

# Civil Engineering FE CBT Exam

## Knowledge: Dynamics with 15 Example Problems

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# 1 Particle Kinematics - Rectilinear Motion

Rectilinear kinematics describes the motion of a particle moving along a straight-line path. The position of the particle at any instant  $t$  is designated by its coordinate  $s$  relative to a fixed origin  $O$ .

## 1.1 Fundamental Kinematic Relations

The three fundamental quantities of rectilinear motion are position ( $s$ ), velocity ( $v$ ), and acceleration ( $a$ ). They are linked by the following differential relationships:

$$v = \frac{ds}{dt} \quad (1)$$

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2} \quad (2)$$

$$a \, ds = v \, dv \quad (\text{derived by eliminating } dt) \quad (3)$$

Integrating these differential relations allows us to determine position and velocity as functions of time when acceleration is known:

$$v(t) = v_0 + \int_0^t a(t) \, dt \quad (4)$$

$$s(t) = s_0 + \int_0^t v(t) \, dt \quad (5)$$

## 1.2 Constant Acceleration Equations

When the acceleration of the particle is constant ( $a = a_c$ ), the integration of the fundamental kinematic relations yields the standard rectilinear motion equations:

$$v = v_0 + a_c t \quad (6)$$

$$s = s_0 + v_0 t + \frac{1}{2} a_c t^2 \quad (7)$$

$$v^2 = v_0^2 + 2a_c(s - s_0) \quad (8)$$

|        |          |                                                                      |
|--------|----------|----------------------------------------------------------------------|
| Where: | $s_0, s$ | Initial and final positions (meters, ft)                             |
|        | $v_0, v$ | Initial and final velocities (meters/sec, ft/s)                      |
|        | $a_c$    | Constant acceleration (meters/sec <sup>2</sup> , ft/s <sup>2</sup> ) |
|        | $t$      | Time interval (seconds)                                              |

## 1.3 Graphical Interpretation of Kinematic Curves

- **Position-Time ( $s$ - $t$ ) Curve:** The slope of the tangent to the curve at any time  $t$  is the velocity ( $v = ds/dt$ ).
- **Velocity-Time ( $v$ - $t$ ) Curve:** The slope of the tangent is the acceleration ( $a = dv/dt$ ). The area under the  $v$ - $t$  curve between  $t_1$  and  $t_2$  represents the displacement:  $\Delta s = s_2 - s_1 = \int_{t_1}^{t_2} v \, dt$ .
- **Acceleration-Time ( $a$ - $t$ ) Curve:** The area under the  $a$ - $t$  curve between  $t_1$  and  $t_2$  represents the change in velocity:  $\Delta v = v_2 - v_1 = \int_{t_1}^{t_2} a \, dt$ .

## 2 Particle Kinematics - Curvilinear & Projectile Motion

Curvilinear kinematics describes the motion of a particle along a curved 2D or 3D path. The position, velocity, and acceleration vectors are represented using distinct coordinate systems depending on the nature of the path.

### 2.1 Coordinate Systems for Planar Curvilinear Motion

1. **Cartesian Coordinates ( $x$ - $y$ ):** Used when motion is naturally divided into horizontal and vertical components.

$$\vec{r} = x\hat{i} + y\hat{j} \quad (9)$$

$$\vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j} = v_x\hat{i} + v_y\hat{j} \quad (10)$$

$$\vec{a} = \ddot{x}\hat{i} + \ddot{y}\hat{j} = a_x\hat{i} + a_y\hat{j} \quad (11)$$

2. **Normal-Tangential Coordinates ( $n$ - $t$ ):** Used when the particle path is known. The tangential axis  $t$  is tangent to the path (direction of velocity), and the normal axis  $n$  points toward the center of curvature.

$$\vec{v} = v\hat{u}_t \quad (12)$$

$$\vec{a} = a_t\hat{u}_t + a_n\hat{u}_n = \frac{dv}{dt}\hat{u}_t + \frac{v^2}{\rho}\hat{u}_n \quad (13)$$

Here,  $a_t = dv/dt$  is the rate of change of speed, and  $a_n = v^2/\rho$  represents the rate of change of direction, where  $\rho$  is the local radius of curvature. The magnitude of total acceleration is  $a = \sqrt{a_t^2 + a_n^2}$ .

### 2.2 Projectile Motion (Constant Gravity Acceleration)

A projectile is a particle launched with an initial velocity  $\vec{v}_0$  at an angle  $\theta$  relative to the horizontal. Neglecting air resistance, the horizontal acceleration is zero and the vertical acceleration is constant ( $a_x = 0, a_y = -g$ ).

