

Environmental Engineering FE CBT Exam

Knowledge: Mathematics with 15 Example Problems

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1 Analytic Geometry & Trigonometry

1.1 Straight Lines and Distance Formulas

Analytic geometry connects algebraic equations with geometric representations on 2D and 3D coordinate axes. In the Cartesian plane, the distance d between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad (1)$$

The slope m of a non-vertical line passing through P_1 and P_2 is defined as:

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad (x_1 \neq x_2) \quad (2)$$

Standard Forms of Linear Equations

- **Slope-Intercept Form:** $y = mx + b$ (m = slope, b = y-intercept)
- **Point-Slope Form:** $y - y_1 = m(x - x_1)$
- **General Linear Form:** $Ax + By + C = 0$
- **Perpendicular Distance from Point** (x_0, y_0) **to** $Ax + By + C = 0$:

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}} \quad (3)$$

1.2 Line Relationships

Two lines with slopes m_1 and m_2 are:

- **Parallel** if $m_1 = m_2$.
- **Perpendicular** if $m_1 \cdot m_2 = -1 \implies m_2 = -\frac{1}{m_1}$.

The angle ψ between two intersecting lines is given by:

$$\tan \psi = \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right| \quad (4)$$

Worked Example A.1: Perpendicular Pipe Alignment Distance

Problem: An environmental monitoring well is located at point $W(4, 8)$. A groundwater sampling pipe runs along the line $3x + 4y - 12 = 0$. Determine the shortest distance from the monitoring well to the pipe line.

Solution: Using the perpendicular distance formula with $A = 3$, $B = 4$, $C = -12$, $x_0 = 4$, and $y_0 = 8$:

$$d = \frac{|3(4) + 4(8) - 12|}{\sqrt{3^2 + 4^2}} = \frac{|12 + 32 - 12|}{\sqrt{25}} = \frac{32}{5} = 6.4 \text{ units}$$

1.3 Conic Sections and General Equations

The general second-degree equation representing any conic section in the xy -plane is:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \quad (5)$$

When $B = 0$ (axes aligned with coordinate axes), the discriminant $\Delta = B^2 - 4AC$ determines the conic classification:

Table 1: Classification of Conic Sections ($B = 0$)

Discriminant Condition	Conic Section Type	Standard Form Equation
$A = C$ ($B^2 - 4AC < 0$)	Circle	$(x - h)^2 + (y - k)^2 = r^2$
$A \neq C$, $AC > 0$	Ellipse	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$
$AC = 0$ ($A = 0$ or $C = 0$)	Parabola	$(y - k)^2 = 2p(x - h)$ or $(x - h)^2 = 2p(y - k)$
$AC < 0$	Hyperbola	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

Circle Geometry Formulas

For a circle with center (h, k) and radius r :

- **General Form Expanded:** $x^2 + y^2 + Dx + Ey + F = 0$ where $h = -D/2$, $k = -E/2$, and $r = \sqrt{h^2 + k^2 - F}$.
- **Length of Tangent Line** from external point (x', y') to circle $(x - h)^2 + (y - k)^2 = r^2$:

$$t = \sqrt{(x' - h)^2 + (y' - k)^2 - r^2} \quad (6)$$

- **Area of Circular Sector** (angle θ in radians): $A_{\text{sector}} = \frac{1}{2}r^2\theta$
- **Area of Circular Segment:** $A_{\text{segment}} = \frac{1}{2}r^2(\theta - \sin \theta)$

Worked Example A.2: Circular Settling Tank Radius & Area

Problem: A circular primary clarification basin equation is expanded as $x^2 + y^2 - 8x + 12y - 29 = 0$ (in meters). Find the basin center, radius, and surface area.

Solution: Complete the squares for x and y :

$$(x^2 - 8x + 16) + (y^2 + 12y + 36) = 29 + 16 + 36$$

$$(x - 4)^2 + (y + 6)^2 = 81 = 9^2$$

Center $C = (4, -6)$ m, Radius $r = 9$ m.

Surface Area $A = \pi r^2 = \pi(9)^2 = 81\pi \approx 254.47 \text{ m}^2$.

1.4 Trigonometry & Polar Coordinate Systems

Trigonometric functions relate angle measurements to ratios of side lengths in right triangles and coordinate systems.

Essential Trigonometric Identities (FE Reference Handbook)

- **Pythagorean Identities:** $\sin^2 \theta + \cos^2 \theta = 1$, $1 + \tan^2 \theta = \sec^2 \theta$, $1 + \cot^2 \theta = \csc^2 \theta$
- **Sine/Cosine Double-Angle Formulas:** $\sin(2\theta) = 2 \sin \theta \cos \theta$, $\cos(2\theta) = \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta$
- **Sum and Difference Identities:** $\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$ $\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$
- **Law of Sines & Law of Cosines (Oblique Triangles):**

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{and} \quad a^2 = b^2 + c^2 - 2bc \cos A \quad (7)$$

1.5 Polar to Cartesian Conversion

Polar coordinates (r, θ) relate to 2D Cartesian coordinates (x, y) by:

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right) \quad (8)$$

Worked Example A.3: Triangulation of Environmental Sampling Site

Problem: Two observation posts A and B are separated by a baseline distance $c = 500$ m. Post A measures angle $A = 40^\circ$ and Post B measures angle $B = 60^\circ$ to an illegal waste discharge point C . Find distance b from Post A to point C .

