Market Segmentation

- Small Group Cooperation in Games and Economies*

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Abstract

In this paper, we provide a framework as well as the toolbox to analyze large segmented markets, in which a continuum of agents with quasi-linear utility functions interact only within small groups. In the first part, we develop a TU-game model to prove that there is a stable state in any segmented market, where agents are partitioned into small groups such that no group of agents can jointly do better. By building up a connection between the stability problem and some symmetric transport problem, we also provide a full characterization for all stable states. Our work provides the only existence result to this problem at our level of generality, a uniform way to understand diverse solution concepts, such as stable matching, fractional core, f-core and epsilon-sized core, as well as the computation feasibility for finding stable states. In the second part, we study a segmented exchange market. With no centralized clearing house, agents trade only within small groups and a classical competitive equilibrium with a linear expenditure function on traded quantities may not exist. However, we found that every stable state is supported by some potentially nonlinear market clearing price. Our result suggests that market segmentation might lead to price nonlinearity.

Keywords: segmentation, small groups, stability, optimal transport, nonlinear price

JEL classification: C6, D5, R1

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Contents

1 Introduction

In large markets, economic agents usually interact with each other in small groups and there are natural bounds on the scales of such groups. For instance, in the marriage market, marriage couples are groups of two people; in the Over-the-Counter market, traders trade privately in a pairwise manner; in the business world, most hotels can transact with at most as many travelers as the number of rooms they have in one day. The purpose of this paper is to provide a framework as well as the toolbox to analyze the large segmented markets.

There are two natural questions in a segmented market. The first question is a centralized question: whether there is a stable state? That is, we want to find a way to partition the continuum of agent into small groups such that no agent could do better. The second question is a decentralized question: in exchange markets, whether such stable states, if exist, can be supported by some market clearing price? That is, we want to post some trading price such that agents compute their demands under this price, form small groups to trade with each other, and the market is clear, and is segmented endogenously into small groups in a stable way.

To study these two questions, we assume that all agents have quasi-linear utility functions. In this paper, we propose a general TU-game model to study the first question, and an exchange economy model to study the second question. In the following, we introduce these two models in order.

In games, a continuum of players form small groups in order to share group surpluses. Group sizes are bounded by natural numbers or percentiles, and group surpluses are determined by the types of its members. We wish to know whether there is a stable state in this game: that is, whether we can partition the continuum of players into small groups such that agents have a way to share group surpluses and no coalition of players can do better by forming a new group. We use the term stable assignments to denote such partitions. Therefore, our question is whether a stable assignment exists. It is worth noting that when the group size is exactly two, the problem is a matching, or roommate, problem.

In the literature, there are some partial answers to the question we posed: when group sizes are bounded by natural numbers, Kaneko and Wooders [1996] proved an approximately stable assignment exists[1]. It is only known that the approximation notion can be dropped when the continuum of players share a finite number of types[2]. When group sizes are bounded by percentiles, Schmeidler [1972a] proved, in exchange

[1]In their definition of stable assignment, there is an additional approximate feasibility condition. In addition, their framework is for games with non-transferable utility. In this paper, when we mention their work, we talk about the application of their result to a game with transferable utility.

[2]See Wooders [2012] for a proof.

economies, core allocations are not blocked by any group of epsilon sizes. However, this observation alone will not lead to the existence of a stable assignment, since core allocations cannot be achieved by reallocation in small groups.

As a result, to my knowledge, when group sizes are bounded by either natural numbers or percentiles, there is no existence result for general models. To make matters worse, even though we know the existence of a stable assignment when group sizes are finitely bounded and the type space is finite, it is not computationally feasible to use the current method, linear programming, to find a stable assignment, as the number of group types becomes astronomical even when group sizes are very small. For example, when there are 1000 types of players and every group contains up to 4 players, there are $\sum_{n=1}^{4} \left(\!\binom{1000}{n}\!\right) \sim 4.2 \times 10^{10}$ types of groups in this game. If we use linear programming to find a stable assignment, we need to solve a maximization problem with 4.2×10^{10} unknowns, which is not a feasible task computationally.

In this paper, conceptually, we prove the existence of a stable assignment for general type spaces and general surplus functions, when group sizes are bounded by either natural numbers or percentiles. Furthermore, when group sizes are bounded by natural numbers, our work provides a parallel yet simpler formulation for the classical assignment problem; when group sizes are bounded by percentiles, our model is new.

Computationally, when there are finitely many types of agents in the market, our method provides the computational feasibility for finding stable assignments by reducing the number of unknowns from about $|I|^N$, in linear programming, to about $|I|$, where $|I|$ is the number of types, N is maximum group size and $|I|$ is much larger than N. In particular, for games with 1000 types of players and group sizes are up to 4, our method reduces the number of unknowns in the maximization problem from about 4.2×10^{10} to 25000.[3]

Methodologically, we prove the existence results by connecting the stability problem to a multi- or "continuum"- marginal transport problem. For games with finite-size groups, our key observation is that any stable assignment can be identified by a *symmetric* transport plan in some replicated agent spaces. When the maximum group size is N, we replicate the agents space $N!$ many times. We choose the number $N!$ as any group, containing no more than N players, can be represented by a vector of length $N!$ via replications. As a consequence of our observation, we are able to reformulate the welfare maximization problem as a symmetric transport problem. This reformulation helps us in two ways. First, the duality theorem of the multi-marginal transport problems helps us prove the existence of a stable assignment. Second, as Friesecke and Vogler [2018] pointed out, the symmetric structure created by our identification trick helps us reduce the number of unknowns. For games with positive-size groups,

[3]See Section 2.5.2 for more details.

we prove the existence of a stable assignment by extending Kantorovich-Koopmans' duality theorem to a "continuum"-marginal setting.

In economic models, we study a continuum of agents who form small groups of bounded finite size in order to exchange commodities with other group members. We want to know whether there is a competitive equilibrium. That is, whether there is a market price such that all agents consume their optimal consumption under the market price, and the large market is endogenously segmented to balance the supply and the demand in every local exchange market.

In the literature, there are some answers to related but different questions: when group sizes are bounded by epsilon percentiles, Schmeidler [1972a], together with the classical existence and core-equivalence results such as Aumann [1966] and Aumann [1964], proved that there is a competitive equilibrium that cannot be blocked by any epsilon sized coalition. When group sizes are finite but unbounded, Hammond et al. [1989] proved a competitive equilibrium approximately exists[4]. When group sizes are finite and bounded, to my knowledge, there is no relevant material.

Indeed, when group sizes are finite and bounded, it is easy to see that a classical competitive equilibrium, where the expenditure on trading quantities is a linear function, does not exist. For example[5], in an economy with 1000 types of agents, each type is endowed with a different type of good and agents can only trade in groups containing at most 4 people. After trading within small groups, each agent can consume no more than four different types of goods. Therefore, every feasible allocation is far from Pareto efficient. On the other hand, by the first welfare theorem, any allocation supported by a linear expenditure function must be Pareto efficient. Consequently, no linear market clearing price exists in this segmented market. A further observation on the nonexistence of a competitive equilibrium is that, in a segmented market, all trade surplus might go to one of the trading parties[6]. Linear expenditure functions are incompatible with such a phenomenon. However, with nonlinear expenditure functions, agents with market power can extract all surpluses from their trade partners. Therefore, we extend the notion of price to be a potentially nonlinear expenditure function on traded quantities.

In this paper, we develop a new general equilibrium model in which agents can trade only within small groups of bounded finite sizes. By introducing nonlinear prices to general equilibrium models, we prove the existence of a competitive equilibrium with a potentially nonlinear market price. Our results suggest: first, there is a stable state in any segmented exchange market. Second, any stable state can be supported by some

[4]The competitive allocation is assumed to be approximately feasible in Hammond et al. [1989].

[5]See Section 4.4.1 for more detailed analysis.

[6]We give an example of such phenomenon in Section 4.4.2.

potentially nonlinear market clearing price. Last, due to the non-existence of the linear market clearing price, market segmentation might lead to price nonlinearity.