Horizontal and vertical kinematic components are solved independently:

$$\text{Horizontal Component: } x(t) = x_0 + (v_0 \cos \theta)t, \quad v_x = v_0 \cos \theta \quad (14)$$

$$\text{Vertical Component: } y(t) = y_0 + (v_0 \sin \theta)t - \frac{1}{2}gt^2, \quad v_y = v_0 \sin \theta - gt \quad (15)$$

### 2.3 Key Projectile Parameters (Level Surface)

- **Time of Flight ( $t_{air}$ ):** Setting  $y = y_0$  and solving for  $t$  (where  $t > 0$ ) yields:

$$t_{air} = \frac{2v_0 \sin \theta}{g} \quad (16)$$

- **Maximum Height ( $H$ ):** Occurs when vertical velocity is zero ( $v_y = 0 \implies t_{peak} = t_{air}/2$ ):

$$H = \frac{v_0^2 \sin^2 \theta}{2g} \quad (17)$$

- **Horizontal Range ( $R$ ):** The total horizontal distance traveled when landing at the launch elevation:

$$R = x(t_{air}) - x_0 = \frac{v_0^2 \sin(2\theta)}{g} \quad (18)$$

### 3 Rigid Body Kinematics - Angular Motion & Relative Velocity

Rigid body kinematics describes the motion of an assembly of particles where the distance between any two points remains constant. Types of rigid body motion include translation, rotation about a fixed axis, and general planar motion.

#### 3.1 Rotation About a Fixed Axis

When a rigid body rotates about a fixed axis passing through point  $O$ , all particles in the body move in circular paths centered on  $O$ .

- **Angular Position ( $\theta$ ):** Measured in radians.
- **Angular Velocity ( $\omega$ ):** The rate of change of angular position:  $\omega = \frac{d\theta}{dt}$  (rad/sec).
- **Angular Acceleration ( $\alpha$ ):** The rate of change of angular velocity:  $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$  (rad/sec<sup>2</sup>).

#### 3.2 Constant Angular Acceleration Equations

When the angular acceleration of the body is constant ( $\alpha = \alpha_c$ ), the angular kinematics equations parallel the rectilinear constant acceleration equations:

$$\omega = \omega_0 + \alpha_c t \quad (19)$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha_c t^2 \quad (20)$$

$$\omega^2 = \omega_0^2 + 2\alpha_c(\theta - \theta_0) \quad (21)$$

#### 3.3 Motion of a Point on a Rotating Rigid Body

For a point located at a distance  $r$  from the fixed axis of rotation  $O$ :

$$\text{Linear Velocity: } v = \omega r \quad (22)$$

$$\text{Tangential Acceleration: } a_t = \alpha r \quad (23)$$

$$\text{Normal (Centripetal) Acceleration: } a_n = \omega^2 r = \frac{v^2}{r} \quad (24)$$

The total linear acceleration of the point is the vector sum:  $\vec{a} = \vec{a}_t + \vec{a}_n$ , with magnitude  $a = \sqrt{a_t^2 + a_n^2}$ .

#### 3.4 Relative Kinematics on a Rigid Link

The velocity of a point  $B$  on a rigid body relative to a point  $A$  on the same body is analyzed using the relative-velocity equation:

$$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A} = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A} \quad (25)$$

For 2D planar motion, the magnitude of relative velocity is  $v_{B/A} = \omega r_{B/A}$ , and its direction is perpendicular to the position vector  $\vec{r}_{B/A}$ .

## 4 Mass Moments of Inertia of Standard Shapes

The mass moment of inertia ( $I$ ) represents an object's resistance to rotational acceleration about a specific axis. It is the rotational analog of mass in linear systems and is defined as:

$$I = \int r^2 dm \quad (26)$$

where  $r$  is the perpendicular distance from the axis of rotation to the differential mass element  $dm$ . The units of mass moment of inertia are  $\text{kg} \cdot \text{m}^2$  (SI) or  $\text{slug} \cdot \text{ft}^2$  (USCS).

### 4.1 Standard Shapes from NCEES Handbook

The following standard shapes are routinely tested on the exam. All formulas below assume a uniform density and a total mass  $M$ .