Solution: First, calculate interior angle C : $C = 180^\circ - (40^\circ + 60^\circ) = 80^\circ$.
Apply the Law of Sines:

$$\frac{b}{\sin B} = \frac{c}{\sin C} \implies b = c \cdot \frac{\sin B}{\sin C} = 500 \cdot \frac{\sin 60^\circ}{\sin 80^\circ} = 500 \cdot \frac{0.8660}{0.9848} \approx 439.69 \text{ m}$$

2 Algebraic Equations & Roots

2.1 Polynomial Equations and Root Extraction

A polynomial equation of degree n has the general form:

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0 \quad (a_n \neq 0) \quad (9)$$

By the Fundamental Theorem of Algebra, $P_n(x) = 0$ has exactly n roots (real or complex).

Quadratic Equation & Roots

For the quadratic equation $ax^2 + bx + c = 0$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (10)$$

- If $b^2 - 4ac > 0$: Two distinct real roots.
- If $b^2 - 4ac = 0$: One repeated real root ($x = -b/2a$).
- If $b^2 - 4ac < 0$: Two complex conjugate roots ($x = \lambda \pm j\mu$).

2.2 Partial Fraction Expansion

Partial fraction decomposition breaks rational functions $R(x) = \frac{P(x)}{Q(x)}$ into simpler sum terms for integration and inverse Laplace transforms.

Table 2: Partial Fraction Expansion Forms

Factor Type in Denominator $Q(x)$	Corresponding Expansion Term
Linear Factor $(ax + b)$	$\frac{A}{ax+b}$
Repeated Linear Factor $(ax + b)^k$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$
Irreducible Quadratic Factor $(ax^2 + bx + c)$	$\frac{Ax+B}{ax^2+bx+c}$

Worked Example B.1: Partial Fractions in Chemical Reactor Kinetics

Problem: Decompose the reaction rate function $f(s) = \frac{5s+12}{s^2+4s}$ into partial fractions.

Solution: Factor denominator: $s^2 + 4s = s(s + 4)$.

$$\frac{5s + 12}{s(s + 4)} = \frac{A}{s} + \frac{B}{s + 4}$$

Multiply both sides by $s(s + 4)$: $5s + 12 = A(s + 4) + Bs$.

Evaluate at $s = 0$: $12 = 4A \implies A = 3$.

Evaluate at $s = -4$: $5(-4) + 12 = B(-4) \implies -8 = -4B \implies B = 2$.

Result: $\frac{5s+12}{s^2+4s} = \frac{3}{s} + \frac{2}{s+4}$.

2.3 Logarithmic and Exponential Systems

Logarithmic and exponential functions model fundamental environmental phenomena such as pollutant decay, BOD exertion, atmospheric pressure drop, and population projections.

Properties of Logarithms & Exponentials

- **Definition:** $y = \log_b x \iff b^y = x \quad (x > 0, b > 0, b \neq 1)$
- **Natural Logarithm:** $\ln x = \log_e x \quad (e \approx 2.71828)$
- **Product / Quotient Rules:** $\ln(uv) = \ln u + \ln v, \quad \ln(u/v) = \ln u - \ln v$
- **Power Rule:** $\ln(u^k) = k \ln u$
- **Change of Base Formula:** $\log_b x = \frac{\ln x}{\ln b} = \frac{\log_{10} x}{\log_{10} b}$

2.4 Population Projection & Demand Calculations

Environmental infrastructure design (water supply, wastewater flow, solid waste generation) relies on mathematical population forecasting models:

Table 3: NCEES Population Projection Mathematical Models

Model Type	Mathematical Equation	Growth Parameter
Arithmetic (Linear) Growth	$P_t = P_0 + k_a \Delta t$	$k_a = \frac{P_2 - P_1}{t_2 - t_1}$
Geometric (Exponential) Growth	$P_t = P_0 e^{k_g \Delta t} \implies \ln P_t = \ln P_0 + k_g \Delta t$	$k_g = \frac{\ln(P_2/P_1)}{t_2 - t_1}$
Logistic (Declining Growth Rate)	$P_t = \frac{P_{\max}}{1 + e^{-(a+b\Delta t)}}$	$P_{\max} = \text{carrying capacity}$

2.5 Environmental Growth and Decay Models

First-order environmental rate laws follow exponential models:

$$C(t) = C_0 e^{\pm kt} \quad \text{or} \quad \ln\left(\frac{C(t)}{C_0}\right) = \pm kt \quad (11)$$

Where C_0 is initial concentration, k is reaction rate constant (time^{-1}), and t is time.

Table 4: Common Environmental Exponential Models

Process Model	Governing Equation	Key Parameters
Pollutant First-Order Decay	$C(t) = C_0 e^{-kt}$	$k = \text{decay constant}$
BOD Exertion Curve	$y(t) = L_0(1 - e^{-kt})$	$L_0 = \text{ultimate BOD}$
Geometric Population Growth	$P_t = P_0 e^{k_g \Delta t}$	$k_g = \text{growth rate}$
Half-Life Period ($t_{1/2}$)	$t_{1/2} = \frac{\ln 2}{k} \approx \frac{0.693}{k}$	$C(t_{1/2}) = 0.5C_0$

Worked Example B.2: Radioactive Isotope Half-Life Decay

Problem: Radioactive Isotope Tritium (^3H) in contaminated leachate has a decay constant $k = 0.056 \text{ year}^{-1}$. Calculate its half-life and the time required for concentration to decrease by 90% ($C/C_0 = 0.10$).