Methodologically, we provide a new constructive proof for the existence of competitive equilibrium. The proof consists of three steps. First, we apply the result in the game part to find a stable segmented market structure. Second, in each local market, we compute an efficient allocation by solving a local welfare maximization problem. Lastly, we construct a nonlinear equilibrium price for this efficient allocation. Due to the nonlinearity of the market price, our construction makes no use of any separation theorem.

This paper is organized as follows. In Section 2, we study games with finite-size groups. In Section 3, we study games with positive-size groups. In Section 4, we study economic models with finite-size groups. We review related literature at the end of each section. In Section 5, we summarize our results.

2 Games with Finite-Size Groups

In this section, we study transferable utility games with a continuum of agents who form small groups in order to share group surpluses. Group sizes are bounded by natural numbers. We use a natural number $N \geq 2$ to denote the upper bound on group sizes and a natural number $N' \leq N$ to denote the lower bound on group sizes.

This section is organized as follows. In Section 2.1, we describe the model. In Section 2.2, we state our results. In Section 2.3, we prove our results. In Section 2.4, we study four examples. In Section 2.5, we discuss three applications of our results. Lastly, in Section 2.6, we review the literature.

2.1 Model

We study a cooperative (transferable utility) game denoted by the tuple $((I, \mu), s, N', N)$. Here, the natural numbers N' and N correspond to the lower and the upper bound on group sizes.

2.1.1 Type Space

The *type space* of players is summarized by a measure space (I, μ). In particular, the set I is a compact metric space[7] representing a set of players' types. The measure $\mu \in \mathcal{M}_+(I)$ is a non-negative finite Borel measure on I representing the distribution of players' types.

[7]The compactness of the type space can be relaxed with no essential changes in the rest of the paper. See Section 2.2.2 for more details.

There are two canonical examples of the type space. Firstly, I is a finite set and μ is an $|I|$-dimensional real vector with positive entries. In this finite type case, the i-th coordinate of μ, $\mu(i) > 0$, denotes the mass of type i players in this game. We will use the finite type case to explain some definitions in the following subsections. Secondly, I is the unit interval $[0, 1]$ and μ is some probability measure on I with or without atoms.

2.1.2 Groups

Groups are the units in which players interact with each other. Since we only distinguish players according to their types, we can only distinguish groups according to group types. To be simple, we abuse the language by calling group types groups.

Formally, for any permitted group size $n \in \mathbb{N}$ such that $N' \leq n \leq N$, *the set of n-person groups* $\mathcal{G}_n$ consists of all multisets of cardinality n with elements taken from I. That is,

$$\mathcal{G}_n = \left\{ G : I \to \mathbb{N} \,\middle|\, \sum_{i \in I} G(i) = n \right\}$$ [8]

In any n-person group $G \in \mathcal{G}_n$, there are $G(i)$ type i players, for any type $i \in I$.

The set of groups $\mathcal{G}$ consists of all groups of permitted sizes. Therefore,

$$\mathcal{G} = \bigcup_{n=N'}^{N} \mathcal{G}_n$$

Naturally, a *group* is an element in $\mathcal{G}$. All groups in this section consist of finitely many types of players. Groups consisting of infinitely many types of players will be discussed in Section 3.

Next, we identify n-person groups by the equivalence classes on the product space I^n. For any natural number $n \geq 2$, we define an equivalence relation $\sim_n$ on I^n: for any type lists $(i_1, ..., i_n), (j_1, ..., j_n) \in I^n$,

$$(i_1, ..., i_n) \sim_n (j_1, ..., j_n) \iff j_k = i_{\sigma(k)} \text{ for all } 1 \leq k \leq n \text{, for some permutation } \sigma \in S_n$$

That is, two type lists $(i_1, ..., i_n), (j_1, ..., j_n)$ in I^n are equivalent if they are the same up to some index permutation. It is easy to verify there is a bijection between the set of n-person groups $\mathcal{G}_n$ and the set of equivalence classes $I^n/\sim_n$[9]. i.e. $\mathcal{G}_n \simeq I^n/\sim_n$. Therefore, we write an n-person group $G \in \mathcal{G}_n$ as

$$G = [i_1, ..., i_n]$$

[8] Implicitly, for any function $G \in \mathcal{G}_n$, G has nonzero values at most finitely many points in I. That is, the support of G, $\operatorname{supp}(G) = \{i \in I : G(i) \neq 0\}$, is a finite set. The summation $\sum_{i \in I} G(i)$ is thus defined as $\sum_{i \in \operatorname{supp}(G)} G(i)$.

[9] The bijective map is defined by $T : I^n/\sim_n \to \mathcal{G}_n$ such that $T([i_1, ..., i_n]) = G$, where $G : I \to \mathbb{N}$ is defined by $G(i) = |\{1 \leq k \leq n : i_k = i\}|$ for all $i \in I$.

by listing all its members' types with repetitions. By identifying n-person groups by equivalence classes $I^n/\sim_n$, we know $\mathcal{G}_n$ is metrizable under the quotient topology.[10].

In the finite type case, our model can be described in the language of hypergraph theory: the type set I corresponds to vertices of a hypergraph, and the set of n-person groups $\mathcal{G}_n$ corresponds to n-uniform hyperedges in the hypergraph. It is well known that the number of n-person groups, or n-uniform hyperedges, is given by,

$$|\mathcal{G}_n| = \left(\!\!\binom{|I|}{n}\!\!\right) = \binom{|I|+n-1}{n} = \frac{(|I|+n-1)!}{n!(|I|-1)!} \tag{1}$$

In particular, for any fixed group size n, there are $\Theta(I^n)$ types of n-person groups.

2.1.3 Surplus Function

A surplus function specifies the total amount of surplus group members can share. Formally, a *surplus function* s is a non-negative valued function on the set of groups $\mathcal{G}$. Its restriction on $\mathcal{G}_n$, $s_n = s|_{\mathcal{G}_n}$, is a surplus function on n-person groups. We impose the following assumption on the surplus function:

(A1) for any permitted group size $N' \leq n \leq N$, s_n is continuous in $\mathcal{G}_n$[11]. i.e. for any $[i_1^k, ..., i_n^k] \to [i_1, ..., i_n]$ in $\mathcal{G}_n$,

$$\lim_{k\to\infty} s_n([i_1^k, ..., i_n^k]) = s_n([i_1, ..., i_n)])$$

Assumption (A1) assumes, for any permitted group size n, the surplus function on n-person groups is continuous. In the finite type case, this assumption impose no restriction on the surplus function.

In addition, we remark that we assume no relation between surplus functions on different group sizes. In particular, the surplus function need not be super-additive: the departure of an agent from a group might increase the surplus of the remaining group members. The non super-additivity helps us to analyze examples such as exchange economies with consumption externalities[12].

2.1.4 Assignments

An assignment is a partition of the continuum of players into groups of permitted sizes.

Rather than studying the partition directly, we define an assignment by a statistical representation of the partition, in which the mass of each group in the partition is

[10] See Appendix B.1 for a proof.

[11] The continuity assumption can be replaced by an upper semi-continuity assumption with no essential changes on the rest of this paper. See Section 2.2.2 for more details.

[12] An example is given in Section 2.4.4.

specified. By using this statistical representation, we simplify the classical definition of assignment in the literature[13] and also obtain some measure structure in the definition.

To describe this statistical representation, we first need to explore the structure of $\mathcal{G}_n$, the set of n-person groups. In particular, we will define a collection of partitions of $\mathcal{G}_n$: for every measurable subset $A \subset I$ and every natural number $0 \leq k \leq n$, the set $\mathcal{G}_n(A, k)$ is defined to be a set consisting of all n-person groups in which there are k agents whose types are in the set A. That is,

$$\mathcal{G}_n(A, k) = \left\{ G \in \mathcal{G}_n : \sum_{i \in A} G(i) = k \right\}$$ [14]

It is routine to check that, for every measurable $A \subset I$, $\{\mathcal{G}_n(A,k) : 0 \leq k \leq n\}$ is a partition of $\mathcal{G}_n$. That is, the collection of partitions we defined is $\{\{\mathcal{G}_n(A,k) : 0 \leq k \leq n\} :$ measurable $A \subset I\}$. Moreover, $\mathcal{G}_n(A,k)$ is a measurable set in $\mathcal{G}_n = I^n / \sim_n$.[15].