#### 1. Thin Slender Rod (length $L$ ):

- About centroidal axis perpendicular to rod ( $I_{cm}$ ):

$$I_{cm} = \frac{1}{12}ML^2 \quad (27)$$

- About an end axis perpendicular to rod ( $I_{end}$ ):

$$I_{end} = \frac{1}{3}ML^2 \quad (28)$$

#### 2. Thin Slender Ring (radius $R$ ):

- About central polar axis ( $I_{zc}$ ):

$$I_{zc} = MR^2 \quad (29)$$

- About centroidal diametrical axes ( $I_{xc}, I_{yc}$ ):

$$I_{xc} = I_{yc} = \frac{1}{2}MR^2 \quad (30)$$

#### 3. Solid Circular Cylinder or Thin Disk (radius $R$ , height $h$ ):

- About longitudinal centroidal axis ( $y$ ):

$$I_{yc} = I_y = \frac{1}{2}MR^2 \quad (31)$$

- About transverse centroidal axes ( $x, z$ ):

$$I_{xc} = I_{zc} = \frac{1}{12}M(3R^2 + h^2) \quad (32)$$

### 4.2 Radius of Gyration

The radius of gyration ( $k$ ) represents the radial distance from the axis of rotation at which the entire mass of the body could be concentrated without altering its moment of inertia:

$$k = \sqrt{\frac{I}{M}} \implies I = Mk^2 \quad (33)$$

## 5 The Parallel Axis Theorem & Composite Inertia

Calculating the moment of inertia about arbitrary axes is a foundational skill in dynamics, especially when analyzing systems that rotate about off-center pins or hinges.

### 5.1 The Parallel Axis Theorem

The Parallel Axis Theorem (also known as the Huygens-Steiner theorem) states that the mass moment of inertia  $I$  of a rigid body about any axis parallel to a centroidal axis is equal to the moment of inertia about the centroidal axis plus the product of the total mass and the square of the perpendicular distance between the two axes:

$$I = I_{cm} + Md^2 \quad (34)$$

Where:

- $I$  Moment of inertia about the new, non-centroidal parallel axis
- $I_{cm}$  Centroidal moment of inertia of the body (passing through the center of mass)
- $M$  Total mass of the body
- $d$  Perpendicular distance between the two parallel axes

**Significance:** For parallel axes,  $I = I_{cm} + Md^2 \geq I_{cm}$ . This comparison uses the same axis direction; different centroidal directions can have different inertias.

### 5.2 Composite Bodies (Assemblies)

For a rigid body composed of several simpler sub-shapes (e.g., a rod and a disk welded together), the total moment of inertia about a specified axis is the sum of the moments of inertia of each sub-shape about that same axis:

$$I_{total} = \sum_{i=1}^n I_i = \sum_{i=1}^n (I_{cm,i} + M_i d_i^2) \quad (35)$$

### 5.3 Recommended Tabular Calculation Layout

To avoid algebraic errors under exam pressure, utilize a structured table to compile components.

| Component Part ( $i$ ) | Mass $M_i$  | Centroidal $I_{cm,i}$    | Distance $d_i$ | Parallel Term $M_i d_i^2$ | Total $I_i$            |
|------------------------|-------------|--------------------------|----------------|---------------------------|------------------------|
| 1. Rod AB              | $M_1$       | $\frac{1}{12} M_1 L_1^2$ | $d_1$          | $M_1 d_1^2$               | $I_{cm,1} + M_1 d_1^2$ |
| 2. Cylinder C          | $M_2$       | $\frac{1}{2} M_2 R_2^2$  | $d_2$          | $M_2 d_2^2$               | $I_{cm,2} + M_2 d_2^2$ |
| Sum ( $\Sigma$ )       | $M_{total}$ |                          |                |                           | $I_{total,A}$          |

## 6 Kinetics of Particles - Force and Acceleration ( $\mathbf{F} = m\mathbf{a}$ )

Kinetics relates the forces acting on a particle to its resulting motion. The fundamental equation of kinetics is Newton's Second Law, which states that the net unbalanced external force acting on a particle is equal to the product of its mass and acceleration:

$$\sum \vec{F} = m\vec{a} \quad (36)$$

### 6.1 Equations of Motion in Rectangular Coordinates

For a particle moving in a 2D plane under rectangular coordinates, Newton's Second Law is resolved into two scalar algebraic equations:

$$\sum F_x = ma_x \quad (37)$$

$$\sum F_y = ma_y \quad (38)$$

If a system of connected particles is analyzed (such as blocks on inclines or pulleys), write free-body diagrams (FBDs) for each particle separately, and then link their accelerations using constraint equations.

### 6.2 Equations of Motion in Curvilinear (Normal-Tangential) Coordinates

When a particle moves along a curved path of radius of curvature  $\rho$ , the equations of motion are best analyzed along the normal and tangential axes:

$$\sum F_t = ma_t = m \frac{dv}{dt} \quad (\text{describes the rate of change of speed}) \quad (39)$$

$$\sum F_n = ma_n = m \frac{v^2}{\rho} \quad (\text{describes the centripetal force pointing inward}) \quad (40)$$

### 6.3 Friction-Limited Motion on Flat Curves

Consider a vehicle of mass  $m$  traversing a horizontal, flat circular curve of radius  $R$  at a speed  $v$ . If there is no road superelevation (banking), the only force preventing the vehicle from sliding laterally off the curve is the static friction force  $F_f$  between the tires and the road.

