and } B)}{P(A)}$? Explain why or why not.

Talk the Talk

Flaunt That Formula!

Use the conditional probability formula to answer each question.

1. A biology teacher gave her students two tests. The probability that a student received a score of 90% or above on both tests is $\frac{1}{10}$. The probability that a student received a score of 90% or above on the first test is $\frac{1}{5}$, and the probability that a student received a score of 90% or above on the second test is $\frac{1}{2}$.

   a. What is the probability that a student who received a score of 90% or above on the first test also received a score of 90% or above on the second test?

b. Are "scoring 90% or above on the first test" and "scoring 90% or above on the second test" independent or dependent events? Explain your reasoning.

2. The probability of having the gene for color blindness and actually being color blind is $\frac{1}{4000}$. The probability of having the color-blind gene is $\frac{1}{3000}$. Seventy out of 100 people are color blind.

   a. What is the probability of actually having the disease given a positive test?

b. Are having the gene for color blindness and being color blind independent events? Explain your reasoning.

Ask Yourself . . .
What tools or strategies can you use to solve this problem?

3. A basketball player makes two out of two free throws 49% of the time. She makes 70% of her free throws.

   a. What is the probability that she will make the second free throw after making the first?

b. Are making the first free throw and making the second free throw independent or dependent events? Explain your reasoning.

Lesson 2 Assignment

Write

Describe a conditional probability in your own words. Distinguish between independent and dependent events.

Remember

Conditional probabilities are used to determine the probability of an event, given that another event has already occurred, and to determine whether or not events are independent. The conditional probability formula is $P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$.

Practice

1. Suppose you have 3 nickels, 1 quarter, and 1 penny in your pocket. You choose a coin, do not replace it, and then choose a second coin.

   a. Write an organized list to represent all of the possible outcomes.

   b. What is the probability of randomly picking a penny first, $P(\text{penny 1st})$?

   c. What is the probability of picking a quarter first and a nickel second?

   d. What is the probability of picking a nickel second, given that a quarter is picked first?

Lesson 2 Assignment

2. Kayla is selling candy outside the supermarket to raise money for new uniforms for the gymnastics team. The probability of a customer stopping to talk to Kayla and buying some candy is $\frac{2}{9}$. The probability of a customer just stopping to talk to Kayla is $\frac{5}{12}$. Fifty out of the 120 customers at the supermarket bought candy from Kayla.

   a. What is the probability of a customer buying candy from Kayla given that they stopped to talk to her?

   b. Are "a customer talking to Kayla" and "a customer buying candy from Kayla" independent or dependent events? Explain your reasoning.

3. A survey was taken to determine the number of students that own a dog and a cat as a pet. When a student from the survey is chosen at random, the probability that the student owns both a dog and a cat is $\frac{1}{12}$. When a student is chosen at random, the probability that the student owns a dog is $\frac{1}{4}$, and the probability the student owns a cat is $\frac{1}{3}$.

   a. What is the probability that a student chosen at random who owns a dog also owns a cat?

Lesson 2 Assignment

b. Are "owning a dog" and "owning a cat" independent or dependent events? Explain your reasoning.

Prepare

Determine the value of each expression.

1. $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

2. $\frac{8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1}$

3. $\frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(3 \cdot 2 \cdot 1)(3 \cdot 2 \cdot 1)}$

3 Permutations and Combinations

OBJECTIVES

- Use permutations to calculate the size of sample spaces.
- Use combinations to calculate the size of sample spaces.
- Apply reading strategies to use permutations to calculate probabilities.
- Apply reading strategies to use combinations to calculate probabilities.
- Calculate permutations with repeated elements.
- Calculate circular permutations.

KEY TERMS

- factorial
- permutation
- circular permutation
- combination

You have been drawing tree diagrams and making organized lists to determine the total number of arrangements that can occur when arranging elements of a set.

How can you determine the number of arrangements when the number of elements is too large to create a list?

Getting Started

No Strings Attached

Calculating large sample spaces can present several challenges, because it can be too time consuming or impractical to list all of the possible outcomes. Even for a relatively small number of options, listing the sample space can be challenging.

1. Using the first four letters of the alphabet, list all of the three-letter strings, such as DBA, that can be formed without using the same letter twice in one string.

2. How many different strings are possible?

3. For each string, how many possible letters could be first? Second? Third?

4. How can your answer to Question 3 help you to calculate the number of possible three-letter strings?

5. How many different four-letter strings can be made using the first four letters of the alphabet? Explain your reasoning.

Review the definition of **explain your reasoning** in the Academic Glossary.

6. When each letter can be used more than once, how many three-letter strings could you list? How many four-letter strings? Explain your reasoning.

7. When each letter can be used only once, how many 10-letter strings could you list using the first 10 letters of the alphabet? Explain your reasoning.

8. When you can use the entire alphabet, how many three-letter strings can be made when each letter was used only once in each string? Explain your reasoning.

9. Calculate the number of 26-letter strings possible without replacement.

10. In general, when there are *n* letters, how many three-letter strings are possible without repetition?

11. In general, when there are *n* letters, how many three-letter strings are possible with repetition?

ACTIVITY 3.1

Factorials

In 1808, French mathematician Christian Kramp introduced the *factorial*, which could help you to perform some of the calculations in the previous activity. The **factorial** of n, which is written with an exclamation mark as $n!$, is the product of all non-negative integers less than or equal to n. For example, $3! = 3 \cdot 2 \cdot 1 = 6$.

1. Calculate each factorial.

   a. $5! =$

   b. $7! =$

   c. $10! =$

2. Nahimana says that $\frac{10!}{5!}$ is $2!$ because $10 \div 5 = 2$. Is Nahimana correct? When yes, explain Nahimana's reasoning. When no, show how to calculate the correct quotient.

3. Write each expression as a ratio of two factorials.

   a. $(5)(4)(3) =$

   b. $7 \cdot 6 \cdot 5 \cdot 4 =$

WORKED EXAMPLE

You can rewrite fractions involving factorials by dividing out common factors. For example:

$$\frac{8!}{6!} = \frac{8 \cdot 7 \cdot \cancel{6!}^{1}}{\cancel{6!}_{1}}$$

$$= 8 \cdot 7$$

$$= 56$$

4. Determine the value of each expression.

   a. $\frac{8!}{6!} =$

   b. $\frac{11!}{9!} =$

   c. $\frac{7!}{4!} =$

   d. $\frac{7!5!}{8!3!} =$

5. Using factorials, rewrite your answers to the following questions from the Getting Started.

   a. How many different four-letter strings can be made using the first four letters of the alphabet?

   b. When each letter could be used only once, how many 10-letter strings could you list using the first 10 letters of the alphabet?

   c. When you use the entire alphabet, how many three-letter strings can be made when each letter was used only once in each string?

   d. In general, when there are *n* letters, how many three-letter strings are possible without repetition?

ACTIVITY 3.2

Permutations

An ordered arrangement of items without repetition is called a **permutation**.

In Questions 1 through 4 of the Getting Started, the different three-letter strings formed from the first four letters of the alphabet are permutations. There are different notations that are used for the permutations of *r* elements taken from a collection of *n* items:

$$_nP_r = P(n, r) = P_r^n$$

1. Suppose you form two-letter strings from the first four letters of the alphabet without using the same letter twice in one string.

   a. How many choices are there for the first letter? The second letter?

   b. How many different permutations are there of two letters from the first four letters of the alphabet, $_4P_2$?

   c. Write the number of permutations as a ratio of factorials.