Now, we define assignments. An *assignment* is a tuple $\tau = (\tau_{N'}, ..., \tau_N)$, where τ_n is a non-negative measure on $\mathcal{G}_n$ satisfying the following consistency condition: for any measurable $A \subset I$,

$$\sum_{n=N'}^{N} \sum_{k=0}^{n} k\tau_n(\mathcal{G}_n(A,k)) = \mu(A) \tag{2}$$

In this equation, $\tau_n(\mathcal{G}_n(A,k))$ is the total mass of all n-person groups containing exactly k players whose types are in A. Thus, $k\tau_n(\mathcal{G}_n(A,k))$ is the total mass of players, whose types are in A, that are assigned to some n-person groups containing exactly k players whose types are in set A. Summing over k, $\sum_{k=0}^{n} k\tau_n(\mathcal{G}_n(A,k))$ is the total mass of players, whose types are in A, that are assigned to some n-person groups. Therefore, $\sum_{n=N'}^{N} \sum_{k=0}^{n} k\tau_n(\mathcal{G}_n(A,k))$ is the total mass of players, whose types are in A, that are assigned by the assignment. Since all players need to be assigned by the assignment, this total mass is equal to $\mu(A)$, which is the total mass of players whose types are in set A. Consequently, the consistency condition means all players are assigned to some group in the partition specified by the assignment.

In the finite type case, an assignment is a vector τ with $|\mathcal{G}|$ non-negative real entries. For any group $G \in \mathcal{G}$, $\tau(G)$ is the mass of group G in the partition represented by τ.

[13] In Kaneko and Wooders [1986] and Kaneko and Wooders [1996], a partition p of the player space I is defined to be measure-consistent if there is a partition of I into N measurable sets $I_1, ..., I_N$ and each set I_n has a partition, consisting of measurable subsets $\{I_{n1}, ..., I_{nn}\}$, with the following property: there are measure preserving isomorphisms $\psi_{n1}, ..., \psi_{nn}$ from I_{n1} to $I_{n1}, ..., I_{nn}$, respectively, such that ψ_{n1} is the identity map and $\{\psi_{n1}, ..., \psi_{nn}\} \in p$ for all $i \in I_{k1}$.

[14] Similar to Footnote 8, $\sum_{i \in A} G(i)$ is defined to be $\sum_{i \in A \cap \mathrm{supp}(G)} G(i)$.

[15] See Appendix B.2 for a proof.

In this case, the consistency condition is: for any type $i \in I$,

$$\sum_{k=0}^{N} \sum_{G \in \mathcal{G}(i,k)} k\tau(G) = \mu(i)$$

where $\mathcal{G}(i,k)$ is the set of groups containing exactly k type i players.

We finish this subsubsection by defining the following notions.

Firstly, we use a set T to denote *the set of assignments*. That is,

$$\mathrm{T} = \left\{ \tau = (\tau_n)_{N' \leq n \leq N} : \tau_n \in \mathcal{M}_+(\mathcal{G}_n), \forall n, \sum_{n=N'}^{N} \sum_{k=0}^{n} k\tau_n(\mathcal{G}_n(A,k)) = \mu(A), \forall A \subset I \right\}$$

Secondly, given any group $G = [i_1, ..., i_n]$, we say *a type i agent is in group G*, written as $i \in G$, if $i = i_k$ for some $1 \leq k \leq n$. In this case, a type i agent in group G has a *group partner* (or a trade partner) of type i_m, for all $m \neq k$.

Thirdly, given any assignment $\tau \in \mathrm{T}$, we say a group $G \in \mathcal{G}$ is *formed*, or is a formed group, under assignment τ if $G \in \mathrm{supp}(\tau) = \cup_{n=N'}^{N} \mathrm{supp}(\tau_n)$. In the finite type case, a group is a formed group under assignment τ if and only if $\tau(G) > 0$. i.e. there is a positive mass of group G in the partition represented by assignment τ.

2.1.5 Stability

In the game, players form small groups in order to share group surplus. We say an assignment is stable if there is a way to split group surpluses such that no group of agents can jointly do better by forming a new group.

Formally, an assignment $\tau \in \mathrm{T}$ is *stable* if there is an *imputation* $u \in L^1(I, \mu)$ satisfying the following two conditions:

1. Feasibility: $\sum_{i \in G} u(i) \leq s(G)$[16], for all formed groups $G \in \mathrm{supp}(\tau)$
2. No-blocking: $\sum_{i \in G} u(i) \geq s(G)$, for all groups $G \in \mathcal{G}$

In this definition, an imputation specifies a payoff for each type of players. In particular, the same type of player has the same payoff, as otherwise the player with a lower payoff has an incentive to form a group with the group partners of the player with a higher payoff such that all members in the new group have higher payoffs. The feasibility condition ensures that players have small enough payoffs such that they can achieve these payoffs by sharing group surpluses. The no-blocking condition ensures that players have high enough payoffs such that no group of players can jointly do better by forming a new group.

[16]In this paper, $\sum_{i \in G} f(i) = \sum_{i \in I} G(i) f(i)$. This definition is well-defined since $G = 0$ at all but finitely many points in its domain. See Footnote 8 for a definition of summation over G.

2.2 Results

In this section, we state our results. The proofs of these results will be given in Section 2.3.

The first main theorem in this paper is the existence of a stable assignment when group sizes are finitely bounded.

Theorem 1. *For any game $((I,\mu),s,N',N)$ satisfying Assumption (A1), there is a stable assignment and its associated imputation is continuous.*

Following the long-term wisdom in works such as Koopmans and Beckmann [1957], Shapley and Shubik [1971], Gretsky et al. [1992], we prove the existence theorem by establishing a duality relation. In our setting, the duality relation is that the maximum social welfare achieved by forming small groups is equal to the minimum social welfare such that no blocking coalition exists. The formal statement of the duality relation is given by the following theorem.

Theorem 2. *For any game $((I,\mu),s,N',N)$ satisfying Assumption (A1), we have*

$$\sup_{\tau\in\mathrm{T}}\sum_{n=N'}^{N}\int_{\mathcal{G}_n} s_n d\tau_n = \inf_{u\in\mathcal{U}}\int_I u d\mu \tag{3}$$

where $\mathcal{U}=\{u\in L^1(I,\mu):\sum_{i\in G}u(i)\geq s(G),\forall G\in\mathcal{G}\}$ and the inifimum can be achieved by a continuous function.

In particular, this duality theorem suggests that, in a stable state, the payoff of a type i player is equal to his marginal contribution to the maximum social welfare. See Section 2.2.1 for more details.

Our key observation in the proof of these two theorems is that any assignment can be identified by a *symmetric* transport plan in some replicated agent space. We will discuss this identification trick in more detail in Section 2.3.1 and Section 2.3.2. As a consequence of the observation, the social welfare maximization problem can be reformulated as a symmetric transport problem.

Proposition 1. *For any game $((I,\mu),s,N',N)$ satisfying Assumption (A1), there is a symmetric[17] upper semi-continuous function $S: I^{N!}\to\mathbb{R}$ such that*

$$\sup_{\tau\in\mathrm{T}}\sum_{n=N'}^{N}\int_{\mathcal{G}_n} s_n d\tau_n = \sup_{\hat{\gamma}\in\hat{\Gamma}_{sym}}\int_{I^{N!}}\frac{S}{N}d\hat{\gamma} \tag{4}$$

where $\hat{\Gamma}_{sym}$ is the set of symmetric measures on $I^{N!}$ such that all marginals are μ. i.e.

$$\hat{\Gamma}_{sym}=\{\hat{\gamma}\in\mathcal{M}_+(I^{N!}):\hat{\gamma}\text{ is symmetric},\hat{\gamma}(A\times I\times...\times I)=\mu(A),\forall\text{measurable }A\subset I\}$$

[17] S is symmetric if, for any permutation $\sigma\in S_{N!}$, any $(i_1,...,i_{N!})\in I^{N!}$, $S(i_1,...,i_{N!})=S(i_{\sigma(1)},...,i_{\sigma(N!)})$.