- Vertical equilibrium:  $N = mg$
- Required centripetal force (normal direction):  $\sum F_n = m \frac{v^2}{R} = F_f$
- Maximum available static friction force:  $F_{f,max} = \mu_s N = \mu_s mg$
- Safety boundary: The required force must not exceed the maximum friction:

$$m \frac{v^2}{R} \leq \mu_s mg \implies v_{max} = \sqrt{\mu_s g R} \quad \text{or} \quad R_{min} = \frac{v^2}{\mu_s g} \quad (41)$$

## 7 Kinetics of Rigid Bodies - Planar Equations of Motion

Rigid body kinetics applies Newton's Second Law to three-dimensional structures with non-negligible dimensions. In planar rigid body kinetics, we analyze the translational acceleration of the center of mass and the rotational acceleration of the body.

### 7.1 Planar Equations of Motion

For a planar rigid body of total mass  $M$  undergoing general planar motion in the  $x$ - $y$  plane, three scalar equations describe its kinetics:

$$\sum F_x = Ma_{cm,x} \quad (42)$$

$$\sum F_y = Ma_{cm,y} \quad (43)$$

$$\sum M_{cm} = I_{cm}\alpha \quad (44)$$

|        |                      |                                                                                            |
|--------|----------------------|--------------------------------------------------------------------------------------------|
| Where: | $a_{cm,x}, a_{cm,y}$ | Acceleration components of the center of mass (m/s <sup>2</sup> or ft/s <sup>2</sup> )     |
|        | $I_{cm}$             | Centroidal mass moment of inertia of the body (kg·m <sup>2</sup> or slug·ft <sup>2</sup> ) |
|        | $\alpha$             | Angular acceleration of the body (rad/s <sup>2</sup> )                                     |
|        | $\sum M_{cm}$        | Sum of moments of all external forces about the center of mass                             |

### 7.2 Rotation About a Fixed Frictionless Pin at Point O

If a rigid body is pinned at point  $O$  and rotates about a fixed axis through  $O$ , we can sum moments directly about  $O$  to eliminate the unknown reaction forces at the pin:

$$\sum M_O = I_O\alpha \quad (45)$$

where  $I_O = I_{cm} + Md^2$  (using the parallel axis theorem).

### 7.3 Dynamic Pin Reactions

Although pin reactions do not appear in the moment equation about the pin  $O$ , they are critical for structural support analysis. The center of mass of the body accelerates along circular paths of radius  $d$  (distance from  $O$  to center of mass):

$$a_{cm,t} = \alpha d \quad (\text{tangential to path}) \quad (46)$$

$$a_{cm,n} = \omega^2 d \quad (\text{normal, pointing directly toward pin } O) \quad (47)$$

Applying Newton's Second Law along normal and tangential coordinate axes gives:

$$\sum F_t = Ma_{cm,t} = M\alpha d \quad (48)$$

$$\sum F_n = Ma_{cm,n} = M\omega^2 d \quad (49)$$

These equations allow direct computation of the dynamic reaction forces at the support pin.

## 8 Work, Energy, and Power of Particles & Rigid Bodies

The principle of work and energy is highly efficient for solving dynamics problems that involve velocity, displacement, and forces, without requiring direct computation of accelerations.

### 8.1 Work of a Force

Work ( $U_{1 \rightarrow 2}$ ) is done when a force  $\vec{F}$  undergoes a displacement  $d\vec{s}$ :

$$U_{1 \rightarrow 2} = \int_{s_1}^{s_2} \vec{F} \cdot d\vec{s} = \int_{s_1}^{s_2} F \cos \theta ds \quad (50)$$

- **Work of Gravity:** Under a vertical displacement  $\Delta y = y_2 - y_1$ :

$$U_g = -W\Delta y = -mg(y_2 - y_1) \quad (51)$$

- **Work of a Spring (Stiffness  $k$ ):** For change in spring stretch  $s$ :

$$U_s = -\left(\frac{1}{2}ks_2^2 - \frac{1}{2}ks_1^2\right) \quad (52)$$

### 8.2 Kinetic Energy (T)

Kinetic energy represents a body's capacity to do work due to its motion.