2. Suppose you form three-letter strings from any five letters of the alphabet. Write the number of permutations as a ratio of factorials.

3. How many four-letter strings can be formed from the first 10 letters of the alphabet? Write the number of permutations as a ratio of factorials.

4. Analyze the ratios of factorials you wrote in Questions 1 through 3. Let *r* be the number of letters in a permutation. Let *n* be the number of letters to choose from. Write a formula to describe the number of permutations, *P*, of *r* letters chosen from *n* letters.

The formula you wrote in Question 4 is the general formula used to determine the number of permutations given *n*, the total number of elements in a set and *r*, the number of elements in a set you are selecting and arranging.

5. Calculate each permutation using the formula ${}_nP_r = \frac{n!}{(n-r)!}$.

   a. ${}_6P_3 =$

   b. ${}_{10}P_1 =$

   c. ${}_5P_2 =$

Graphing calculators have a permutations key: ${}_nP_r$.

6. A sundae shop sells yogurt sundaes with two layers, four layers, or five layers of yogurt flavors. The shop carries 65 flavors in all.

   a. How many different four-layer sundaes could you create without repeating flavors?

b. Which type of sundae would give you the greatest number of choices when you do not repeat flavors?

7. A briefcase lock has three dials, each with the digits 0–9. The three-digit code that unlocks the briefcase has no digits repeating.

   a. When you tried one code every five seconds, what is the greatest amount of time it could take you to open the briefcase?

   b. Suppose the briefcase has 10 dials, each with the digits 0–9. How many different 10-digit codes are possible with no repeated digits?

   c. What conclusion can you make about the value of 0!? Explain your reasoning.

ACTIVITY 3.3

Permutations with Repeated Elements

Sarah has four books on her shelf: two novels (N_1 and N_2), one science book (S), and one reference book (R).

1. How many permutations of all four books are there? Write an organized list to verify your answer.

2. Suppose that Sarah wants to arrange her books by type. The order of the two novels doesn't matter to her.

   a. Does this change the number of permutations? Explain your reasoning.

b. When necessary, modify the organized list in Question 1 to show the new number of permutations.

3. Suppose Sarah has three novels (N_1, N_2, N_3) and one reference book (R) on her shelf.

   a. How many permutations of all four books are there?

   b. How many permutations are there when the order of the novels doesn't matter? Write an organized list to determine your answer.

4. Suppose all four of Sarah's books are novels (N_1, N_2, N_3, N_4).

   a. How many permutations of all four books are there?

b. How many permutations are there when the order of the novels doesn't matter? Write an organized list to determine your answer.

When the order of all of the elements in an arrangement matters, the arrangement contains one copy of each element. For example, the arrangement of Sarah's books N_1N_2RS contains one copy of each element.

When the order of some or all of the elements in an arrangement does not matter, the arrangement contains two or more copies of one or more of the elements. For example, the arrangement NNRS of Sarah's four books contains two copies of the element N (two novels).

5. Use your answers from Questions 1 through 4 to complete the first three rows of the table.

List of Elements	Number of Permutations (order of novels matters)	Repeated Elements	Factorial of Number of Repeated Elements	Number of Permutations (order of novels doesn't matter)
N_1, N_2, S, R	24			
N_1, N_2, N_3, R	24			
N_1, N_2, N_3, N_4	24			

6. Refer to the table in Question 5 to describe how to calculate the number of permutations when the order of the novels doesn't matter.

7. Write a formula to describe the number of permutations of *n* elements with *k* copies of an element.

Think About . . .
How many repeated elements are there now?

8. Suppose Sarah has two novels and two science books. The order of each type of book doesn't matter. Write an organized list of all the different permutations of Sarah's books.

9. Suppose Sarah has three reference books and two science books. The order of each type of book doesn't matter. Write an organized list of all the different permutations of Sarah's books.

10. Use your answers from Questions 8 and 9 to complete the table.

List of Elements	Number of Permutations (order of repeated elements matters)	First Repeated Element	Factorial of First Number of Repeated Elements	Second Repeated Element	Factorial of Second Number of Repeated Elements	Product of Factorials	Number of Permutations (order of repeated elements does not matter)
N_1, N_2, S_1, S_2							
R_1, R_2, R_3, S_1, S_2							

11. Compare the value that you divided by each time to the number of copies. What do you notice?

12. Write a formula to describe the number of permutations of *n* elements with *k* copies of one element and *h* copies of another element.

13. Suppose Sarah has three novels, a science book, and a reference book. She chooses two books. The order of the novels doesn't matter. How many different permutations of two books are possible?
Diego and Emily tried to solve this problem. Explain why Diego is correct and Emily is incorrect.

Emily

There are 5 elements in the collection, and Sofie chooses 2 with 3 elements repeating, so I think the number of permutations is

$${}_nP_r = \frac{n!}{(n-r)!(r!)} = \frac{5!}{(5-2)!(3!)}$$

$$= \frac{120}{36} \approx 3.3\ldots$$

There are approximately 3.3 permutations.

Diego

The organized list looks like this:

NN NS NR

SN SR

RN RS

There are 7 permutations.

14. Does $\frac{n!}{k!}$ describe the correct number of permutations in the situation in Question 13? Explain your reasoning.

15. Calculate the number of six-letter strings that can be formed from the word *cannon*.

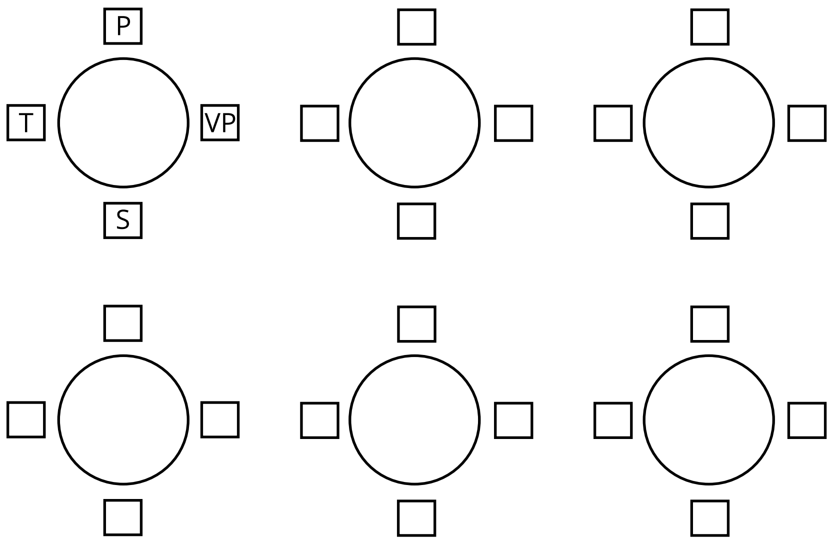
16. Calculate the number of seven-letter strings that can be formed from the word *Alabama*.

ACTIVITY 3.4

Circular Permutations

A club consists of four officers: a President (P), a Vice President (VP), a Secretary (S), and a Treasurer (T).

1. List the different ways that the four officers could be seated at a round table.



2. List the different ways that the officers could be arranged in a line.

3. Which elements from Question 2 are equivalent to the table seating of President, Vice President, Secretary, and Treasurer?

