Duration, Hiding in A Taylor Series

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Introduction

Duration has been touted as a tool for measuring the sensitivity of the price, or value, of an asset, or liability, whose cash flows are fairly determinable, to changes in interest rates. This paper seeks to describe the above relationship in a concrete fashion by expressing the value of an asset or liability as a function of the current interest rate. This function is then expanded in a Taylor series to illustrate just where the duration concept fits in. After this presentation is made, the Taylor series is further employed to illustrate that one may obtain a level of immunization as close to complete as desired by essentially matching successive terms in the Taylor series, the second of which reflects duration.

The Fundamental Relationships

The formula below presents the price of a known stream of cash flows given an interest rate *i*. This paper will assume a flat yield curve for ease of presentation.

$$P(i) = \sum CF_t / (1+i)^t$$

P*(i)* is the price of this cash flow and is expressed as a function of the interest rate *i*. CF_t is the cash flow at time *t*.

The Taylor series for the price at a new interest rate may be expressed as follows:

$$P(i + \Delta i) = P(i) + P'(i)\,\Delta i + \frac{P''(i)(\Delta i)}{2!} + \ldots$$

The change in the interest rate, Δi, has produced a change in the price of $P(i + \Delta i) - P(i)$. It is this change in price that is frequently estimated using duration.

The duration, D(*i*), of a stream of cash flows as a function of the interest rate *i* is:

$$D(i) = \frac{\sum t\, CF_t/(1+i)^t}{\sum CF_t/(1+i)^t}$$

Note the denominator is the price of the cash flow. The second term in the Taylor series, $P'(i)\Delta i$, can be shown to consist of duration multiplied by a constant and the change in i.

$$P'(i) = \frac{d}{di}\sum CF_t(1+i)^{-t} = \frac{-1}{(1+i)}\sum tCF_t/(1+i)^t$$

$$P'(i) = -D(i)P(i)/(1+i)$$

Therefore, using only the first two terms of the Taylor series; the change in the price of the instrument, $P(i+\Delta i) - P(i)$, is often approximated by $-D(i)\, P(i)\, \Delta i\,/(1+i)$.

This approximation is refined when the third term is considered. However, this term essentially reflects the quantity known as convexity. Convexity is defined as:

$$C(i) = \frac{\sum t^2 CF_t/(1+i)^t}{\sum CF_t/(1+i)^t}$$

The relation to the Talyor series is revealed by determining the second derivative of the price as follows:

$$P''(i) = \frac{d}{di}(P'(i)) = -\frac{d}{di}\sum tCF_t(1+i)^{-(t+1)}$$

This equals:

$$P''(i) = \frac{1}{(1+i)^2} \text{ X } [\sum t^2 CF_t / (1+i)^t + \sum t \; CF_t / (1+i)^t]]$$

OR

$$P''(i) = [C(i) + D(i)] \, P(i)/(1+i)^2$$

Therefore, the price of the instrument after a change in interest rates of Δi can be approximated by:

Original Price X {1 - Duration x$\Delta i/(1+i)$ + [Convexity + Duration] x $(\Delta i)^2$ X $.5/(1+i)^2$}

The use of duration, in the second term of the Taylor series, to determine the change in the instrument value is only an approximation. As more terms of the Taylor series are added the accuracy improves (note the limit of the series must exist).

By matching the cash flows of an asset to the cash flow of a liability one is assured that the gain or loss on the asset [illegible] to changes in the interest r[illegible] will be exactly offset by changes in the value of the liability. [illegible]sets and liabilities [illegible]id to be completely matched or immunized against changes in i[illegible]t rates. Th[illegible]umes that the cash flows of the asset and the liability are fixed.

Often the duration of assets is match[illegible] th[illegible]ation of liabilities in an attempt to gain a level of immunization when the cash fl[illegible]re not ex[illegible]matched. One of the primary purposes of matching duration rather tha[illegible]ntire cash flows [illegible] some of the assets held can be of longer maturities to take a[illegible]tage of the higher yields. [illegible]n this is done it is often not realized that there is a trade-off. As duration matching principally accounts for only the first two terms in the Taylor series, the immunization is not complete. Therefore, the price of the investment gain from the higher yields is the potential loss resulting from the asset liability mismatch.