This proposition helps us in two ways. Conceptually, it helps us to prove the existence of a stable assignment. In particular, it helps us to prove the duality theorem (Theorem 2), which further implies the existence of a stable assignment in the game (Theorem 1). Computationally, by generating symmetries to the problem, it helps us to reduce the number of unknowns for finding stable assignments significantly. In particular, this proposition enables the possibility of applying a dimensional reduction technique developed in Friesecke and Vogler [2018]. We will discuss this computational improvement in more detail in a production problem context in Section 2.5.2.

In addition, the reformulation has the potential to answer the following questions:

1. the uniqueness of stable assignments
2. the uniqueness of imputations
3. whether players of the same type have the same group partners

These questions are related to the uniqueness and purification properties of multi-marginal transport problems. To my knowledge, no existing work can be applied directly beyond the two marginal case. We refer to Pass [2011] and Pass [2015b] for some related results.

2.2.1 Remarks

The duality relation in Theorem 2 suggests that the payoff of a type i player in any stable assignment is equal to his marginal contribution to the maximum social welfare. For instance, in the finite type case, the maximum social welfare, as a function of type distribution, is defined by a function $\Pi : \mathbb{R}_+^{|I|} \to \mathbb{R}$, where

$$\Pi(\mu) = \sup_{\tau \in \mathrm{T}} \sum_{G \in \mathcal{G}} s(G)\tau(G) \tag{5}$$

Since there are a continuum of players of each type, the maximum social surplus function Π is a concave function on $\mathbb{R}_+^{|I|}$.[18] Therefore, the super-derivative of the maximum social surplus is well-defined.

On the other hand, the duality theorem, Theorem 2, suggests,

$$\Pi(\mu) = \inf_{u \in \mathcal{U}} \sum_{i \in I} u(i)\mu(i) \tag{6}$$

where $\mathcal{U} = \{u \in \mathbb{R}_+^{|I|} : \sum_{i \in G} u(i) \geq s(G), \forall G \in \mathcal{G}\}$. Therefore, the imputation u associated with a stable assignment is equal to the super-derivative of the concave

[18]For any type distributions μ_1 and μ_2, we fix two arbitrary assignments τ_1 and τ_2 representing the partitions of players with a type distribution μ_1 and μ_2 respectively. Any convex combination $c\tau_1 + (1-c)\tau_2$ is an assignment representing a partition of players with a type distribution $c\mu_1 + (1-c)\mu_2$.

maximum social welfare function. Formally, by Danskin's theorem (Proposition B.25 in Bertsekas [1999]),

$$\partial\Pi(\mu) = \left\{ u \in \mathcal{U} : \sum_{i \in I} u(i)\mu(i) = \Pi(\mu) \right\}$$

That is, the payoff of a type i player is equal to his marginal contrition to the maximum social surplus, provided the maximum social surplus function is differentiable. More generally, the imputations provide separating hyperplanes for the set of feasible social surplus.

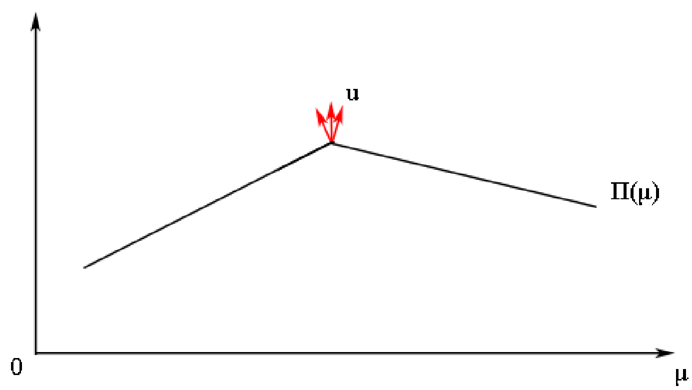

Figure 1: Maximum Social Welfare and The Imputation of a Stable Assignment

By Figure 1, it is clear that, given a stable assignment, the corresponding imputation is unique if and only if the maximum social surplus is differentiable. Unfortunately, the differentiability requires stronger assumptions on the surplus function. An example of a game with a non-differentiable maximum social surplus function is given in Section 2.4.1. On the other hand, as the maximum social welfare function Π is concave, Π is differentiable almost everywhere. That is, in the finite type case, for almost all initial distributions of types, the payoff at the stable state is unique.

2.2.2 Extensions

As remarked in Footnote 7 and Footnote 11, our model and results in this section can be extended to the case where the type set is not compact and the surplus function is only upper semi-continuous. In this subsubsection, we discuss two types of extensions.

Firstly, when I is a (possibly non-compact) Polish space and the surplus function is Lipchitz continuous, i.e. every s_n is Lipchitz continuous, with all definitions in this section remain the same, all results in this section hold, provided the surplus function satisfies Assumption ($A2^c$): for any permitted group size n, there is a function $a_n \in C(I)$ such that $s_n([i_1, ..., i_n]) \leq \sum_{k=1}^n a_n(i_k)$ for all $i_1, ..., i_n \in I$. In particular, Assumption ($A2^c$) is automatically satisfied when s_n is bounded or I is compact.

More generally, if I is a (possibly non-compact) Polish space and the surplus function is upper semi-continuous, i.e. every s_n is upper semi-continuous, we need to impose the following boundedness assumption on the surplus function:

(A2) for any permitted group size n, there is a function $a_n \in LSC(I)$ such that $s_n([i_1, ..., i_n]) \leq \sum_{k=1}^n a_n(i_k)$ for all $i_1, ..., i_n \in I$.

Again, Assumption ($A2$) is automatically satisfied when s_n is bounded or I is compact. With Assumption ($A2$) and the following changes on the definition of "formed groups", all results in this section hold:

1. in the definition of stable assignment, replace the term "for all formed groups $G \in \text{supp}(\tau)$" in the feasibility condition by the term "for τ-almost all $G \in \text{supp}(\tau)$"[19]
2. Theorem 1 holds if "and its associated imputation is continuous" is removed.
3. Theorem 2 holds if "by a continuous function" is removed.
4. Proposition 1 holds without any change.

2.3 Proof

Firstly, we prove Proposition 1 which states the social welfare maximization problem can be reformulated as a symmetric transport problem. Then, we apply the duality theorem for multi-transport problem in Kellerer [1984] to prove the duality theorem Theorem 2. Lastly, we show the optimizers of the duality theorem can be attained and are equivalent to a stable assignment and its corresponding imputation.

To start with, we reformulate the social welfare maximization problem as a symmetric transport problem in two steps.

In the first step, we transform a problem with multiple permitted group sizes up to N to a problem with a unique permitted group size $N!$. We use the number $N!$ as it

[19] Equivalently, we can redefine a notion for formed groups in order to keep the term "for all formed groups" in the sentence. The definition is as follows: a measurable set $\mathcal{F} \subset \text{supp}(\tau)$ is a set of formed groups if $\mathcal{F}$ has a full τ-measure. Formally, it means $\tau_n(\mathcal{F} \cap \mathcal{G}_n) = \tau_n(\mathcal{G}_n)$ for all permitted group size n. Then, in the definition of stable assignment, we just need to replace "for all formed groups $G \in \text{supp}(\tau)$" by "for all formed groups $G \in \mathcal{F}$ where $\mathcal{F}$ is a set of formed groups". This subtlety about measure zero set comes from the measure theoretic language we choose to use.

is a common multiple of all group sizes. The same argument works if we replace $N!$ by any common multiple of group sizes[20].

In the second step, we extend the domain of assignment from an unordered list of types to an ordered list of types. On the technical level, we get rid of equivalent classes by defining a pull-back measure carefully.

The proof is proceeded as follows. Firstly, we introduce these two steps formally in Section 2.3.1 and Section 2.3.2. Next, we reformulate the social welfare maximization problem as a symmetric transport problem in Section 2.3.3. Lastly, we prove the existence of a stable assignment in Section 2.3.4.

2.3.1 Unifying Group Sizes

To unify the group sizes, we use fractional groups to identify groups of different sizes.