4. Which elements from Question 2 are equivalent to the table seating of President, Secretary, Vice President, and Treasurer?

5. Which elements from Question 2 are equivalent to the table seating of President, Treasurer, Secretary, and Vice President?

6. Based on Questions 3 through 5, how many equivalent elements are included for each seating arrangement? How does this number appear in the original problem?

A **circular permutation** is a permutation in which there is no starting point and no ending point.

The seating arrangement problem is a *circular permutation*. The circular permutation of n objects is $(n - 1)!$.

7. Calculate the number of table arrangements for each number of officers.

   a. Five officers

   b. Six officers

   c. Ten officers

ACTIVITY 3.5

Combinations

The three-letter permutations of DEFG are shown.

$${}_4P_3 = \frac{4!}{(4-3)!} = 24 \text{ permutations}$$

1. Consider the list.

   a. Start at the top left (DEF). Cross out all the outcomes that contain the exact same elements as DEF, but in a different order. One example is DFE.

   b. Move to the next outcome in the first column that is not crossed out, DEG, and repeat the process. Continue until you cannot cross out any outcomes.

The arrangements that you did not cross off are all the *combinations* of three letters from the collection of four letters. A **combination** is an unordered collection of items. Different notations can be used for the combinations of r elements taken from a collection of n elements:

$$\binom{n}{r} = {}_nC_r = C(n, r) = C_r^n$$

Permutations involve ordered items and combinations involve unordered items.

2. Consider the three-letter permutation list of DEFG.

   a. How many combinations of three-letter strings are there, ${}_4C_3$?

b. The notation ${}_4C_3$ represents the number of combinations of three-letter strings formed from the four letters DEFG. Write a value in factorial notation to make the equation true.

$${}_4C_3 = \frac{{}_4P_3}{\boxed{}}$$

3. The two-letter permutations of DEFG are shown.

a. How many combinations of two-letter strings are there, ${}_4C_3$?

DE ED FD GD
DF EF FE GE
DG EG FG GF

$${}_4P_2 = \frac{4!}{(4-2)!} = \frac{24}{2} = 12 \text{ permutations}$$

b. Write a value in factorial notation to make the equation true.

$${}_4C_3 = \frac{{}_4P_2}{\boxed{}}$$

4. The three-letter permutations of WXY are shown.

a. How many combinations of three-letter strings are there, ${}_4C_3$?

WXY XWY YWX
WYX XYW YXW

$${}_3P_3 = \frac{3!}{(3-3)!} = \frac{6}{1} = 6 \text{ permutations}$$

b. Write a value in factorial notation to make the equation true.

$${}_3C_3 = \frac{{}_3P_3}{\boxed{}}$$

5. The two-letter permutations of RSTVX are shown.

   a. How many combinations of two-letter strings are there, ${}_5C_2$?

RS	SR	TR	VR	XR
RT	ST	TS	VS	XS
RV	SV	TV	VT	XT
RX	SX	TX	VX	XV

$${}_5P_2 = \frac{5!}{(5-2)!} = \frac{120}{6} = 20 \text{ permutations}$$

   b. Write a value in factorial notation to make the equation true.

$${}_5C_2 = \frac{{}_5P_2}{\boxed{}}$$

6. Use your answers in Questions 2 through 5 to complete the table.

	n	r	${}_nP_r$	Divided By	${}_nC_r$
Question 2	4				
Question 3	4				
Question 4	3				
Question 5	5				

7. Compare the value that you divided by each time to the value of *r*. What do you notice?

8. Study the table. Write a value in factorial notation to make this general statement true.

$${}_nC_r = \frac{{}_nP_r}{\boxed{}}$$

Graphing calculators also have a combinations key: $_nC_r$.

9. Calculate each combination using the formula $_nC_r = \frac{_nP_r}{r!}$ or $_nC_r = \frac{n!}{(n-r)!r!}$.

   a. $_6C_3$

   b. $_{10}C_1$

   c. $\binom{7}{3}$

10. Use an organization of 10 members and the formula for combinations to answer each question.

    a. How many four-member committees can be chosen?

    b. How many six-member committees can be chosen?

    c. How many ten-member committees can be chosen?

Talk the Talk

Never Comb a Perm

1. State whether each question uses permutations or combinations. Then, calculate the answer.

   a. Using a standard deck of playing cards, how many different five-card hands can be dealt without replacement?

   b. How many different numbers can be made using any three digits of 12,378?

   c. How many different ways can you arrange 10 CDs on a shelf?

   d. A professional basketball team has 12 members, but only five can play at any one time. How many different groups of players can be on the court at one time?

2. Calculate each probability.

   a. Using a standard deck of playing cards, what is the probability that a person is randomly dealt a five-card hand containing an ace, a king, a queen, a jack, and a ten?

   b. Consider the number 12,378. Using any three digits, what is the probability of making a three-digit number whose value is greater than 700?

3. Explain the usefulness of each formula.

   a. ${}_nC_r = \frac{n!}{(n-r)!r!}$

   b. $\frac{n!}{k!h!}$

c. ${}_nP_r = \dfrac{n!}{(n-r)!}$

d. $\dfrac{n!}{k!}$

e. $(n-1)!$

f. $n(n-1)(n-2)\ldots$

Lesson 3 Assignment

Write

1. A ______ is an ordered arrangement of items without repetition.

2. The ______ of n, written as $n!$, is the product of all non-negative integers less than or equal to n.

3. A ______ is an unordered collection of items.

4. The ______ of n objects is $(n - 1)!$.

Remember

When the order in which the items in a set matters, each arrangement is called a *permutation*. When the order of the items in a set does not matter, each arrangement is called a *combination*.

Practice

1. State whether each question uses permutations or combinations. Then calculate the answer.

   a. The Debate Club has 13 members. They need to elect three members to the executive board: a president, vice president, and secretary. How many different executive boards are possible?

   b. Lucas used seven websites during research for a report. How many different ways can he list the websites in his bibliography?

Lesson 3 Assignment

c. Mason has 28 songs on his computer. He is transferring eight songs to his MP3 player. How many different ways can the songs be chosen?

d. Jasmine is making a pattern with two squares, one triangle, and one circle. How many different patterns can she make using two shapes?

e. Luna works at a kennel. She takes the eight dogs at the kennel out for a walk in groups of two. How many different groups of dogs can Luna take?

2. Calculate each probability.

   a. A field hockey team has 10 members and the coach randomly selects two players as captains for each game. What is the probability that coach chooses you and your best friend as captains?

   b. Sebastian has two quarters, one nickel, and one penny that he is randomly placing on the table in a line. What is the probability that the order of the coins will be quarter, nickel, penny, and quarter?

Lesson 3 Assignment

3. Calculate the number of arrangements.

   a. How many different eight-digit numbers can be written using the digits 1, 1, 2, 7, 7, 7, 8, and 9?

   b. How many different ways can the letters in the word GEOMETRY be arranged?

4. Mrs. Nguyen is a kindergarten teacher. She asks her students to sit in a circle. Calculate the number of arrangements for each number of students.

   a. Four students

   b. Six students

Lesson 3 Assignment

Prepare

Describe the probability of each event occurring as *impossible*, *unlikely*, *even chance*, *likely*, or *certain*.