Solution: 1. Half-life: $t_{1/2} = \frac{\ln 2}{k} = \frac{0.69315}{0.056} \approx 12.38 \text{ years}$.

2. Time for 90% reduction ($C = 0.10C_0$):

$$0.10C_0 = C_0 e^{-0.056t} \implies \ln(0.10) = -0.056t \implies -2.3026 = -0.056t$$

$$t = \frac{-2.3026}{-0.056} \approx 41.12 \text{ years}$$

3 Calculus: Differential & Integral

3.1 Differential Calculus Rules

Differentiation measures instantaneous rate of change and curve slopes.

Fundamental Differentiation Rules

Let $u(x)$ and $v(x)$ be differentiable functions and c be a constant:

- **Derivative of Constant & Power:** $\frac{d}{dx}(c) = 0$, $\frac{d}{dx}(x^n) = nx^{n-1}$
- **Product Rule:** $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
- **Quotient Rule:** $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
- **Chain Rule:** $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$
- **Exponentials & Logarithms:** $\frac{d}{dx}(e^{kx}) = ke^{kx}$, $\frac{d}{dx}(\ln x) = \frac{1}{x}$
- **Trigonometric Derivatives:** $\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(\cos x) = -\sin x$, $\frac{d}{dx}(\tan x) = \sec^2 x$

3.2 L'Hopital's Rule for Indeterminate Limits

When evaluating $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ resulting in $\frac{0}{0}$ or $\frac{\infty}{\infty}$:

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} \quad (\text{provided limit exists}) \quad (12)$$

Worked Example C.1: Critical Point & Minimum Stream DO Deficit

Problem: The Dissolved Oxygen (DO) deficit $D(t)$ in a river downstream of a wastewater discharge is given by $D(t) = 10(e^{-0.2t} - e^{-0.8t})$ mg/L (where t is travel time in days). Find the time t_c of maximum DO deficit.

Solution: Set derivative $\frac{dD}{dt} = 0$:

$$\frac{dD}{dt} = 10(-0.2e^{-0.2t} - (-0.8e^{-0.8t})) = 0 \implies -0.2e^{-0.2t} + 0.8e^{-0.8t} = 0$$

$$0.8e^{-0.8t} = 0.2e^{-0.2t} \implies \frac{e^{-0.2t}}{e^{-0.8t}} = \frac{0.8}{0.2} = 4$$

$$e^{0.6t} = 4 \implies 0.6t = \ln(4) \approx 1.3863 \implies t_c = \frac{1.3863}{0.6} \approx 2.31 \text{ days}$$

3.3 Integral Calculus and Integration Techniques

Integration calculates accumulated quantities, areas under curves, volumes, and mean values over intervals.

Core Integration Rules & Techniques

- **Power Rule:** $\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
- **Logarithmic Integration:** $\int \frac{1}{x} dx = \ln |x| + C$
- **Exponential Integration:** $\int e^{kx} dx = \frac{1}{k} e^{kx} + C$
- **Integration by Substitution (u-substitution):** $\int f(g(x))g'(x)dx = \int f(u)du$
- **Integration by Parts:** $\int u dv = uv - \int v du$
- **Mean Value Theorem for Definite Integrals:**

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx \quad (13)$$

3.4 Area Between Curves and Environmental Accumulation

The net accumulation or area between $f(x)$ and $g(x)$ from $x = a$ to $x = b$ is:

$$A = \int_a^b |f(x) - g(x)| dx \quad (14)$$

Worked Example C.2: Average Diurnal Stormwater Flow Rate

Problem: Stormwater inflow to a detention basin follows $Q(t) = 20 + 10 \sin\left(\frac{\pi t}{12}\right)$ m³/s over $t = 0$ to $t = 24$ hours. Calculate the total daily water volume V and the average inflow rate Q_{avg} .

Solution: 1. Total volume V :

$$V = \int_0^{24} \left(20 + 10 \sin\left(\frac{\pi t}{12}\right) \right) dt = \left[20t - 10 \left(\frac{12}{\pi} \right) \cos\left(\frac{\pi t}{12}\right) \right]_0^{24}$$

$$V = \left(20(24) - \frac{120}{\pi} \cos(2\pi) \right) - \left(0 - \frac{120}{\pi} \cos(0) \right) = 480 - \frac{120}{\pi}(1) + \frac{120}{\pi}(1) = 480 \text{ m}^3/\text{s} \cdot \text{hr}$$

Converting hours to seconds: $V = 480 \times 3600 = 1,728,000 \text{ m}^3$.

2. Average flow rate $Q_{\text{avg}} = \frac{1}{24-0} \int_0^{24} Q(t) dt = \frac{480}{24} = 20 \text{ m}^3/\text{s}$.

3.5 Ordinary Differential Equations (ODEs)

Differential equations express relationships between changing physical quantities and their rates of change. They form the basis of environmental reactor modeling, hydrology, and contaminant transport.