Formally, a fractional group is an element $\hat{G} = [i_1, ..., i_{N!}] \in I^{N!}/\sim_{N!}$. Intuitively, a fractional group $\hat{G} = [i_1, ..., i_{N!}]$ represents a set of players of total mass N consisting of $\frac{1}{(N-1)!}$ unit mass of type i_n players for all permitted group size n.

For any permitted group size n, an n-person group can be identified by some fractional group via replication. Therefore, the set of n-person groups corresponds to a collection of fractional groups. Formally, for any permitted group size $N' \leq n \leq N$, we define a subset $K_n \subset I^{N!}/\sim_{N!}$, where

$$K_n = \left\{[i_1, ..., i_{N!}] \in I^{N!}/\sim_{N!}: |k \in \mathbb{N} : i_k = i| \text{ is divisible by } \frac{N!}{n}, \forall i \in I\right\} \tag{7}$$

Intuitively, every fractional group in K_n corresponds to an $N!/n$-fold replication of some n-person group. We define an identification map $P_n : K_n \to I^n/\sim_n$ by

$$P_n([\underbrace{i_1, ..., i_1}_{N!/n \text{ many}}, ..., \underbrace{i_n, ..., i_n}_{N!/n \text{ many}}]) = [i_1, ..., i_n]$$

It is easy to verify that the identification map P_n is bijective. Therefore, $\mathcal{G}_n \simeq K_n$ and $\mathcal{G}_n = P_n(K_n)$.

In addition, we extend the domain of surplus function: for any $N' \leq n \leq N$, we define a function $\hat{s}_n : I^{N!}/\sim_{N!} \to \mathbb{R}$ by

$$\hat{s}_n(\hat{G}) = \begin{cases} s_n(P_n(\hat{G})), & \hat{G} \in K_n \\ 0, & \hat{G} \notin K_n \end{cases}$$

Next, we define a surplus function on the set of fractional groups. We note that the collection of subsets $\{K_n\}_{N' \leq n \leq N}$ is not a partition of the set of fractional groups

[20] This replacement will help the computation significantly. However, it makes the notations messier. So we stick with the choice $N!$ in this paper.

$I^{N!}/\sim_{N!}$. Therefore, we cannot combine the functions $\hat{s}_n$ to define this surplus function directly. However, we note that a welfare maximizing assignment assigns positive masses to two groups, which correspond to the same fractional group, if and only if the average surplus of these two groups are the same[21]. Therefore, we define the surplus function $\hat{s} : I^{N!}/\sim_{N!} \to \mathbb{R}$ by

$$\hat{s} = \max\left(\frac{N}{n}\hat{s}_n : n \in \{N', ..., N\}\right)^{22} \tag{8}$$

The surplus function $\hat{s}$ on fractional groups induces a partition $\{R_n\}_{n=0,N',...,N}$ of the space $I^{N!}/\sim_{N!}$ such that

- $R_0 \cap K_n = \emptyset, \forall n \in \{N', ..., N\}$
- $R_n \subset K_n$ and $\hat{s}_n \geq \frac{n}{m}\hat{s}_m$ on $R_n, \forall N' \leq n, m \leq N$

Such partition $\{R_n\}_{n=0,N',...,N}$ can be constructed iteratively:

- define $R_0 = I^{N!}/\sim_{N!} - \bigcup_{n=N'}^{N} K_n$, and $R_n = \emptyset$ for all $n \in \{N', ..., N\}$
- set $n = N'$
- update R_n to be $\{\hat{G} \in K_n - \cup_{k=n}^{N} R_k : \hat{s}(\hat{G}) = \frac{N}{n}\hat{s}_n(\hat{G})\}$ for all $n \in \{N', ..., N\}$
- if $n < N$, set n to be $n+1$ and repeat the previous step, otherwise stops

Moreover, the surplus function $\hat{s}$ on fractional groups inherits the properties of the surplus function s:

Lemma 1. *If the surplus function s satisfies Assumption (A1) and Assumption (A2), then $\hat{s}$ is upper semi-continuous and there is a lower semi-continuous function $\hat{a} \in L^1(I, \mu)$ such that*

$$\hat{s}([i_1, ..., i_{N!}]) \leq \sum_{k=1}^{N!} \hat{a}(i_k)$$

Proof. See Appendix B.4. ■

2.3.2 A Change of Variable Trick

Next, we extend the domain of a measure. In particular, a non-negative measure τ on $I^n/\sim_n$ will be identified by a symmetric measure γ on I^n. Here, n is a natural number larger than or equal to 2.

Formally, given any non-negative measure τ_n on $I^n/\sim_n$, we define a non-negative measure γ_n on I^n by

$$\gamma_n = c_n Q_n^{\#} \tau_n \tag{9}$$

[21]Otherwise, only the group with a higher average surplus will be formed in any welfare maximized assignment.

[22]In the numerator, it is N rather than $N!$ as total mass of players in a fractional group is normalized to be N.

where $c_n : I^n \to \mathbb{R}$ is a measurable function[23] defined by the combinatorial numbers

$$c_n(i_1, ..., i_n) = \frac{1}{(n-1)!} \prod_{i \in I} n_i! \quad ^{24} \tag{10}$$

where $n_i = |\{k \in \mathbb{N} : i_k = i\}|$ is the number of type i players in group $[i_1, ..., i_n] \in \mathcal{G}_n$, and $Q_n : I^n \to I^n / \sim_n$ is the quotient map such that

$$Q_n(i_1, ..., i_n) = [i_1, ..., i_n]$$

In order to define the pullback measure $Q_n^{\#}\tau_n$ on I^n, we partition I^n into small components such that Q_n is an injective map on each component.

Lemma 2. *There is a partition* $\{J_\alpha\}_{\alpha \in \mathrm{A}}$ *of* I^n *such that*

- *the index set* A *is a finite set*
- *each component* J_α *is Borel measurable in* I^n
- *the quotient map* Q_n *is injective on* J_α

Proof. We partition I^n in two steps.

Firstly, we partition I^n into $|\mathcal{M}|$ components, where the index set $\mathcal{M}$ is defined by

$$\mathcal{M} = \bigcup_{k=1}^{n} \{m = (m_1, ..., m_k) \in \mathbb{N}^k : m_1 \geq ... \geq m_k \geq 1, m_1 + ... + m_k = n\}$$

A component J_m with a index $m = (m_1, ..., m_k) \in \mathcal{M}$ consists of all elements in the set

$$\{(\underbrace{i_1, ..., i_1}_{m_1 \text{ many}}, \underbrace{i_2, ..., i_2}_{m_2 \text{ many}}, ..., \underbrace{i_k, ..., i_k}_{m_k \text{ many}}) \in I^n : i_1, ..., i_k \text{ are disjoint}\}$$

and all their permutations. We know J_m is a measurable set[25].

Secondly, for each index $m \in \mathcal{M}$, we partition J_m into $\frac{n!}{m_1! ... m_k!}$ components such that Q_n is injective on each component. The construction is as follows: for each permutation $\sigma \in S_n$, we define

$$J_{m,\sigma} = \{(j_{\sigma(1)}, ..., j_{\sigma(n)}) \in I^n : j = (\underbrace{i_1, ..., i_1}_{m_1 \text{ many}}, \underbrace{i_2, ..., i_2}_{m_2 \text{ many}}, ..., \underbrace{i_k, ..., i_k}_{m_k \text{ many}}) \in I^n, i_1, ..., i_k \text{ are disjoint}\}$$

$J_{m,\sigma}$ is a measurable set[26]. Moreover, for any permutations $\sigma_1, \sigma_2 \in S_n$, we have either $J_{m,\sigma_1} = J_{m,\sigma_2}$ or $J_{m,\sigma_1} \cap J_{m,\sigma_2} = \emptyset$. Moreover, $\cup_{\sigma \in S_n} J_{m,\sigma} = J_m$. By deleting the

[23] See Appendix B.3 for the proof of measurability.

[24] There are finitely many $i \in I$ such that $n_i \neq 0$. The product over an infinite set $\prod_{i \in I} n_i!$ is defined to be $\prod_{i \in I : n_i \neq 0} n_i!$

[25] See Appendix B.3 for a proof.