1. The probability of flipping a coin that results in heads.

2. The probability of flipping a coin that results in heads or tails.

3. The probability of rolling a 3 on a six-sided number cube.

4. The probability of not rolling a 3 on a six-sided number cube.

4 Independent Trials

OBJECTIVES

- Calculate the probability of two trials of two independent events.
- Calculate the probability of multiple trials of two independent events and describe them in writing.
- Determine the formula for calculating the probability of multiple trials of independent events.

You have calculated the probability of two independent events both occurring.

What would you do when those independent events could happen in more than one way?

Getting Started

Make or Miss?

Isaiah is a basketball player who is constantly working at improving his free throw shooting. He's worked his way up to consistently making 2 out of 3 free throws.

1. Isaiah shoots 2 free throws. Carlos, Mei, and Angelina determined the probability of Isaiah making 1 out of 2 free throws.

Carlos

The probability of Isaiah making 1 out of 2 free throws is $\frac{2}{4}$, or $\frac{1}{2}$ because 2 out of the 4 outcomes in the sample space show one made free throw.

make-make	miss-miss
make-miss	miss-make

Mei

The probability of Isaiah making 1 out of 2 free throws is $\frac{2}{9}$.

I calculated the probability by multiplying the probability of a make and the probability of a miss.

$$P(1 \text{ make and } 1 \text{ miss}) = P(1 \text{ make}) \cdot P(1 \text{ miss})$$
$$= \frac{2}{3} \cdot \frac{1}{3}$$
$$= \frac{2}{9}$$

Angelina

The probability of Isaiah making 1 out of 2 free throws is $\frac{4}{9}$.

I answered the question by multiplying the probability of a make and the probability of a miss. Then, I multiplied that result by 2 because Isaiah can make 1 out of 2 in two different ways. He could make the first and miss the second, or miss the first and make the second.

$$P(1 \text{ make and } 1 \text{ miss}) = 2[P(1 \text{ make}) \cdot P(1 \text{ miss})]$$
$$= 2\left[\frac{2}{3} \cdot \frac{1}{3}\right]$$
$$= 2\left[\frac{2}{9}\right]$$
$$= \frac{4}{9}$$

a. Why is Carlos's response incorrect? Explain your reasoning.

b. Why is Mei's response incorrect? Explain your reasoning.

c. Explain why Angelina's method is correct and Mei's is incorrect.

2. Calculate each probability.

a. What is the probability of Isaiah making 2 out of 2 free throws?

b. How many different outcomes result in Isaiah making 0 out of 2 free throws? What is the probability of Isaiah making 0 out of 2 free throws?

ACTIVITY 4.1

Calculating Probability Using Combinations

Recall that Isaiah consistently makes 2 out of 3 free throws.

1. Isaiah shoots 3 free throws. Determine the probability of the following events.

   a. What is the probability of Isaiah making 0 out of 3 free throws?

   b. What is the probability of Isaiah making 1 out of 3 free throws?

   c. What is the probability of Isaiah making 2 out of 3 free throws?

d. What is the probability of Isaiah making 3 out of 3 free throws?

2. Kayla claims that she can calculate the probability of Isaiah making free throws more efficiently, using combinations.

For determining the "3 ways" Isaiah can make 1 out of 3 free throws, I multiplied ${}_3C_1$ and P(1 make).

$$\begin{aligned} P(1 \text{ make and } 2 \text{ misses}) &= {}_3C_1 \cdot [P(\text{make}) \cdot P(\text{miss}) \cdot P(\text{miss})] \\ &= 3 \cdot [P(\text{make}) \cdot P(\text{miss}) \cdot P(\text{miss})] \\ &= 3 \cdot \left[\frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}\right] \\ &= 3 \cdot \left[\frac{2}{27}\right] \\ &= \frac{6}{27} \\ &= \frac{2}{9} \end{aligned}$$

Verify that Kayla's method works for the calculations in Question 1 by using combinations to calculate each probability.

a. Isaiah making 0 out of 3 free throws.

b. Isaiah making 1 out of 3 free throws.

c. Isaiah making 2 out of 3 free throws.

d. Isaiah making 3 out of 3 free throws.

3. Isaiah shoots 4 free throws. Use combinations to determine the probability of each event.

   a. Isaiah making 0 out of 4 free throws.

   b. Isaiah making 1 out of 4 free throws.

   c. Isaiah making 2 out of 4 free throws.

d. Isaiah making 3 out of 4 free throws.

e. Isaiah making 4 out of 4 free throws.

4. Complete the table to summarize the combination calculation that determines the number of ways each outcome can occur.

Number of Free Throws Attempted	Outcome 1	Outcome 2	Outcome 3	Outcome 4	Outcome 5	Outcome 6
1	0 makes, 1 miss occurs 1 way $_1C_0 = 1$	1 make, 0 misses occur 1 way $_1C_1 = 1$				
2	0 makes, 2 misses occur 1 way $_2C_0 = 1$	1 make, 1 miss occurs 2 ways $_2C_1 = 2$	2 makes, 0 misses occur 1 way $_2C_2 = 1$			
3						
4						
5						

Remember . . .
The triangle of numbers is Pascal's Triangle, named after Blaise Pascal, a 17th century mathematician. In the free throw shooting scenario, the numbers in Pascal's Triangle represent the number of ways an outcome can occur.

5. The following is a different way to organize the number of ways each outcome may occur. Analyze any patterns you see. Describe these patterns and use them to write the next two rows.

$$
\begin{array}{ccccccccccc}
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

6. Summarize the probability of Isaiah making each number of free throws.

   a. The probability of Isaiah making 0 out of 1 free throw is provided. Complete the table to show the additional probabilities.

Number of Free Throws Isaiah Attempts	Number of Free Throws Isaiah Makes	Probability of Isaiah Making Each Free Throw	Probability of Isaiah Missing Each Free Throw	Probability of Isaiah Making *r* out of *n* Free Throw Attempts
n	r	p	$(1 - p)$	
1	0	$\frac{2}{3}$	$\frac{1}{3}$	${}_1C_0\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^1$
	1	$\frac{2}{3}$	$\frac{1}{3}$	
2	0	$\frac{2}{3}$	$\frac{1}{3}$	
	1	$\frac{2}{3}$	$\frac{1}{3}$	
	2	$\frac{2}{3}$	$\frac{1}{3}$	
3	0	$\frac{2}{3}$	$\frac{1}{3}$	
	1	$\frac{2}{3}$	$\frac{1}{3}$	
	2	$\frac{2}{3}$	$\frac{1}{3}$	
	3	$\frac{2}{3}$	$\frac{1}{3}$	

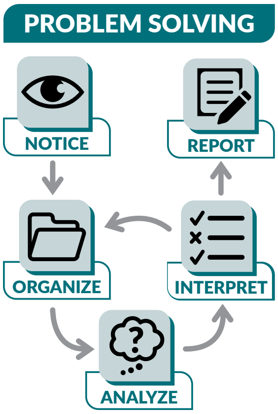
b. Describe how to calculate the probability of Isaiah making r free throws out of n free throw attempts.

c. Write an expression to calculate the probability of Isaiah making r free throws out of n free throw attempts.

7. Isaiah's free throw shooting was not as good in the past. He used to make 2 out of every 5 free throws. What was the probability of Isaiah making 8 out of 10 free throws?

8. Isaiah is working to improve his free throw shooting. His goal is to consistently make 9 out of every 10 free throws. What is the probability of Isaiah making 8 out of 10 free throws?