3.5.1 First-Order Ordinary Differential Equations

A first-order linear ODE has the general canonical form:

$$\frac{dy}{dx} + P(x)y = Q(x) \quad (15)$$

This is solved using the **Integrating Factor** $I(x) = e^{\int P(x)dx}$:

$$y(x) = \frac{1}{I(x)} \left[\int I(x)Q(x)dx + C \right] \quad (16)$$

Separable First-Order ODEs

If $\frac{dy}{dx} = g(x)h(y)$, separate variables and integrate directly:

$$\int \frac{1}{h(y)} dy = \int g(x) dx + C \quad (17)$$

3.6 Environmental Reactor Mass Balances

Mass balances for continuous flow systems follow the dynamic state equation:

$$\text{Accumulation Rate} = \text{Input Rate} - \text{Output Rate} \pm \text{Reaction Rate} \quad (18)$$

$$V \frac{dC}{dt} = QC_{\text{in}} - QC_{\text{out}} \pm rV \quad (19)$$

Table 5: Steady-State Reactor Performance Comparison ($r = -kC$)

Reactor Type	Effluent Concentration C_{out}	Hydraulic Retention Time $\tau = V/Q$
Ideal Batch Reactor	$C(t) = C_0 e^{-kt}$	$t = \frac{\ln(C_0/C_t)}{k}$
Ideal Plug Flow Reactor (PFR)	$C_{\text{out}} = C_{\text{in}} e^{-k\tau}$	$\tau = \frac{\ln(C_{\text{in}}/C_{\text{out}})}{k}$
Continuously Stirred Tank (CSTR)	$C_{\text{out}} = \frac{C_{\text{in}}}{1+k\tau}$	$\tau = \frac{C_{\text{in}} - C_{\text{out}}}{kC_{\text{out}}}$

Worked Example C.3: CSTR Unsteady-State Pollutant Build-up

Problem: A 1000 m³ aerated lagoon (V) initially contains fresh water ($C(0) = 0$). An inflow $Q = 100$ m³/day containing pollutant $C_{\text{in}} = 50$ mg/L enters the basin. The pollutant decays with $k = 0.1$ day⁻¹. Find $C(t)$.

Solution: Set up ODE: $V \frac{dC}{dt} = QC_{\text{in}} - QC - kVC \Rightarrow \frac{dC}{dt} + \left(\frac{Q}{V} + k\right)C = \frac{Q}{V}C_{\text{in}}$.

$\frac{Q}{V} = \frac{100}{1000} = 0.1$ day⁻¹, so $\frac{dC}{dt} + (0.1 + 0.1)C = 0.1(50) \Rightarrow \frac{dC}{dt} + 0.2C = 5$.

Integrating factor $I(t) = e^{0.2t} \Rightarrow \frac{d}{dt}[Ce^{0.2t}] = 5e^{0.2t}$.

Integrate: $C(t)e^{0.2t} = \frac{5}{0.2}e^{0.2t} + C_1 = 25e^{0.2t} + C_1 \Rightarrow C(t) = 25 + C_1e^{-0.2t}$.

Apply $C(0) = 0 \Rightarrow 0 = 25 + C_1 \Rightarrow C_1 = -25$.

Result: $C(t) = 25(1 - e^{-0.2t})$ mg/L.

3.7 Streeter-Phelps Dissolved Oxygen Sag Equation

In stream water quality modeling, the Streeter-Phelps equation models the dissolved oxygen deficit $D(t) = DO_{\text{sat}} - DO(t)$ resulting from simultaneous biochemical oxygen demand (BOD) deoxygenation and stream reaeration:

$$\frac{dD}{dt} = k_d L(t) - k_r D(t) \quad (20)$$

Where $L(t) = L_0 e^{-k_d t}$ is the remaining ultimate BOD, k_d is the deoxygenation rate constant (day^{-1}), and k_r is the reaeration rate constant (day^{-1}).

Solving this linear first-order ODE yields the classic DO Sag equation:

$$D(t) = \frac{k_d L_0}{k_r - k_d} (e^{-k_d t} - e^{-k_r t}) + D_0 e^{-k_r t} \quad (21)$$

Critical Travel Time (t_c) and Critical Deficit (D_c)

The critical time t_c downriver where the minimum dissolved oxygen concentration occurs is found by setting $\frac{dD}{dt} = 0$:

$$t_c = \frac{1}{k_r - k_d} \ln \left[\frac{k_r}{k_d} \left(1 - \frac{D_0(k_r - k_d)}{k_d L_0} \right) \right] \quad (22)$$

The corresponding maximum deficit D_c at time t_c satisfies:

$$D_c = \frac{k_d}{k_r} L_0 e^{-k_d t_c} \quad (23)$$

3.8 Second-Order Linear Homogeneous ODEs

Second-order linear ODEs with constant coefficients model physical oscillations, groundwater flow, and transient dispersion:

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0 \quad (24)$$

Substitute $y = e^{rx}$ to form the **Characteristic Polynomial**:

$$ar^2 + br + c = 0 \quad (25)$$

Classification of Solutions based on Roots r_1 and r_2

- **Case 1: Overdamped** ($b^2 - 4ac > 0$): Two distinct real roots $r_1 \neq r_2$.

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} \quad (26)$$

- **Case 2: Critically Damped** ($b^2 - 4ac = 0$): One repeated real root $r = -b/2a$.

$$y(x) = (C_1 + C_2 x) e^{rx} \quad (27)$$

- **Case 3: Underdamped** ($b^2 - 4ac < 0$): Complex conjugate roots $r = \alpha \pm j\beta$.

$$y(x) = e^{\alpha x} (C_1 \cos(\beta x) + C_2 \sin(\beta x)) \quad (28)$$

Worked Example C.4: Solving a 2nd-Order Initial-Value Problem

Problem: Solve $y'' + 5y' + 6y = 0$ with initial conditions $y(0) = 2$ and $y'(0) = -1$.