[26] See Appendix B.3 for a proof.

repeated components from $\{J_{m,\sigma}\}_{\sigma \in S_n}$, we have a partition of the set J_m such that Q_n is injective on each component.

In sum, by putting the indices (m, σ) together as a single index set A, we have the partition. ■

With the partition $(J_\alpha)_{\alpha \in \mathrm{A}}$ of I^n, we define the pullback measure $Q_n^{\#}\tau_n$ on I^n by

$$Q_n^{\#}\tau_n(S) = \sum_{\alpha \in \mathrm{A}} \tau_n(Q_n(S \cap J_\alpha)), \forall \text{measurable } S \subset I^n \tag{11}$$

We verify our definition of pullback measure by proving that τ_n is the push-forward measure of γ_n under the map Q_n/n:

Lemma 3. *If γ_n is defined by τ_n according to Equation 9 and Equation 11, we have*

$$n\tau_n = (Q_n)_{\#}\gamma_n \tag{12}$$

Proof. See Appendix B.5. ■

In the finite type case, Lemma 3 implies, for any list of types $(i_1, ..., i_n) \in I^n$,

$$n\tau_n([i_1, ..., i_n]) = \gamma_n(Q_n^{-1}([i_1, ..., i_n])) = \sum_{i_1,...,i_n \in I} \gamma_n(i_1, ..., i_n)$$

That is, for all groups $G = [i_1, ..., i_n]$, the sum of the γ values over all permutations of the group G is equal to the total mass of players assigned to group G. Since γ_n is a symmetric function, we have

$$\gamma_n(i_1, ..., i_n) = c_n(i_1, ..., i_n)\tau_n([i_1, ..., i_n])$$

which gives us Equation 9. In particular, $\gamma_n(i_1, ..., i_n)$ is the product of some positive constant and the mass of the n-person group $[i_1, ..., i_n]$ in assignment $\tau \in \mathrm{T}$. Recall that a group $G = [i_1, ..., i_n]$ is formed when $\tau([i_1, ..., i_n]) > 0$. Therefore, an n-person group $G \in \mathcal{G}_n$ is formed if and only if $\gamma_n(Q_n^{-1}(G)) > 0$ for some permitted group size n. That is, a preimage $Q_n^{-1}(G)$ is in the support of γ_n for some permitted group size n. Consequently, the positive constant c plays no role in our definition of stability.

Other properties of γ_n is given as follows:

Lemma 4. *For any measure γ_n defined by some measure τ_n and Equation 12, we have*

1. *γ_n is symmetric: for any measurable sets $A_1, ..., A_n \subset I$,*

$$\gamma_n(A_1, ..., A_n) = \gamma_n(A_{\sigma(1)}, ..., A_{\sigma(n)}), \forall \sigma \in S_n$$

2. *$\gamma_n(A, I, I..., I) = \sum_{k=0}^{n} k\tau_n(\mathcal{G}_n(A, k))$ for all measurable $A \subset I$*

Proof. See Appendix B.6. ■

An immediate consequence of Lemma 4 is that, when there is a unique group size, i.e. $N' = N$, γ_N, defined by an assignment $\tau_N \in \mathrm{T}$, is a symmetric measure such that all its marginals are μ. That is, γ_N is a symmetric transport plan in the N-fold replicated player space.

Lastly, we illustrate the change of variable trick in a game with three types of agents. That is, $I = \{i_1, i_2, i_3\}$. In this game, the unique permitted group size is 2.

Recall that an assignment $\tau \in \mathrm{T}$ is a function defined on $\mathcal{G} = \mathcal{G}_2 = I^2/\sim_2$ satisfying the consistency condition

$$\sum_{m \neq n} \tau_{nm} + 2\tau_{nn} = 1, \forall n \in \{1, 2, 3\}$$

where $\tau_{nm} = \tau([i_n, i_m])$ for any $n \leq m$. In particular, τ_{nm} is the mass of groups consisting of one type i_n player and one type i_m player in assignment τ. Therefore, an assignment τ can be represented by the upper triangular entries of a 3 by 3 matrix

$$\tau = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ & \tau_{22} & \tau_{23} \\ & & \tau_{33} \end{pmatrix}$$

Naturally, we wish to extend the domain of τ to the full domain I^2 such that an assignment can be presented by a matrix. There are infinitely many ways to extend the domain. Among them, one of the most natural ways is to extend the upper triangular entries symmetrically:

$$Q_3^{\#}\tau = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}$$

While $Q_3^{\#}\tau$ is a symmetric matrix, it is not a transport plan as it does not satisfy the marginal condition. In this case, it means the matrix is not bi-stochastic. To obtain a bi-stochastic matrix, we multiply the diagonal entries by 2:

$$\gamma = cQ_3^{\#}\tau = \begin{pmatrix} 2\tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & 2\tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & 2\tau_{33} \end{pmatrix}$$

By the consistency condition of the assignment τ, γ is bi-stochastic. Therefore, we obtain a symmetric transport plan γ in a two-folds replication of agent space.

A symmetric transport plan γ is represented in Figure 2. For any $s, t \in \{1, 2, 3\}$, the mass transported from a source i_s on the left to a destination i_t on the right is γ_{st}. By the symmetry of the transport plan γ, the mass transported from a source i_s on the left to a destination i_t on the right is equal to the mass transported from a source i_t on the left to a destination i_s on the right.

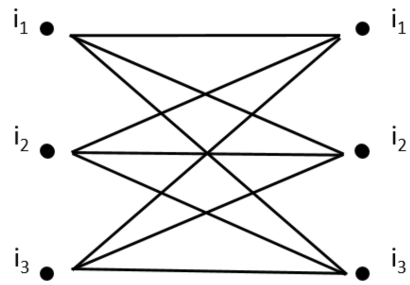

Figure 2: The symmetric transport plan γ in a two-folds replicated agent space

2.3.3 Symmetric Transport Problem

Next, we prove Proposition 1 by using the two tricks introduced in the previous two subsubsections.

Firstly, we extend the domain of surplus function on fractional groups to the whole domain by defining $S : I^{N!} \to \mathbb{R}$ where

$$S(i_1, ..., i_{N!}) = \hat{s}([i_1, ..., i_{N!}]) \tag{13}$$

where $\hat{s}$ is defined in Equation 8. By Lemma 1, when the surplus function satisfies assumption Assumption (A1) and Assumption (A2), S is upper semi-continuous in $I^{N!}$ and there is a lower semi continuous $\hat{a} \in L^1(I, \mu)$ such that

$$S(i_1, ..., i_{N!}) \leq \sum_{k=1}^{N!} \hat{a}(i_k), \forall i_1, ..., i_{N!} \in I \tag{14}$$

Now, we prove Proposition 1 by proving Equation 4 holds. i.e.

$$\sup_{\tau \in \mathrm{T}} \sum_{n=N'}^{N} \int_{\mathcal{G}_n} s_n d\tau_n = \sup_{\hat{\gamma} \in \hat{\Gamma}_{sym}} \int_{I^{N!}} \frac{S}{N} d\hat{\gamma}$$

Firstly, we show the left hand side is not larger than the right hand side in Equation 4: for any fixed $\tau \in \mathrm{T}$, we construct $\hat{\gamma} \in \hat{\Gamma}_{sym}$ such that

$$\sum_{n=N'}^{N} \int_{\mathcal{G}_n} s_n d\tau_n \leq \int_{I^{N!}} \frac{S}{N} d\hat{\gamma}$$

Given an assignment $\tau \in \mathrm{T}$, we define a measure $\hat{\tau}_n$ on $I^{N!}/\sim_{N!}$ such that for any $S \subset I^{N!}/\sim_{N!}$,

$$\hat{\tau}_n(S) = \tau_n(P_n(S \cap K_n))$$

where K_n is defined in Equation 7. Applying the change of variable trick in Section 2.3.2, we have a measure $\hat{\gamma}_n$ on $I^{N!}$ where

$$\hat{\gamma}_n = c \cdot Q_{N!}^{\#} \hat{\tau}_n$$

where $c : I^{N!} \to \mathbb{R}$ is a measurable function defined by the combinatorial numbers

$$c(i_1, ..., i_{N!}) = \frac{1}{(N!-1)!} \prod_{i \in I} n_i!$$

where $n_i = |\{k \in \mathbb{N} : i_k = i\}|$. By Lemma 4,

$$\begin{aligned}
\hat{\gamma}_n(A \times I \times ... \times I) =& \sum_{k=0}^{N!} k \hat{\tau}_n(\mathcal{G}_{N!}(A, k)) \\
=& \sum_{k=0}^{n} \frac{N!k}{n} \hat{\tau}_n(\mathcal{G}_{N!}(A, \frac{N!k}{n})) \\
=& \frac{N!}{n} \sum_{k=0}^{n} k \tau_n(\mathcal{G}_n(A, k))
\end{aligned}$$

Therefore, we define

$$\hat{\gamma} = \frac{1}{N!} \sum_{n=N'}^{N} n \hat{\gamma}_n$$

By Lemma 4, $\hat{\gamma}$ is symmetric. By Equation 2, $\hat{\gamma}_n(A \times I \times ... \times I) = \mu(A)$ for any Borel measurable $A \subset I$. Consequently, $\hat{\gamma} \in \hat{\Gamma}_{sym}$.