Ask Yourself . . .
How can you use combinations in other situations in life?



ACTIVITY 4.2 Using a Formula for Multiple Trials

Suppose you roll a number cube with 4 red faces and 2 blue faces.

1. Calculate each probability.

   a. Rolling 4 reds and 1 blue when the number cube is rolled five times.

   b. Rolling 3 reds and 3 blues when the number cube is rolled six times.

   c. Rolling 1 red and 6 blues when the number cube is rolled seven times.

2. Calculate each probability.

   a. Rolling 3 reds and 7 blues when the number cube is rolled ten times.

   b. Rolling 3 reds and 4 blues when the number cube is rolled seven times.

A regular tetrahedron is a four-sided solid with each face an equilateral triangle. A regular tetrahedron has three sides painted blue (*B*) and one side painted red (*R*).

3. Calculate each probability.

   a. What is *P*(*R*), the probability of rolling a tetrahedron that lands on a red face?

   b. What is *P*(*B*), the probability of rolling a tetrahedron that lands on a blue face?

   c. What is *P*(5*B* and 2*R*), the probability of rolling a tetrahedron that lands on five blues and two reds in seven rolls?

d. What is $P(4B \text{ and } 1R)$, the probability of rolling a tetrahedron that lands on four blues and one red in five rolls?

e. What is $P(5B \text{ and } 5R)$, the probability of rolling a tetrahedron that lands on five blues and five reds in ten rolls?

Talk the Talk

It's a D8!

A regular octahedron is an eight-sided solid with each face an equilateral triangle. A regular octahedron has six sides painted yellow and two sides painted green.

1. What is the probability of rolling a yellow?

2. What is the probability of rolling a green?

3. What is the probability of rolling seven yellows and one green in eight rolls?

4. What is the probability of rolling six yellows and two greens in eight rolls?

5. What is the probability of rolling five yellows and two greens in seven rolls?

Lesson 4 Assignment

Write

Describe a scenario that might use the formula ${}_nC_r\,(p)^r\,(1-p)^{n-r}$. What do each of the variables represent in the formula?

Remember

The arrangements represented by ${}_nC_r$ can also be computed using Pascal's Triangle.

Practice

1. A number cube has 5 green sides and 1 orange side.

    a. What is the probability of 4 green outcomes and 1 orange outcome when the number cube is rolled 5 times?

    b. What is the probability of 2 green outcomes and 2 orange outcomes when the number cube is rolled 4 times?

    c. What is the probability of 3 green outcomes and 4 orange outcomes when the number cube is rolled 7 times?

2. A bag contains 3 nickels and 1 penny. Coins are replaced in the bag after every choice.

    a. What is the probability of randomly choosing 4 nickels and 1 penny in 5 trials?

Lesson 4 Assignment

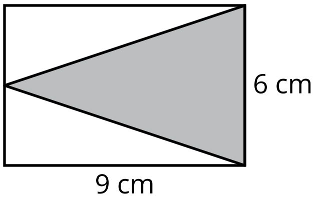
b. What is the probability of randomly choosing 2 nickels and 3 pennies in 5 trials?

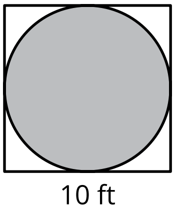
c. What is the probability of randomly choosing 1 nickel and 5 pennies in 6 trials?

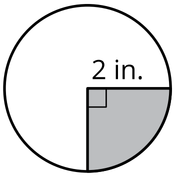
d. What is the probability of randomly choosing 4 nickels and 3 pennies in 7 trials?

Prepare

Determine the area of each unshaded section.

1. 

2. 

3. 

5 Expected Value

OBJECTIVES

- Determine geometric probability.
- Calculate the expected value of an event and describe it verbally and in writing.

KEY TERMS

- geometric probability
- expected value

You have worked with the formulas for permutations and combinations.

How can these formulas be used to calculate the total number of ways that outcomes can occur, especially when they can occur many times?

Getting Started

Let's Make a Deal

Suppose you are a contestant on a game show. The host shows you 10 doors and tells you there is a car behind one of the doors and the remaining nine doors have zonks. You choose door number 3 because it's your lucky number. The host decides to open eight doors that he knows will all reveal zonks.

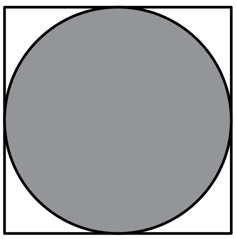
The host asks you one more time, do you want to keep your door or switch with the remaining closed door.

1. Should you keep your door or switch doors? Use probabilities to defend your decision.

ACTIVITY 5.1 Geometric Probability

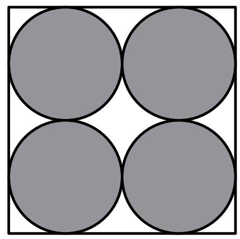
A class is hosting a dart competition to raise money. Three different designs under consideration are shown. All of the dartboards are squares that measure 50 cm by 50 cm. For this situation, you have to assume that a dart is thrown randomly at a dartboard and that the dart will land somewhere on the dartboard.

Big Bullseye



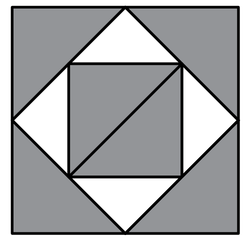
1 shaded circle

Four Circles



4 congruent shaded circles

Tricky Triangles



6 congruent shaded triangles

1. Which dartboard will give the player the greatest chance of hitting a shaded section? Why?

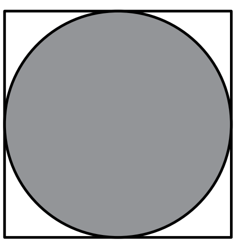
2. You can use what you know about probability to determine the chance a person has of hitting a shaded section on a dartboard.

   a. What value would you use to represent the number of possible outcomes?

   b. What value would you use to represent the number of desired outcomes?

3. Lucas says that the number of desired outcomes for Big Bullseye is 1 because there is one shaded section. That means that Four Circles has four desired outcomes, and Tricky Triangles has six desired outcomes. Is Lucas's reasoning correct? Explain why or why not.

Big Bullseye



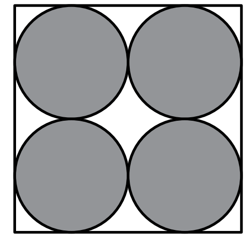
1 shaded circle

4. Determine the probability that a dart lands in a shaded section of each dartboard.

   a. Big Bullseye

b. Four Circles

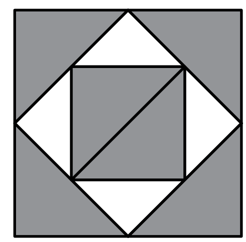
Four Circles



4 congruent shaded circles

c. Tricky Triangles

Tricky Triangles

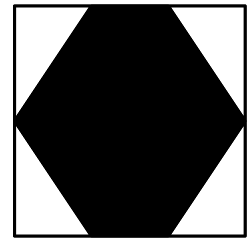


6 congruent shaded triangles

5. Which dartboard should the class use to make the competition the most difficult? Explain your reasoning.

To determine the probability of hitting a shaded section on a dartboard, you used *geometric probability*. **Geometric probability** is probability that involves a geometric measure, such as length, area, volume, and so on.