Solution: Characteristic equation: $r^2 + 5r + 6 = 0 \implies (r + 2)(r + 3) = 0 \implies r_1 = -2, r_2 = -3$.

General solution: $y(x) = C_1 e^{-2x} + C_2 e^{-3x}$.

Derivative: $y'(x) = -2C_1 e^{-2x} - 3C_2 e^{-3x}$.

Apply initial conditions:

$$y(0) = C_1 + C_2 = 2$$

$$y'(0) = -2C_1 - 3C_2 = -1$$

From $C_1 = 2 - C_2$, substitute into second equation:

$$-2(2 - C_2) - 3C_2 = -1 \implies -4 + 2C_2 - 3C_2 = -1 \implies -C_2 = 3 \implies C_2 = -3$$

Thus $C_1 = 2 - (-3) = 5$.

Final solution: $y(x) = 5e^{-2x} - 3e^{-3x}$.

4 Numerical Methods & Precision

4.1 Numerical Integration (Quadrature Rules)

Numerical integration approximates definite integrals $\int_a^b f(x)dx$ when closed-form antiderivatives do not exist or when data consists of discrete field measurements. Divide interval $[a, b]$ into n subintervals of uniform width $\Delta x = \frac{b-a}{n}$.

Standard Quadrature Formulas (FE Reference Handbook)

- **Euler's / Rectangular Rule:**

$$\int_a^b f(x)dx \approx \Delta x \sum_{k=0}^{n-1} f(a + k\Delta x) \quad (29)$$

- **Trapezoidal Rule:**

$$\int_a^b f(x)dx \approx \frac{\Delta x}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)] \quad (30)$$

- **Simpson's 1/3 Rule (Requires n to be EVEN):**

$$\int_a^b f(x)dx \approx \frac{\Delta x}{3} \left[f(x_0) + 4 \sum_{k \text{ odd}} f(x_k) + 2 \sum_{k \text{ even}} f(x_k) + f(x_n) \right] \quad (31)$$

Table 6: Order of Accuracy & Truncation Error for Quadrature Rules

Numerical Rule	Polynomial Exactness	Global Truncation Error Bound
Trapezoidal Rule	Exact for linear polynomials (1 st degree)	$E_T = -\frac{(b-a)^3}{12n^2} f''(\xi) \quad \mathcal{O}(\Delta x^2)$
Simpson's 1/3 Rule	Exact for cubics (3 rd degree)	$E_S = -\frac{(b-a)^5}{180n^4} f^{(4)}(\xi) \quad \mathcal{O}(\Delta x^4)$

Worked Example D.1: River Cross-Sectional Area Integration

Problem: River depth measurements $h(x)$ (m) at $x = 0, 2, 4, 6, 8$ meters across a channel are 0.0, 1.8, 2.4, 1.6, 0.0. Estimate the cross-sectional area using Simpson's 1/3 Rule.

Solution: Subinterval width $\Delta x = 2$ m, number of panels $n = 4$ (even).

$h_0 = 0.0$, $h_1 = 1.8$, $h_2 = 2.4$, $h_3 = 1.6$, $h_4 = 0.0$.

Apply Simpson's 1/3 Rule:

$$\begin{aligned} A &\approx \frac{\Delta x}{3} [h_0 + 4(h_1 + h_3) + 2(h_2) + h_4] \\ A &\approx \frac{2}{3} [0.0 + 4(1.8 + 1.6) + 2(2.4) + 0.0] = \frac{2}{3} [0 + 4(3.4) + 4.8 + 0] \\ A &\approx \frac{2}{3} [13.6 + 4.8] = \frac{2}{3} (18.4) = 12.27 \text{ m}^2 \end{aligned}$$

4.2 Root Extraction & Newton-Raphson Method

Newton's Method (Newton-Raphson) is an efficient iterative technique for finding roots of non-linear algebraic equations $f(x) = 0$.

Newton-Raphson Iteration Formula

Given an initial estimate x_0 , subsequent approximations are computed recursively via:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \quad (f'(x_k) \neq 0) \quad (32)$$

- **Convergence Rate:** Quadratic convergence near simple roots ($e_{k+1} \approx Ce_k^2$).
- **Stopping Criteria:** $|x_{k+1} - x_k| < \epsilon$ or $|f(x_k)| < \epsilon$.

Worked Example D.2: Pipe Friction Factor (Colebrook Equation Root)

Problem: Use Newton-Raphson iteration to find the root of $f(x) = x^3 - 5x + 1 = 0$ starting with $x_0 = 2.0$. Perform 2 iterations.

Solution: 1. Derivative: $f'(x) = 3x^2 - 5$.
2. First Iteration ($k = 0$):

$$f(x_0) = f(2.0) = (2)^3 - 5(2) + 1 = 8 - 10 + 1 = -1.0$$

$$f'(x_0) = f'(2.0) = 3(2)^2 - 5 = 12 - 5 = 7.0$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2.0 - \frac{-1.0}{7.0} = 2.0 + 0.14286 = 2.14286$$

3. Second Iteration ($k = 1$):

$$f(x_1) = (2.14286)^3 - 5(2.14286) + 1 = 9.83969 - 10.71430 + 1 = 0.12539$$

$$f'(x_1) = 3(2.14286)^2 - 5 = 8.77555$$

$$x_2 = x_1 - f(x_1)/f'(x_1) = 2.14286 - 0.12539/8.77555 = 2.12857$$

4.3 Taylor Series Expansion

A continuous, infinitely differentiable function $f(x)$ expanded about point $x = a$ is:

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + R_n(x) \quad (33)$$

4.4 Precision Limits & Error Propagation

Measurements in environmental monitoring inherently possess uncertainty and limits of precision.