Moreover, by Equation 8,

$$\sum_{n=N'}^{N} \int_{\mathcal{G}_n} s_n d\tau_n = \sum_{n=N'}^{N} \int_{I^{N!}/\sim_{N!}} \hat{s}_n d\hat{\tau}_n \leq \sum_{n=N'}^{N} n \int_{I^{N!}/\sim_{N!}} \frac{\hat{s}}{N} d\hat{\tau}_n$$

Applying Lemma 3, we have

$$\int_{I^{N!}/\sim_{N!}} \frac{\hat{s}}{N} d\hat{\tau}_n = \frac{1}{N!} \int_{I^{N!}} \frac{S}{N} d\hat{\gamma}_n$$

Therefore,

$$\sum_{n=N'}^{N} \int_{\mathcal{G}_n} s_n d\tau_n \leq \sum_{n=N'}^{N} \frac{n}{N!} \int_{I^{N!}} \frac{S}{N} d\hat{\gamma}_n = \int_{I^{N!}} \frac{S}{N} d\hat{\gamma}$$

Conversely, we show the left hand side is not less than the right hand side in Equation 4: for any maximizer $\hat{\gamma} \in \hat{\Gamma}_{sym}$ of the right hand side of Equation 4[27], we construct a $\tau \in \mathrm{T}$ such that

$$\sum_{n=N'}^{N} \int_{\mathcal{G}_n} s_n d\tau_n = \int_{I^{N!}} \frac{S}{N} d\hat{\gamma} \tag{15}$$

Firstly, since $\hat{\gamma}$ is a maximizer, the support of $\hat{\gamma}$ cannot intersect the set $Q_{N!}^{-1}(R_0)$. Otherwise, a normalized version of $\hat{\gamma}|_{Q_{N!}^{-1}(R_0)^c}$ will give a higher value for the maximization problem.

[27] The existence of a maximizer is proved in the next subsubsection.

For every permitted group size $n \in \{N', ..., N\}$, we define a measure $\hat{\gamma}_n$ on $I^{N!}$ by

$$\hat{\gamma}_n(S) = \frac{N!}{n}\hat{\gamma}(S \cap Q_{N!}^{-1}(R_n)), \forall \text{measurable } S \subset I^{N!}$$

In particular, the support of γ_n lies in the set $Q_{N!}^{-1}(R_n)$. By the definition of $\hat{\Gamma}_{sym}$ and the fact that the support of $\hat{\gamma}$ is in $\cup_{n=N'}^{N} Q_{N!}^{-1}(R_n)$, we have

$$\begin{aligned}
\mu(A) &= \hat{\gamma}(A \times I \times ... \times I) \\
&= \hat{\gamma}(A \times I \times ... \times I \cap (\cup_{n=N'}^{N} Q_{N!}^{-1}(R_n))) \\
&= \sum_{n=N'}^{N} \hat{\gamma}(A \times I \times ... \times I \cap Q_{N!}^{-1}(R_n)) \\
&= \sum_{n=N'}^{N} \frac{n}{N!}\hat{\gamma}_n(A \times I \times ... \times I)
\end{aligned} \tag{16}$$

Next, for every permitted group size n, we define a measure $\hat{\tau}_n$ on the set of fractional groups $I^{N!}/\sim_{N!}$ by

$$\hat{\tau}_n = (Q_{N!})_{\#}\hat{\gamma}_n$$

Applying Lemma 4 to Equation 16, we have

$$\mu(A) = \sum_{n=N'}^{N} \frac{n}{N!}\hat{\gamma}_n(A \times I \times ... \times I) = \sum_{n=N'}^{N}\sum_{k=0}^{N!} k\frac{n}{N!}\hat{\tau}_n(\mathcal{G}_{N!}(A, k))$$

Recall the identification map $P_n : K_n \subset I^{N!}/\sim_{N!} \to \mathcal{G}_n$, we define a measure τ_n on the set of n-person groups $\mathcal{G}_n \simeq I^n/\sim_n$ by

$$\tau_n(S) = \hat{\tau}_n(P_n^{-1}(S)), \forall \text{ measurable } S \subset \mathcal{G}_n$$

Therefore,

$$\sum_{k=0}^{N!} k\hat{\tau}_n(\mathcal{G}_{N!}(A, k)) = \sum_{k=0}^{n} \frac{N!}{n}k\hat{\tau}_n(\mathcal{G}_{N!}(A, \frac{N!}{n}k)) = \sum_{k=0}^{n} \frac{N!}{n}k\tau_n(\mathcal{G}_n(A, k))$$

Consequently, we have

$$\mu(A) = \sum_{n=N'}^{N}\sum_{k=0}^{n} k\tau_n(\mathcal{G}_n(A, k))$$

i.e. $\tau = (\tau_{N'}, ..., \tau_N) \in \mathrm{T}$ is an assignment.

Lastly, we check that Equation 15 holds:

$$
\begin{aligned}
\int_{I^{N!}} \frac{S}{N} d\hat{\gamma} &= \sum_{n=N'}^{N} \int_{Q_{N!}^{-1}(R_n)} \frac{S}{N} d\hat{\gamma} \\
&= \sum_{n=N'}^{N} \int_{Q_{N!}^{-1}(R_n)} \frac{S}{N} \frac{n}{N!} d\hat{\gamma}_n \\
&= \sum_{n=N'}^{N} \int_{R_n} \frac{n}{N} \hat{s} d\hat{\tau}_n \\
&= \sum_{n=N'}^{N} \int_{R_n} \hat{s}_n d\hat{\tau}_n \\
&= \sum_{n=N'}^{N} \int_{\mathcal{G}_n} s_n d\tau_n
\end{aligned}
$$

Hence, we proved Equation 4 holds.

2.3.4 Existence

Lastly, we prove the existence of a stable assignment. We will first prove the duality relation stated in Theorem 2 holds and the optimizers can be achieved. Then, we establish the relationship between the duality relation and the stable assignment.

The proof of the duality relation is proceeded in three steps:

1. reformulate the social welfare maximization problem as a symmetric transport problem, as stated in Proposition 1
2. apply the duality result in multi-marginal transport problem in Kellerer [1984]
3. prove the dual problem of the symmetric tranport problem is the same as a social welfare minimization problem such that no blocking coalition exists

To prove Theorem 2, we state two lemmas. Lemma 5 states that the theorem holds when there is a unique group size. Lemma 6 implies the dual problem of the transport problem is the same as the social welfare minimization problem such that no blocking coalition exists. We need the following definitions to state these two lemmas:

1. $\hat{\Gamma}$ is the set of measures on $I^{N!}$ such that all marginals are μ:

$$\hat{\Gamma} = \left\{ \hat{\gamma} \in \mathcal{M}_+(I^{N!}) : \hat{\gamma}(A \times I \times ... \times I) = ... = \hat{\gamma}(I \times ... \times I \times A) = \mu(A), \forall A \subset I \right\}$$

2. $\hat{\mathcal{U}}$ is the set of imputations such that there is no blocking fractional group.

$$\hat{\mathcal{U}} = \left\{ u \in L^1(I, \mu) : \sum_{k=1}^{N!} u(i_k) \geq (N-1)! S(i_1, ..., i_{N!}), \forall i_1, ..., i_{N!} \in I \right\}$$

For comparison, we recall that

$$\mathcal{U}_n = \left\{ u \in L^1(I,\mu) : \sum_{k=1}^{n} u(i_k) \geq s([i_1,...,i_n]), \forall i_1,...,i_n \in I \right\}$$

and

$$\mathcal{U} = \left\{ u \in L^1(I,\mu) : \sum_{i \in G} u(i) \geq s(G), \forall G \in \mathcal{G} \right\}$$

Lemma 5. *For any symmetric upper semi-continuous function S on $I^{N!}$ satisfying Equation 14,*

$$\sup_{\hat{\gamma} \in \hat{\Gamma}_{sym}} \int_{I^{N!}} \frac{S}{N} d\hat{\gamma} = \inf_{u \in \hat{\mathcal{U}}} \int_I u d\mu$$

and the infimum can be achieved.