- **Translating Particle:**  $T = \frac{1}{2}mv^2$
- **Rigid Body (General Planar Motion):** Translational and rotational components:

$$T = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2 \quad (53)$$

- **Rigid Body Rotating about Fixed Pivot  $O$ :** Expressed as a single term:

$$T = \frac{1}{2}I_O\omega^2 \quad (54)$$

### 8.3 Principle of Work and Energy & Conservation

The work-energy theorem states that the final kinetic energy of a system equals its initial kinetic energy plus the net work done by all external forces:

$$T_1 + \sum U_{1 \rightarrow 2} = T_2 \quad (55)$$

If all acting forces are conservative (such as gravity and ideal springs), energy is conserved:

$$T_1 + V_1 = T_2 + V_2 \quad (56)$$

where  $V_g = mgh$  and  $V_e = \frac{1}{2}ks^2$  are gravitational and elastic potential energies.

### 8.4 Power and Mechanical Efficiency

**Power ( $P$ ):** The rate at which work is performed:

$$P = \frac{dU}{dt} = \vec{F} \cdot \vec{v} \quad (\text{linear}) \quad \text{or} \quad P = M\omega \quad (\text{rotational}) \quad (57)$$

Units: Watts (1 W = 1 J/s = 1 N · m/s) or Horsepower (1 hp = 550 ft · lb/s).

**Mechanical Efficiency ( $\eta$ ):**  $\eta = \frac{P_{out}}{P_{in}} = \frac{\text{Energy Output}}{\text{Energy Input}} \leq 1.0$

## 9 Linear Impulse, Momentum, and Collisions (Impact)

Impulse-momentum formulations are optimized for problems involving force, velocity, and time. This approach is highly useful for impulsive loadings, sudden impacts, and collisions.

### 9.1 Linear Impulse and Momentum

Linear momentum is defined as the product of mass and velocity ( $\vec{L} = m\vec{v}$ ). Linear impulse is the integral of force over time ( $\vec{I} = \int \vec{F}dt$ ). They are linked by the impulse-momentum principle:

$$m\vec{v}_1 + \sum \int_{t_1}^{t_2} \vec{F}dt = m\vec{v}_2 \quad (58)$$

### 9.2 Conservation of Linear Momentum

If the sum of external forces acting on a system of particles is zero, the total linear momentum of the system is conserved:

$$\sum m_i \vec{v}_{i,1} = \sum m_i \vec{v}_{i,2} \quad (59)$$

### 9.3 Collisions (Impact)

In a collision, system momentum is conserved when external impulse is negligible over the impact interval. Kinetic energy can be lost to deformation and heat.

$$m_A v_{A1} + m_B v_{B1} = m_A v_{A2} + m_B v_{B2} \quad (60)$$

### 9.4 The Coefficient of Restitution (e)

The coefficient of restitution is the ratio of relative separation velocity to relative approach velocity along the line of impact:

$$e = \frac{v_{B2} - v_{A2}}{v_{A1} - v_{B1}} \quad (61)$$

Depending on the value of  $e$ , collisions are categorized into three regimes:

1. **Perfectly Elastic Collision** ( $e = 1$ ): No kinetic energy is lost.
2. **Perfectly Plastic Central Impact** ( $e = 0$ ): In a one-dimensional central impact, both bodies have the same final line-of-impact velocity. This does not imply sticking in every oblique collision.

$$v_2 = \frac{m_A v_{A1} + m_B v_{B1}}{m_A + m_B} \quad (62)$$

3. **Inelastic Collision** ( $0 < e < 1$ ): Some energy is lost to heat and deformation. This is the realistic physical behavior of colliding objects.

## 10 Example Problems and Solutions

### Problem 1: Rectilinear Stopping distance Uphill

A vehicle moves uphill at 30 mph (44 ft/s) on a +2% grade. Its constant braking-only deceleration is  $10.0 \text{ ft/s}^2$ , excluding gravity. Find its stopping time.

#### Solution

1. The grade is small ( $\theta \approx 0.02$ ). The gravity component slowing the car is  $a_g = g \sin \theta \approx 32.2 \text{ ft/s}^2 \times 0.02 = 0.644 \text{ ft/s}^2$ . 2. The total deceleration is  $a_{total} = a_{braking} + a_g = 10.0 + 0.644 = 10.644 \text{ ft/s}^2$ . 3. Since acceleration is constant, use  $v = v_0 - at \implies 0 = 44 - 10.644t \implies t = 44/10.644 = \mathbf{4.13}$  seconds.

### Problem 2: Variable Force and Calculus Integration

A 550 kg mass initially at rest is acted upon by a time-varying force  $F(t) = 50e^t$  Newtons. Determine the displacement of the mass after 4.0 seconds.