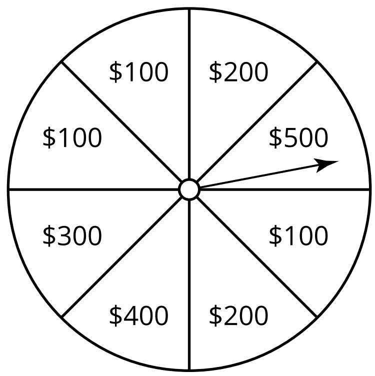
6. The dartboard shown is a hexagon inscribed in a square. Two of the vertices of the hexagon bisect two of the sides of the square. The remaining four vertices trisect the other two sides of the square. What is the probability that a dart will hit the shaded area of the dartboard? Show your work.



ACTIVITY 5.2

Exploring Expected Value

You are a contestant on a game show. In the first round, the host gives you \$200. You can choose to keep the money or give the money back and spin the wheel to determine your winnings. All the sections on the wheel are equal in size.



1. Would you choose to keep the \$200 or spin the wheel? Why?

2. When you spin the wheel, what is the probability that you will win each amount?

   a. \$100

   b. \$200

   c. \$300

   d. \$400

   e. \$500

3. When you spin the wheel, how often would you expect to win each amount?

   a. $100

   b. $200

   c. $300

   d. $400

   e. $500

4. Let's calculate the expected value for the game.

   a. Multiply each amount by the probability of winning that amount.

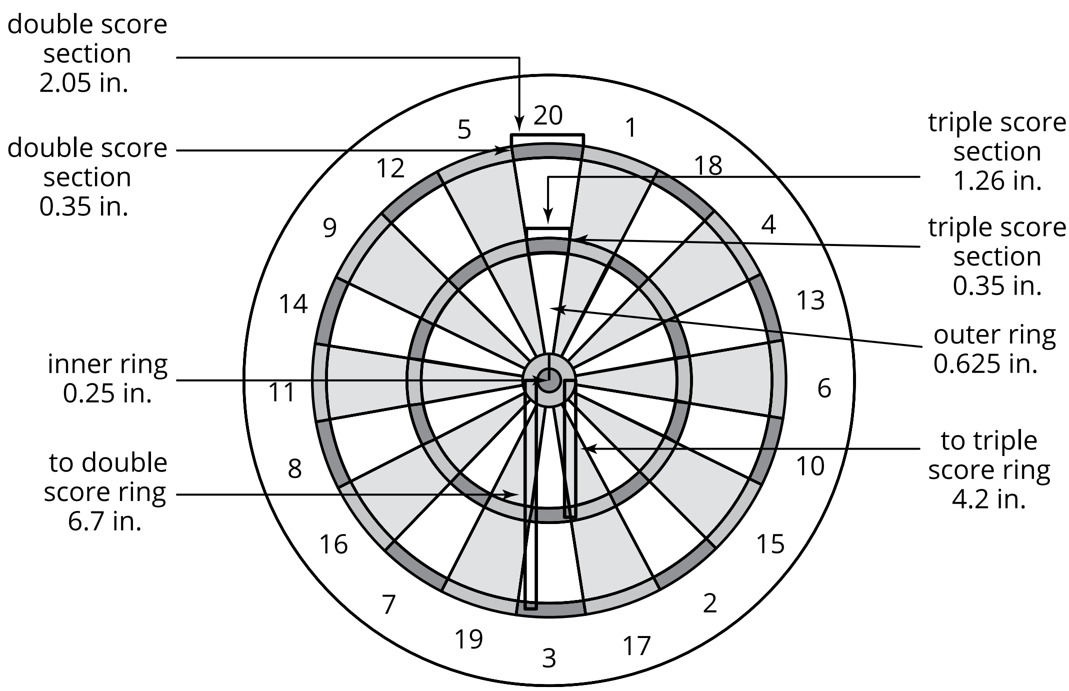
   b. Calculate the sum of the values from part (a)

The sum in Question 4, part (b) is the *expected value*. The **expected value** is the average value when the number of trials is large. In this problem, the expected value represents the amount that you could expect to receive from a single spin.

5. Based on the expected value, should you keep the $200 or spin the wheel? Explain your reasoning.

Using Geometric Probability to Determine Expected Value

The dartboard shown has a diameter of 17.75 inches and 20 congruent sections in a circular scoring area. Players score points by throwing darts inside the scoring area. A dart that lands outside the scoring area scores no points.



1. Determine each of the approximate probabilities, assuming that one dart is thrown at random at the board. Explain your reasoning and show your work. Round values to the nearest ten-thousandth.

   a. Probability of getting any score.

b. Probability of landing anywhere on the section labeled 20.

c. Probability of landing within the outer ring but not the inner ring of the bullseye.

d. Probability of landing within the inner ring of the bullseye.

e. Probability of getting a triple 20.

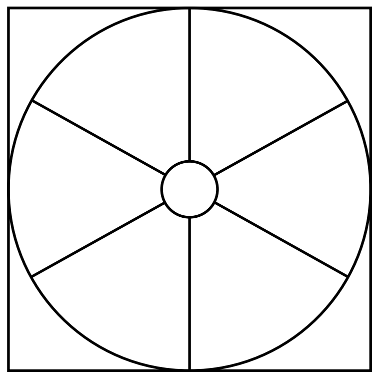
f. Probability of getting a double 20.

g. Probability of landing on 20 without a double or triple score.

Talk the Talk

Calculating Expected Value

The square dartboard shown has a side length of 42 centimeters, 6 congruent sections inside an outer circle, and an inner circle with a diameter of 6 centimeters. Players score points by throwing darts at the dartboard.



1. Suppose the inner circle is worth 50 points. Each section in the outer circle is worth 25 points. Each section outside the outer circle but inside the square is worth 5 points. What is the expected value for one dart throw? Show your work. When necessary, round each probability to the nearest thousandth.

Lesson 5 Assignment

Write

1. ______________ is the likelihood of an event occurring based on geometric relationships, such as length, area, or volume.

2. ______________ is the sum of the values of a random variable with each value multiplied by its probability of occurrence.

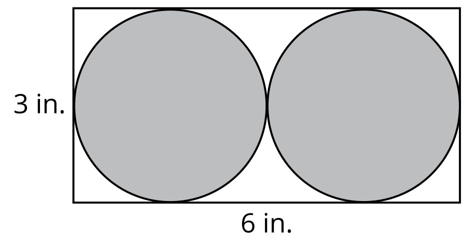
Remember

Geometric probability helps solve real-world problems, such as determining the probability of a thrown dart landing in the inner ring of a dartboard.

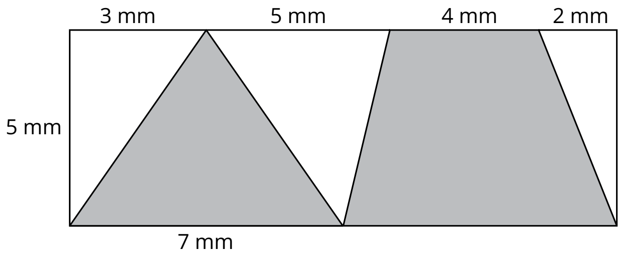
Practice

1. A dart is thrown and lands on a random spot on each target. Determine the probability of hitting the shaded region. Write your answers as a percent rounded to the nearest tenth.