Error Definitions & Multivariable Propagation

- **Absolute Error:** $E_{\text{abs}} = |x_{\text{measured}} - x_{\text{true}}|$
- **Relative Error:** $E_{\text{rel}} = \frac{|x_{\text{measured}} - x_{\text{true}}|}{x_{\text{true}}}$
- **Percent Error:** $E_{\text{pct}} = E_{\text{rel}} \times 100\%$
- **Kline-McClintock Multivariable Error Propagation:** If a calculated result Y depends on independent measured variables $X_1, X_2, \dots, X_n$ via $Y = f(X_1, X_2, \dots, X_n)$, with individual uncertainties $u_{x_1}, u_{x_2}, \dots, u_{x_n}$, the total result uncertainty u_Y is:

$$u_Y = \sqrt{\sum_{i=1}^n \left(\frac{\partial f}{\partial X_i} u_{x_i} \right)^2} = \sqrt{\left(\frac{\partial f}{\partial X_1} u_{x_1} \right)^2 + \left(\frac{\partial f}{\partial X_2} u_{x_2} \right)^2 + \dots + \left(\frac{\partial f}{\partial X_n} u_{x_n} \right)^2} \quad (34)$$

Worked Example D.3: Uncertainty in Flume Discharge Measurement

Problem: Discharge through a rectangular weir is $Q = 1.84BH^{3/2}$. Channel width $B = 2.0 \text{ m} \pm 0.05 \text{ m}$ ($u_B = 0.05$) and head $H = 1.0 \text{ m} \pm 0.02 \text{ m}$ ($u_H = 0.02$). Calculate nominal discharge Q and its absolute uncertainty u_Q .

Solution: 1. Nominal Discharge: $Q = 1.84(2.0)(1.0)^{1.5} = 3.68 \text{ m}^3/\text{s}$.

2. Partial Derivatives:

$$\frac{\partial Q}{\partial B} = 1.84H^{3/2} = 1.84(1.0)^{1.5} = 1.84$$

$$\frac{\partial Q}{\partial H} = 1.84B \left(\frac{3}{2} H^{1/2} \right) = 2.76BH^{1/2} = 2.76(2.0)(1.0)^{0.5} = 5.52$$

3. Propagated Uncertainty u_Q :

$$u_Q = \sqrt{(1.84 \times 0.05)^2 + (5.52 \times 0.02)^2} = \sqrt{(0.092)^2 + (0.1104)^2}$$

$$u_Q = \sqrt{0.008464 + 0.012188} = \sqrt{0.020652} \approx 0.1437 \text{ m}^3/\text{s}$$

Result: $Q = 3.68 \pm 0.14 \text{ m}^3/\text{s}$ (relative error $\approx 3.9\%$).

5 FE Environmental Math Reference Summary

Table 7: Comprehensive Formula Reference Matrix for Quick Exam Lookup

Domain / Topic	Governing Formula / Relationship	Handbook Page / Context
Perpendicular Distance	$d = \frac{ Ax_0 + By_0 + C }{\sqrt{A^2 + B^2}}$	Mathematics – Analytic Geometry
General Conic Discriminant	$\Delta = B^2 - 4AC$	Conic Sections Classification
Circle Tangent Length	$t = \sqrt{(x' - h)^2 + (y' - k)^2 - r^2}$	Circular Geometry
Law of Cosines	$a^2 = b^2 + c^2 - 2bc \cos A$	Trigonometry
Quadratic Formula	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Polynomial Roots
Half-Life Period	$t_{1/2} = \frac{\ln 2}{k} \approx \frac{0.693}{k}$	Exponential Decay
Product Rule	$(uv)' = uv' + vu'$	Differential Calculus
Quotient Rule	$(u/v)' = \frac{vu' - uv'}{v^2}$	Differential Calculus
Mean Value of Function	$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx$	Integral Calculus
Integrating Factor (1st ODE)	$I(x) = e^{\int P(x) dx}$	Differential Equations
CSTR Effluent	$C_{\text{out}} = \frac{C_{\text{in}}}{1 + k\tau}$	Environmental Engineering Reactors
Trapezoidal Rule	$\int_a^b f(x) dx \approx \frac{\Delta x}{2} [f_0 + 2 \sum f_i + f_n]$	Numerical Integration
Simpson's 1/3 Rule	$\int_a^b f(x) dx \approx \frac{\Delta x}{3} [f_0 + 4 \sum_{\text{odd}} f_i + 2 \sum_{\text{even}} f_i + f_n]$	Numerical Integration (n even)
Newton-Raphson Step	$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$	Root Extraction
Kline-McClintock Error	$u_Y = \sqrt{\sum \left(\frac{\partial f}{\partial X_i} u_{X_i} \right)^2}$	Precision & Measurement

6 15 Example Problems

Example Problem 1: Analytic Geometry

A straight access road to an environmental treatment facility passes through points $A(1, 2)$ and $B(5, 10)$. What is the equation of the line perpendicular to this road that passes through the site office at $P(3, 1)$?