#### Solution

1. Find acceleration:  $a(t) = F(t)/M = 50e^t/550 = \frac{1}{11}e^t \text{ m/s}^2$ . 2. Integrate for velocity  $v(t)$ :  $v(t) = \int \frac{1}{11}e^t dt = \frac{1}{11}e^t + C_1$ . Since  $v(0) = 0 \implies C_1 = -\frac{1}{11}$ . Thus,  $v(t) = \frac{1}{11}(e^t - 1) \text{ m/s}$ . 3. Integrate for position  $x(t)$ :  $x(t) = \int \frac{1}{11}(e^t - 1) dt = \frac{1}{11}(e^t - t) + C_2$ . Since  $x(0) = 0 \implies C_2 = -\frac{1}{11}$ . Thus,  $x(t) = \frac{1}{11}(e^t - t - 1) \text{ m}$ . 4. At  $t = 4 \text{ s}$ :  $x(4) = \frac{e^4 - 4 - 1}{11} = \frac{54.598 - 5}{11} = \mathbf{4.51}$  meters.

### Problem 3: Multi-Stage Particle Motion

A particle starting from rest accelerates uniformly at  $3 \text{ m/s}^2$  for 2 s. It then immediately decelerates uniformly to a stop over an additional distance of 8 m. What is the total elapsed time?

#### Solution

1. Phase 1: Velocity at  $t_1 = 2 \text{ s}$  is  $v_1 = at_1 = 3 \times 2 = 6 \text{ m/s}$ . 2. Phase 2: Starts with  $v_i = 6 \text{ m/s}$  and stops ( $v_f = 0$ ) in  $\Delta s = 8 \text{ m}$ . Find deceleration:  $v_f^2 = v_i^2 + 2a\Delta s \implies 0 = 6^2 + 2a(8) \implies a = -2.25 \text{ m/s}^2$ . 3. Phase 2 time:  $t_2 = (v_f - v_i)/a = (0 - 6)/(-2.25) = 2.67 \text{ s}$ . Total time  $t = 2 + 2.67 = \mathbf{4.67}$  seconds.

### Problem 4: Projectile Landing Distance

A motorcycle launches off a ramp at  $20^\circ$  at a speed of  $17.5 \text{ m/s}$ . If a landing ramp of the same height is placed a horizontal distance  $X$  away, find the required distance  $X$ .

#### Solution

1. Total flight time:  $t_{air} = \frac{2v_0 \sin \theta}{g} = \frac{2 \times 17.5 \sin 20^\circ}{9.81} = 1.22 \text{ seconds}$ . 2. Horizontal range:  $X = (v_0 \cos \theta)t_{air} = 17.5 \cos 20^\circ \times 1.22 = \mathbf{20.06}$  meters.

### Problem 5: Projectile Radius of Curvature

For the projectile in Problem 4, calculate the radius of curvature of its path at its peak trajectory.

#### Solution

1. At the peak, vertical velocity  $v_y = 0$ . Total velocity is purely horizontal:  $v_x = v_0 \cos \theta = 17.5 \cos 20^\circ = 16.445 \text{ m/s}$ . 2. The only acceleration acting on the projectile at the peak is gravity ( $a = g = 9.81 \text{ m/s}^2$ , pointing downward), which acts perpendicular to the path. Hence,  $a_n = g$ . 3.  $a_n = v^2/\rho \implies g = v_x^2/\rho \implies \rho = v_x^2/g = 16.445^2/9.81 = \mathbf{27.57}$  meters.

**Problem 6: Constant Angular Deceleration**

A rotating disk is slowing down with a constant angular deceleration. At  $t = 2$  s, the angular velocity is 12 rad/s and at  $t = 4$  s, it is 9 rad/s. Find the time (from  $t = 0$ ) to come to a stop.

**Solution**

1. Angular acceleration  $\alpha = \frac{\omega_4 - \omega_2}{t_4 - t_2} = \frac{9 - 12}{4 - 2} = -1.5 \text{ rad/s}^2$ . 2. Find initial angular velocity:  $\omega_2 = \omega_0 + \alpha t_2 \implies 12 = \omega_0 - 1.5(2) \implies \omega_0 = 15 \text{ rad/s}$ . 3. Solve for stopping time  $t$  ( $\omega_t = 0$ ):  $0 = \omega_0 + \alpha t \implies 0 = 15 - 1.5t \implies t = \mathbf{10.0}$  seconds.

**Problem 7: Gear Total Acceleration**

A gear rotates about its center. The angular position of line OP ( $r = 2$  ft) is given by  $\theta(t) = 2t^3 - 7t^2$  rad. Calculate the magnitude of the total linear acceleration of point P at  $t = 2$  s.