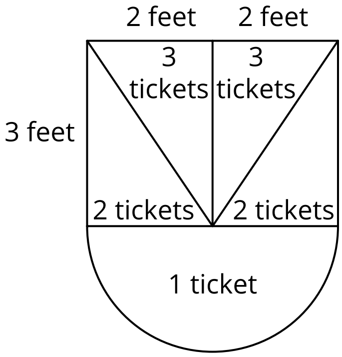
   a. The board shown is two circles inscribed inside a rectangle.



b. The board shown is a triangle and a trapezoid inside of a rectangle.



2. At a carnival, Jasmine is playing a game with the dartboard shown below. The dartboard has a rectangular top and a semicircle along the bottom. She throws a dart that lands randomly on the dartboard and wins the number of tickets shown in each region.



a. Determine the areas you will need to know to calculate the expected value for the number of tickets Jasmine will win.

Lesson 5 Assignment

b. Determine the probabilities you will need to know to calculate the expected value for the number of tickets Jasmine will win.

c. What is the expected value for the number of tickets Jasmine will win?

TOPIC 2 SELF-REFLECTION

Name: ______________________

Computing Probabilities

When you reflect on what you are learning, you develop your understanding and know when to ask for help.

Reflect on these statements. Place a number in each circle from 1–3, where 1 represents **the skill is new to me,** 2 represents **I am building proficiency of the skill,** and 3 represents **I have demonstrated proficiency of the skill**.

I can demonstrate an understanding of the standards in the *Computing Probabilties* topic by:

TOPIC 2: *Computing Probabilities*	Beginning of Topic	Middle of Topic	End of Topic
constructing a two-way frequency table for a data set.	○	○	○
calculating compound and conditional probabilities from a two-way table.	○	○	○
using the sample space in a two-way table to decide whether events are independent.	○	○	○
defining *dependent events* and *conditional probability*.	○	○	○
explaining that Events A and B are independent if the occurrence of Event A does not affect the probability of the occurrence of Event B.	○	○	○
calculating the conditional probability of Event A occurring, given the occurrence of Event B, using the formula $P(A\|B) = \frac{P(A \text{ and } B)}{P(B)}$.	○	○	○
using real-world examples of dependent events to illustrate the concept of conditional probability.	○	○	○
using real-world examples of independent events to illustrate the concept of independent probability.	○	○	○

continued on the next page

TOPIC 2 SELF-REFLECTION *continued*

TOPIC 2: *Computing Probabilities*	Beginning of Topic	Middle of Topic	End of Topic
interpreting probability based on the context of a given problem.			
defining *factorial*, *permutation*, and *combination*, and distinguishing between situations that require permutations and those that require combinations.			
applying the permutation and combination formulas, as appropriate, to determine the number of outcomes in an event.			
solving problems involving permutations and combinations.			
using probabilities as a method for making decisions.			
using expected values to analyze the results of a game or experiment.			
analyzing data to determine whether the best decision was made.			

continued on the next page

TOPIC 2 SELF-REFLECTION *continued*

At the end of the topic, take some time to reflect on your learning and answer the following questions.

1. Describe a new strategy you learned in the *Computing Probabilities* topic.

2. What mathematical understandings from the topic do you feel you are making the most progress with?

3. Are there any concepts within this topic that you find challenging to understand? How do you plan to improve your understanding of these concepts?

TOPIC 2 SUMMARY

Computing Probabilities

LESSON 1 Compound Probability for Data Displayed in Two-Way Tables

A **two-way table** shows the relationship between two data sets; one data set is organized in rows, and the other data set is organized in columns.

For example, the two-way table below represents all possible sums that will result from rolling two number cubes one time.

2nd Number Cube

1st Number Cube	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

A **frequency table** shows the frequency of an item, number, or event appearing in a sample space.

Outcome (sum of number cubes)	Frequency
2	1
3	2
4	3
5	4
6	5
7	6
8	5

KEY TERMS

- two-way table
- frequency table [tabla de frecuencia(s)]
- two-way frequency table
- categorical data [datos categóricos]
- relative frequency [frecuencia relativa]
- two-way relative frequency table
- factorial [factorial]
- permutation [permutación]
- circular permutation [permutación circular]
- combination [combinación]
- geometric probability [probabilidad geométrica]
- expected value [valor esperado]

Outcome (sum of number cubes)	Frequency
9	4
10	3
11	2
12	1

You have learned that if two events A and B are independent, the probability of both events occurring, $P(A \text{ and } B)$, is $P(A) \cdot P(B)$. The converse is also true: If the probability of two events, A and B, $P(A \text{ and } B)$, occurring together is $P(A) \cdot P(B)$, then the two events are independent.

A **two-way frequency table** shows the number of data points and their frequencies for two variables. One variable is divided into rows, and the other variable is divided into columns.

Categorical data are data for which each piece of data fits into exactly one of several different groups or categories.

For example, consider a survey which asks participants about their favored hand and whether the participant plays certain types of sports or no sports at all. The results are shown in the two-way table.

Favored Hand	Sports Participation: Individual	Team	Does Not Play	Total
Left	3	13	8	24
Right	6	23	4	33
Mixed	1	3	2	6
Total	10	39	14	63

A **relative frequency** is the ratio of occurrences within a category to the total number of occurrences. To determine the ratio for each category, determine the part-to-whole ratio for each category. A **two-way relative frequency table** displays the relative frequencies for two categories of data.

Sports Participation

	Individual	Team	Does Not Play	Total
Left	$\frac{3}{63} \approx 4.8\%$	$\frac{13}{63} \approx 25.4\%$	$\frac{8}{63} \approx 12.7\%$	$\frac{24}{63} \approx 38.1\%$
Right	$\frac{6}{63} \approx 9.5\%$	$\frac{23}{63} \approx 36.5\%$	$\frac{4}{63} \approx 6.3\%$	$\frac{33}{63} \approx 52.4\%$
Mixed	$\frac{1}{63} \approx 1.6\%$	$\frac{3}{63} \approx 4.8\%$	$\frac{2}{63} \approx 3.2\%$	$\frac{6}{63} \approx 9.5\%$
Total	$\frac{10}{63} \approx 15.9\%$	$\frac{39}{63} \approx 61.9\%$	$\frac{14}{63} \approx 22.2\%$	$\frac{63}{63} = 100\%$

You can use a two-way relative frequency table to calculate the probability of events occuring.

For example, $P(\text{left handed}) = \frac{24}{63} \approx 38.1\%$, $P(\text{right handed in a team sport}) = \frac{23}{63} \approx 36.5\%$, or $P(\text{mixed handed or does not play}) = P(\text{mixed handed}) + P(\text{does not play}) - P(\text{mixed handed and does not play}) = \frac{6}{63} + \frac{14}{63} - \frac{2}{63} = \frac{18}{63} \approx 28.6\%$.

LESSON 2 Conditional Probability

A conditional probability is the probability of event *B*, given that event *A* has already occurred. Consider a number cube that is rolled. The probability of rolling a 4 or less on the second roll of the number cube (*B*), given that a 5 is rolled first (*A*), is an example of a conditional probability.

The notation for conditional probability is $P(B|A)$, which is read, "the probability of event *B*, given event *A*."

When two events are independent, $P(B|A) = P(B)$. When two events are dependent, $P(B|A) \neq P(B)$. Thus, two events are independent if and only if $P(B|A) = P(B)$.