- (A) $x + 2y - 5 = 0$
- (B) $2x + y - 7 = 0$
- (C) $x - 2y - 1 = 0$
- (D) $2x - y - 5 = 0$

Solution

Slope of line through $A(1, 2)$ and $B(5, 10)$: $m_{AB} = \frac{10-2}{5-1} = \frac{8}{4} = 2$.

Perpendicular line slope $m_{\perp} = -\frac{1}{2}$.

Point-slope form at $P(3, 1)$: $y - 1 = -\frac{1}{2}(x - 3) \implies 2y - 2 = -x + 3 \implies x + 2y - 5 = 0$.

Correct Answer: (A)

Example Problem 2: Analytic Geometry

What is the radius of the circular boundary represented by $x^2 + y^2 - 6x + 8y - 11 = 0$?

- (A) 5.0
- (B) 6.0
- (C) 7.0
- (D) 36.0

Solution

Complete squares for $x^2 - 6x + y^2 + 8y = 11$:

$$(x - 3)^2 - 9 + (y + 4)^2 - 16 = 11 \implies (x - 3)^2 + (y + 4)^2 = 11 + 9 + 16 = 36 = 6^2.$$

Radius $r = \sqrt{36} = 6.0$.

Correct Answer: (B)

Example Problem 3: Trigonometry

In an oblique triangle forming a wetland drainage basin, side $a = 10$ m, side $b = 14$ m, and the included angle $C = 60^\circ$. What is the length of side c ?

- (A) 12.49 m
- (B) 14.00 m
- (C) 15.62 m
- (D) 21.00 m

Solution

Apply Law of Cosines: $c^2 = a^2 + b^2 - 2ab \cos C$

$$c^2 = 10^2 + 14^2 - 2(10)(14) \cos(60^\circ) = 100 + 196 - 280(0.5) = 296 - 140 = 156.$$

$$c = \sqrt{156} \approx 12.49 \text{ m} \approx 12.49 \text{ m}$$

Correct Answer: (A)

Example Problem 4: Algebraic Equations

Find the real roots of the cubic equation $x^3 - 7x^2 + 12x = 0$.

- (A) $x = 0$ only
- (B) $x = 3, 4$
- (C) $x = 0, 3, 4$
- (D) $x = -3, -4$

Solution

Factor out x : $x(x^2 - 7x + 12) = 0$.

Factor quadratic: $x(x - 3)(x - 4) = 0 \implies x = 0, 3, 4$.

Correct Answer: (C)

Example Problem 5: Partial Fractions

In evaluating an adsorption column mass transfer expression, the function

$F(s) = \frac{8}{s^2+2s-8}$ is decomposed into partial fractions $\frac{A}{s-2} + \frac{B}{s+4}$. What is the value of coefficient A ?

- (A) 1.33
- (B) 2.00
- (C) -1.33
- (D) 4.00

Solution

Decomposition: $\frac{8}{(s-2)(s+4)} = \frac{A}{s-2} + \frac{B}{s+4} \implies 8 = A(s+4) + B(s-2)$.

Evaluate at $s = 2$: $8 = A(6) \implies A = 8/6 = 1.333$.

Correct Answer: (A)

Example Problem 6: Logarithmic Decay

A river contaminated with industrial organic waste undergoes first-order biodegradation with rate constant $k = 0.25 \text{ day}^{-1}$. How many days are required for the concentration to drop to 5% of its initial level?

- (A) 4.00 days
- (B) 9.21 days
- (C) 11.98 days
- (D) 20.00 days

Solution

$$C/C_0 = 0.05 \implies e^{-0.25t} = 0.05 \implies -0.25t = \ln(0.05) = -2.9957.$$

$$t = \frac{-2.9957}{-0.25} \approx 11.98 \text{ days}.$$

Correct Answer: (C)

Example Problem 7: Differential Calculus

The concentration of a dissolved gas in a water sample as a function of temperature T ($^{\circ}\text{C}$) is given by $C(T) = 50Te^{-0.1T}$ mg/L. At what temperature T is the concentration maximized?

- (A) 5.0°C
- (B) 10.0°C
- (C) 15.0°C
- (D) 20.0°C

Solution

Product rule derivative: $C'(T) = 50e^{-0.1T} + 50T(-0.1e^{-0.1T}) = 50e^{-0.1T}(1 - 0.1T) = 0$.

Since $50e^{-0.1T} \neq 0 \implies 1 - 0.1T = 0 \implies T = 10.0^{\circ}\text{C}$.

Correct Answer: (B)

Example Problem 8: Integral Calculus

Evaluate the definite integral $\int_0^3 (4x^3 - 3x^2 + 2)dx$.

- (A) 54
- (B) 60
- (C) 81
- (D) 108

Solution

Antiderivative: $\int_0^3 (4x^3 - 3x^2 + 2)dx = [x^4 - x^3 + 2x]_0^3 = (81 - 27 + 6) - (0) = 60$.

Correct Answer: (B)

Example Problem 9: Mean Value Theorem

What is the average value of the daily solar radiation function $I(t) = 100 \sin\left(\frac{\pi t}{12}\right)$ W/m² over the 12-hour daylight period $t = 0$ to $t = 12$?