**Solution**

1. Angular velocity:  $\omega(t) = d\theta/dt = 6t^2 - 14t$ . At  $t = 2$ :  $\omega = 6(2^2) - 14(2) = -4 \text{ rad/s}$ . 2. Angular acceleration:  $\alpha(t) = d\omega/dt = 12t - 14$ . At  $t = 2$ :  $\alpha = 12(2) - 14 = 10 \text{ rad/s}^2$ . 3. Tangential acceleration:  $a_t = \alpha r = 10 \times 2 = 20 \text{ ft/s}^2$ . 4. Normal acceleration:  $a_n = \omega^2 r = (-4)^2 \times 2 = 32 \text{ ft/s}^2$ . 5. Total acceleration:  $a = \sqrt{a_t^2 + a_n^2} = \sqrt{20^2 + 32^2} = \sqrt{1424} = \mathbf{37.74 \text{ ft/s}^2}$ .

**Problem 8: Welded Rod-Cylinder Assembly Inertia**

A slender rod ( $M = 2.5 \text{ kg}$ ,  $L = 1.0 \text{ m}$ ) is welded end-to-end to a solid cylinder ( $M = 12.0 \text{ kg}$ ,  $D = 10 \text{ cm}$ ,  $h = 20 \text{ cm}$ ) with the cylinder axis collinear with the rod. The assembly rotates in the plane about pin A, so the rotation axis is transverse to the cylinder and its center is  $L + h/2$  from A. Find the mass moment of inertia about A.

**Solution**

1. **Rod (Part 1):** Rotating about its end.  $I_{rod,A} = \frac{1}{3} M_1 L^2 = \frac{1}{3} (2.5)(1.0)^2 = 0.833 \text{ kg}\cdot\text{m}^2$ . 2. **Cylinder (Part 2):** Perpendicular to transverse centroidal axis.  $I_{cyl,cm} = \frac{1}{12} M_2 (3R^2 + h^2) = \frac{1}{12} (12)[3(0.05)^2 + 0.2^2] = 0.0475 \text{ kg}\cdot\text{m}^2$ . Center of cylinder is at distance  $d = L + h/2 = 1.0 + 0.1 = 1.1 \text{ m}$  from pin A. 3. Apply Parallel Axis Theorem for cylinder:  $I_{cyl,A} = I_{cyl,cm} + M_2 d^2 = 0.0475 + 12(1.1)^2 = 14.568 \text{ kg}\cdot\text{m}^2$ . 4. Total Inertia:  $I_{total,A} = I_{rod,A} + I_{cyl,A} = 0.833 + 14.568 = \mathbf{15.40 \text{ kg}\cdot\text{m}^2}$ .

**Problem 9: Pinned Rod-Disk Inertia**

A disk ( $M = 8 \text{ kg}$ ,  $D = 0.6 \text{ m}$ ) is connected at B to a slender rod AB ( $M = 1 \text{ kg}$ ,  $L = 2.5 \text{ m}$ ). The rod is tangent to the disk at B, so  $BC = R$  is perpendicular to AB and  $AC = \sqrt{L^2 + R^2}$ . The rigid assembly rotates about A. Find its mass moment of inertia about A.

**Solution**

1. Rod:  $I_{r,A} = \frac{1}{3} (1)(2.5)^2 = 2.0833 \text{ kg}\cdot\text{m}^2$ . 2. Disk:  $I_{d,C} = \frac{1}{2} (8)(0.3)^2 = 0.36 \text{ kg}\cdot\text{m}^2$ . Use  $d^2 = 2.5^2 + 0.3^2 = 6.34 \text{ m}^2$  without rounding  $d$ . 3. Total:  $I_A = 2.0833 + 0.36 + 8(6.34) = \mathbf{53.1633 \text{ kg}\cdot\text{m}^2}$ .

**Problem 10: Circular Road Curve Lateral Slip**

A 1000 kg car travels around a flat, unbanked circular curve. The coefficient of static friction is 0.35. If the car's speed is 90 km/h (25 m/s), find the minimum curve radius required to prevent sliding.

**Solution**

1. Friction must supply the centripetal acceleration:  $F_f = \frac{mv^2}{R} \leq \mu_s mg$ . 2. Rearranging gives:  $R_{min} = \frac{v^2}{\mu_s g} = \frac{25^2}{0.35 \times 9.81} = \frac{625}{3.4335} = \mathbf{182.0 \text{ meters}}$ .