The conditional probability, $P(B|A)$, for independent events can be represented as $\frac{\text{desired outcomes}}{\text{total outcomes}}$.

$$\begin{aligned} P(B|A) &= \frac{\text{desired outcomes}}{\text{total outcomes}} \\ &= \frac{A \text{ and } B}{A} \\ &= \frac{A \text{ and } B}{A} \cdot \frac{\frac{1}{\text{total}}}{\frac{1}{\text{total}}} \\ &= \frac{\frac{A \text{ and } B}{\text{total}}}{\frac{A}{\text{total}}} \\ &= \frac{P(A \text{ and } B)}{P(A)} \end{aligned}$$

For example, consider the possible sums when rolling two number cubes.

Let A = rolling a 5

Let B = rolling a sum of 10

$P(B|A) = \frac{\frac{1}{72}}{\frac{1}{6}} = \frac{1}{12}$

You can check if these two events are independent by also calculating $P(B) = \frac{3}{36} = \frac{1}{12}$, thus they are independent events.

LESSON 3 Permutations and Combinations

The **factorial** of n, which is written with an exclamation mark as $n!$, is the product of all non-negative integers less than or equal to n. For example, $3! = 3 \cdot 2 \cdot 1 = 6$.

You can simplify fractions involving factorials by dividing out common factors.

For example, $\frac{8!}{6!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 8 \cdot 7 = 56$.

An ordered arrangement of items without repetition is called a **permutation**. Being able to determine the different arrangement of three letters from the first four letters of the alphabet is an example of a permutation. There are different notations that are used for the permutations of r elements taken from a collection of n items.

$$_nP_r = P(n, r) = P^n_r$$

You can calculate permutations using the formula $_nP_r = \frac{n!}{(n-r)!}$.

For example, consider the number of permutations of two letters from the first four letters of the alphabet.

$$_4P_2 = \frac{4!}{(4-2)!} = \frac{4!}{2!} = 4 \cdot 3 = 12$$

There are 12 different combinations of two letters using only the first four letters of the alphabet.

You must also consider permutations in which there are repeated elements in the sample space.

To determine the number of permutations of a sample space with repeated elements, you can use the formula $\frac{n!}{k!}$, where n is the number of elements in the set and k is the number of copies of an element.

For example, Sofie has four books on her shelf: two novels (N_1 and N_1), one science book (S), and one reference book (R).

The possible combinations of the four books with the two novels grouped together is $\frac{4!}{2!} = 4 \cdot 3 = 12$.

Another type of permutation is called a *circular permutation*. A **circular permutation** is a permutation in which there is no starting point and no ending point. The circular permutation of n objects is $(n - 1)!$. A common example of a circular permutation is a seating arrangement for a group of people.

A **combination** is an unordered collection of items. Different notations can be used for the combinations of r elements taken from a collection of n elements.

$$_{n}C_{r} = C(n, r) = C^{n}_{r}$$

In permutations, the order that the objects are put into is relevant, but in a combination, the order is not relevant. For example, all of the three-letter permutations from the letters DEFG have $_{4}P_{3} = \frac{4!}{(4-3)!} = 24$ outcomes. To eliminate any outcomes that repeated the same three letters that were just in a different order is to determine the combination of three-letter strings from the letters DEFG.

You can calculate combinations using the formula $_{n}C_{r} = \frac{n!}{(n-r)!r!}$.

For example, the possible combinations of three-letter strings from the letters DEFG is $_{4}C_{3} = \frac{4!}{(4-3)!3!} = \frac{4!}{1!3!} = 4$.

LESSON 4 Independent Trials

Independent events can happen in more than one way when performing multiple trials. The probability of the events happening is multiplied by the number of different ways the events can happen.

For example, the table below summarizes the probability of a basketball player making each number of free throws, using combinations.

Number of Free Throws	Number of Free Throws	Probability of Making	Probability of Missing	Probability of Making *r*
Attempts	Makes	Each Free Throw	Each Free Throw	out of *n* Attempts
n	r	p	$(1 - p)$	$_{n}C_{r}(p)^{r}(1 - p)^{n-r}$
1	0	$\frac{2}{3}$	$\frac{1}{3}$	tt $_{1}C_{0}\left(\frac{2}{3}\right)^{0}\left(\frac{1}{3}\right)^{1}$
	1	$\frac{2}{3}$	$\frac{1}{3}$	$_{1}C_{1}\left(\frac{2}{3}\right)^{1}\left(\frac{1}{3}\right)^{0}$

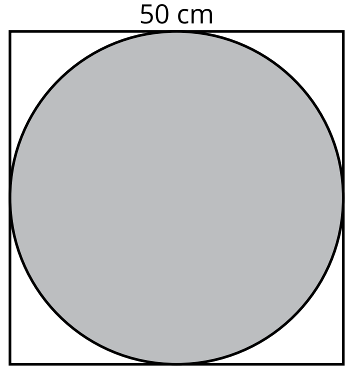
Number of Free Throws	Number of Free Throws	Probability of Making	Probability of Missing	Probability of Making *r*
2	0	$\frac{2}{3}$	$\frac{1}{3}$	${}_2C_0\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^2$
	1	$\frac{2}{3}$	$\frac{1}{3}$	${}_2C_1\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^1$
	2	$\frac{2}{3}$	$\frac{1}{3}$	${}_2C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^0$
3	0	$\frac{2}{3}$	$\frac{1}{3}$	${}_3C_0\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^3$
	1	$\frac{2}{3}$	$\frac{1}{3}$	${}_3C_1\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^2$
	2	$\frac{2}{3}$	$\frac{1}{3}$	${}_3C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^1$
	3	$\frac{2}{3}$	$\frac{1}{3}$	${}_3C_3\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^0$

LESSON 5

Expected Value

To determine the probability of hitting a shaded section on a dartboard, you can use *geometric probability.* **Geometric probability** is probability that involves a geometric measure, such as length, area, volume, and so on.

For example, consider the dartboard shown, which is a square that measures 50 cm by 50 cm.



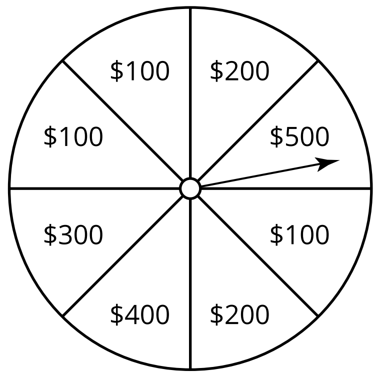
The probability of a dart hitting the shaded region of the dartboard is $\frac{\text{area of shaded region}}{\text{area of dartboard}}$.

$$\frac{\text{area of shaded region}}{\text{area of dartboard}} = \frac{\pi(25)^2}{50 \cdot 50} \approx \frac{1962.5}{2500}$$

The probability that the dart hits the shaded region is $\frac{1962.5}{2500}$, or about 78.5%.

An **expected value** is the average value when the number of trials is large.

For example, suppose you are a contestant on a game show. In the first round, the host gives you \$200. You can choose to keep the money or to give the money back and spin the wheel to determine your winnings.



In this problem, the expected value represents the amount that you could expect to receive from a single spin, or $\frac{3}{8}(100) + \frac{2}{8}(200) + \frac{1}{8}(300) + \frac{1}{8}(400) + \frac{1}{8}(500) = \$237.50$.

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