- (A) 50.0 W/m²
- (B) 63.66 W/m²
- (C) 70.71 W/m²
- (D) 100.0 W/m²

Solution

$$I_{\text{avg}} = \frac{1}{12-0} \int_0^{12} 100 \sin\left(\frac{\pi t}{12}\right) dt = \frac{100}{12} \left[-\frac{12}{\pi} \cos\left(\frac{\pi t}{12}\right)\right]_0^{12} = \frac{100}{\pi} [-\cos(\pi) + \cos(0)] = \frac{100}{\pi} (1 + 1) = \frac{200}{\pi} \approx 63.66 \text{ W/m}^2.$$

Correct Answer: (B)

Example Problem 10: First-Order ODE Reactor

A 500 m³ CSTR reactor processes wastewater at flow rate $Q = 50 \text{ m}^3/\text{hr}$. If a first-order decay reaction $k = 0.2 \text{ hr}^{-1}$ occurs, what is the steady-state removal efficiency $\eta = (1 - C_{\text{out}}/C_{\text{in}}) \times 100\%$?

- (A) 33.3%
- (B) 50.0%
- (C) 66.7%
- (D) 80.0%

Solution

Hydraulic retention time $\tau = V/Q = 500/50 = 10 \text{ hr}$.

CSTR effluent ratio: $\frac{C_{\text{out}}}{C_{\text{in}}} = \frac{1}{1+k\tau} = \frac{1}{1+(0.2)(10)} = \frac{1}{3}$.

Efficiency $\eta = (1 - 1/3) \times 100\% = 66.7\%$.

Correct Answer: (C)

Example Problem 11: Characteristic Roots of a Second-Order ODE

The characteristic equation for a groundwater tracer dispersion model is $r^2 + 6r + 25 = 0$. What type of roots does this equation have?

- (A) Two distinct real roots
- (B) One repeated real root
- (C) A complex-conjugate pair
- (D) One zero root and one nonzero root

Solution

For $ar^2 + br + c = 0$, the discriminant is:

$$\Delta = b^2 - 4ac = 6^2 - 4(1)(25) = 36 - 100 = -64 < 0$$

A negative discriminant means the characteristic roots are a complex-conjugate pair:

$$r = (-6 \pm \sqrt{-64})/2 = -3 \pm 4i$$

For a second-order linear ODE with constant coefficients, these roots yield a solution containing an exponential factor and sinusoidal terms. This mathematical root pattern alone does not establish a physical damping classification for a groundwater model.

Correct Answer: (C)

Example Problem 12: Trapezoidal Integration

The flow rate Q (cfs) through a drainage channel measured at 1-hour intervals is given below:

Time t (hr)	0	1	2	3	4
Flow Q (cfs)	10	25	40	30	15

Estimate the total volume of runoff (in cfs·hr) using the Trapezoidal Rule.

- (A) 95 cfs · hr
- (B) 107.5 cfs · hr
- (C) 120 cfs · hr
- (D) 215 cfs · hr

Solution

Trapezoidal Rule ($\Delta t = 1$): $V \approx \frac{1}{2}[Q_0 + 2(Q_1 + Q_2 + Q_3) + Q_4] =$
 $\frac{1}{2}[10 + 2(25 + 40 + 30) + 15] = \frac{1}{2}[10 + 190 + 15] = \frac{215}{2} = 107.5 \text{ cfs} \cdot \text{hr}.$

Correct Answer: (B)

Example Problem 13: Simpson's 1/3 Integration

Using Simpson's 1/3 Rule with $n = 4$ subintervals ($\Delta x = 1$), evaluate

$$\int_0^4 x^2 dx.$$

- (A) 21.33
- (B) 22.00
- (C) 22.67
- (D) 24.00

Solution

Simpson's 1/3 Rule ($\Delta x = 1$): $\int_0^4 x^2 dx \approx \frac{1}{3}[0^2 + 4(1^2 + 3^2) + 2(2^2) + 4^2] =$
 $\frac{1}{3}[0 + 4(10) + 2(4) + 16] = \frac{1}{3}[40 + 8 + 16] = \frac{64}{3} = 21.333$. (Note: Exact integral is $4^3/3 = 21.333$).

Correct Answer: (A)

Example Problem 14: Newton-Raphson Method

An engineer applies Newton's Method to find a root of $f(x) = x^2 - 5 = 0$. Starting with initial guess $x_0 = 2.0$, what is the value of x_1 after one iteration?

- (A) 2.10
- (B) 2.25
- (C) 2.40
- (D) 2.50

Solution

$$f(x) = x^2 - 5 \implies f'(x) = 2x.$$

$$\text{Iteration 1: } x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2.0 - \frac{2.0^2 - 5}{2(2.0)} = 2.0 - \frac{-1}{4.0} = 2.0 + 0.25 = 2.25.$$

Correct Answer: (B)

Example Problem 15: Precision & Error Propagation

The head loss in a pipe is modeled as $h_L = KQ^2$. If the measured flow rate $Q = 4.0 \text{ cfs} \pm 0.2 \text{ cfs}$ (5% uncertainty) and $K = 0.5$ is exact, what is the absolute uncertainty in h_L ?

- (A) 0.40 ft
- (B) 0.80 ft
- (C) 1.60 ft
- (D) 8.00 ft

Solution

$$h_L = KQ^2 = 0.5(4.0)^2 = 8.0 \text{ ft.}$$

$$\text{Derivative: } \frac{dh_L}{dQ} = 2KQ = 2(0.5)(4.0) = 4.0.$$

$$\text{Uncertainty } u_{h_L} = \left| \frac{dh_L}{dQ} \right| u_Q = (4.0)(0.2) = 0.80 \text{ ft.}$$

Correct Answer: (B)