**Problem 11: Connected Blocks on Incline with Friction**

Block A (10 kg) on a  $15^\circ$  incline is connected by a cable over a frictionless pulley to Block B (20 kg) hanging vertically. The friction coefficient between Block A and the incline is 0.3. Released from rest, find the time it takes Block B to descend 1.0 meter.

**Solution**

1. Draw FBDs. For Block A, normal force is  $N = m_A g \cos 15^\circ = 10 \times 9.81 \cos 15^\circ = 94.76$  N. 2. Friction force on Block A:  $F_f = \mu N = 0.3 \times 94.76 = 28.43$  N. Gravity component along incline:  $F_{gx} = m_A g \sin 15^\circ = 25.39$  N. 3. For the combined system, the pulling force is Block B's weight:  $F_{pull} = m_B g = 20 \times 9.81 = 196.2$  N. 4. System equation of motion:  $\sum F = M_{total} a \Rightarrow m_B g - m_A g \sin 15^\circ - F_f = (m_A + m_B) a$   $196.2 - 25.39 - 28.43 = 30a \Rightarrow 142.38 = 30a \Rightarrow a = 4.746$  m/s<sup>2</sup>. 5. Time to travel 1.0 m:  $s = \frac{1}{2} a t^2 \Rightarrow 1.0 = \frac{1}{2} (4.746) t^2 \Rightarrow t = \sqrt{2/4.746} = \mathbf{0.65}$  seconds.

**Problem 12: Rotational Swing Work-Energy**

The welded rod-cylinder assembly in Problem 8 ( $I_A = 15.4$  kg-m<sup>2</sup>) is released from rest in a horizontal position and swings downward. Determine the angular velocity when the assembly reaches a vertical orientation.

**Solution**

1. With the pivot as datum,  $V_1 = 0$  initially and  $V_2 < 0$  at the bottom. The rod centroid drops 0.5 m and the cylinder centroid drops 1.1 m. 2. Released energy:  $V_1 - V_2 = 2.5(9.81)(0.5) + 12(9.81)(1.1) = 141.7545$  J. 3.  $\frac{1}{2} I_A \omega^2 = 141.7545$  with  $I_A = 15.4008$  kg · m<sup>2</sup>, giving  $\omega = \sqrt{2(141.7545)/15.4008} = \mathbf{4.2905}$  rad/s.

**Problem 13: Swinging Slender Rod**

A slender rod (2 m long, mass 10 kg) is released from rest in a horizontal position and swings downward about pivot O at one end. Calculate its angular velocity at the vertical position.

**Solution**

1. Centroidal inertia  $I_{cm} = \frac{1}{12} M L^2$ . About pivot O:  $I_O = \frac{1}{3} M L^2 = \frac{1}{3} (10)(2^2) = 13.33$  kg-m<sup>2</sup>. 2. Center of mass drops  $h = L/2 = 1.0$  m.  $\Delta V = Mgh = 10 \times 9.81 \times 1.0 = 98.1$  Joules. 3. Kinetic energy vertical:  $T_2 = \frac{1}{2} I_O \omega^2 = 6.67 \omega^2$ . 4.  $6.67 \omega^2 = 98.1 \Rightarrow \omega = \sqrt{98.1/6.67} = \mathbf{3.84}$  rad/sec.

**Problem 14: Linear Conservation of Momentum**

A 2 kg clay ball moving at 40 m/s collides head-on with a 5 kg clay ball moving directly toward it at 10 m/s. If they stick together after collision, determine their final velocity.

**Solution**

1. Define direction of first ball as positive:  $v_1 = +40$  m/s,  $v_2 = -10$  m/s. 2. Total initial momentum:  $P_1 = m_1 v_1 + m_2 v_2 = 2(40) + 5(-10) = 80 - 50 = 30$  kg · m/s. 3. Since they stick, the collision is perfectly plastic ( $e = 0$ ). Final mass  $M_f = 2 + 5 = 7$  kg. 4.  $v_f = P_1/M_f = 30/7 = \mathbf{4.29}$  m/s (in the positive direction).

**Problem 15: Centripetal Tension in a String**

A child flies a 1.0 kg toy airplane in a horizontal circular path of radius 1.5 m at 10.0 m/s. The string lies horizontally in the plane of motion, and aerodynamic lift balances the airplane's weight. Find the string tension.

**Solution**

1. The tension force must supply 100% of the required centripetal acceleration. 2. Centripetal acceleration  $a_n = v^2/R = 10^2/1.5 = 66.67$  m/s<sup>2</sup>. 3. Tension force  $T = m a_n = 1.0 \text{ kg} \times 66.67 \text{ m/s}^2 = \mathbf{66.67}$  Newtons.