Proof. See Appendix B.7. ∎

Lemma 6.

$$\hat{\mathcal{U}} = \mathcal{U} = \bigcap_{n=N'}^{N} \mathcal{U}_n$$

Proof. See Appendix B.8. ∎

Now, we are ready to prove the duality theorem, Theorem 2.

Proof of the duality theorem. By Proposition 1,

$$\sup_{\tau \in \mathrm{T}} \sum_{n=N'}^{N} \int_{\mathcal{G}_n} s_n d\tau_n = \sup_{\hat{\gamma} \in \hat{\Gamma}_{sym}} \int_{I^{N!}} \frac{S}{N} d\hat{\gamma}$$

By Lemma 5,

$$\sup_{\hat{\gamma} \in \hat{\Gamma}_{sym}} \int_{I^{N!}} \frac{S}{N} d\hat{\gamma} = \inf_{u \in \hat{\mathcal{U}}} \int_I u d\mu$$

and the infimum on the right can be achieved. By Lemma 6,

$$\inf_{u \in \hat{\mathcal{U}}} \int_I u d\mu = \inf_{u \in \mathcal{U}} \int_I u d\mu$$

Therefore, we have the duality relation:

$$\sup_{\tau \in \mathrm{T}} \sum_{n=N'}^{N} \int_{\mathcal{G}_n} s_n d\tau_n = \inf_{u \in \hat{\mathcal{U}}} \int_I u d\mu$$

See Appendix B.9 for a proof of the existence of a continuous minimizer. ∎

Next, we show the maximum social welfare can be achieved:

Lemma 7. *There is a maximizer solving the maximization problem*

$$\sup_{\tau \in \mathrm{T}} \sum_{n=N'}^{N} \int_{\mathcal{G}_n} s_n d\tau_n$$

Proof. See Appendix B.10. ■

In sum, we have proved the duality relation in Theorem 2 and the fact that the optima in the duality relation Equation 3 can be achieved. To show the existence of a stable assignment, it remains to the show the equivalence relation between the stable assignment and the duality relation.

Lemma 8. *In the duality relation, Equation 3, any maximizer τ is a stable assignment and a continuous minimizer u gives a corresponding imputation.*

Conversely, any stable assignment τ solves the maximization problem in the duality relation, Equation 3, and any imputation u associated with this stable assignment solves the minimization problem in the duality relation, Equation 3.

Proof. See Appendix B.11. ■

2.4 Examples

2.4.1 Non-Differentiablity of The Maximum Social Welfare Function

Firstly, we illustrate the maximum social welfare function may not be differentiable. In the finite type case, the maximum social welfare function $\Pi : \mathbb{R}_+^{|I|} \to \mathbb{R}$ is given by

$$\Pi(\mu) = \sup_{\tau \in \mathrm{T}} \sum_{G \in \mathcal{G}} s(G)\tau(G)$$

We study a game with two types of players. e.g. $I = \{1, 2\}$. The mass of type $1, 2$ agents are μ_1 and μ_2 respectively. In this game, the only permitted group size is 2. That is, $N' = N = 2$. In addition, all groups have zero surplus except the groups consisting both types of players, which have one unit of surplus. i.e.

$$s([i, j]) = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases}$$

It is easy to see that the social surplus is higher if there are more groups consisting of both types of players. Therefore, we have

$$\Pi(\mu_1, \mu_2) = \min(\mu_1, \mu_2)$$

Consequently, the maximum social surplus is not differentiable at $(1, 1)$ since

$$\lim_{\varepsilon \to 0^+} \Pi(1+\varepsilon, 1) = 1 = \Pi(1, 1)$$
$$\lim_{\varepsilon \to 0^+} \Pi(1-\varepsilon, 1) = 1 - \varepsilon = \Pi(1, 1) - \varepsilon$$

That is, the directional derivative of Π at $(1, 1)$ is 1 along the direction $(1, 0)$, but is 0 along the direction $(-1, 0)$.

However, the set of imputations coincide with the superderivatives of Π: any stable assignment assigns $\min(\mu_1, \mu_2)$ unit mass of the groups consisting players of both types, and assigns the remaining players pairwisely. Players' payoffs at $\mu = (\mu_1, \mu_2)$ is given by

$$\begin{cases} \text{if } \mu_1 = \mu_2, & u_1 + u_2 = 1, u_1 \geq 0, u_2 \geq 0 \\ \text{if } \mu_1 > \mu_2, & u_1 = 0, u_2 = 1 \\ \text{if } \mu_1 < \mu_2, & u_1 = 1, u_2 = 0 \end{cases}$$

2.4.2 Unifying Group sizes

Next, we illustrate the trick of reformulating a problem with multiple group sizes to a problem with a unique group size.

We study a game with three types of players. e.g. $I = \{1, 2, 3\}$. There are a unit mass of each type of players in the game. i.e. $\mu_1 = \mu_2 = \mu_3 = 1$. Permitted group sizes are 2 and 3. i.e. $N' = 2, N = 3$. Group surpluses are given by

$$\begin{cases} s([1, 2, 2]) = 9, s([3, 3, 3]) = 4 \\ s([1, 2]) = 2, s([3, 3]) = 3 \\ \text{all other groups have surplus } 0 \end{cases}$$

To reformulate a problem with multiple group sizes as a problem with a unique group size, we treat a 2-person group as 2/3 unit of some fractional group. For instance, the 2-person group $[i, j]$ corresponds to 2/3 unit of the fractional group $[i, i, i, j, j, j]$, which consists of 1.5 units of type i players and 1.5 units of type j players.

Therefore, to find an optimal partition of the players into small groups of size 2 and 3, it is sufficient to find the optimal partition of the players into fractional groups. Formally, we use a 6-tuple $[i_1, ..., i_6] \in I^6/\sim_6$ to denote a fractional group. Intuitively, a fractional group $[i_1, ..., i_6]$ is a subset of players of total mass 3, which consists of 1/2 unit of type i_k players for each $1 \leq k \leq 6$. In this example, there are 13 types of fractional groups.

$$\begin{aligned}\hat{\mathcal{G}} = \{&[1,1,1,1,1,1],[2,2,2,2,2,2],[3,3,3,3,3,3],\\ &[1,1,1,2,2,2],[1,1,1,3,3,3],[2,2,2,3,3,3],\\ &[1,1,1,1,2,2],[1,1,1,1,3,3],[2,2,2,2,1,1],\\ &[2,2,2,2,3,3],[3,3,3,3,1,1],[3,3,3,3,2,2],[1,1,2,2,3,3]\}\end{aligned}$$

In particular, a fractional group might correspond to two groups. For instance, both the 2-person group $[1,1]$ and the 3-person group $[1,1,1]$ correspond to the fractional group $[1,1,1,1,1,1]$. However, when social welfare is maximized, both the 2-person group $[1,1]$ and the 3-person group $[1,1,1]$ are formed if and only if $1.5s[1,1] = s[1,1,1]$. When $1.5s[1,1] > s[1,1,1]$, only the 2-person group $[1,1]$ will be formed as we can break the group $[1,1,1]$ into 1.5 units of the 2-person group $[1,1]$ and the total surplus will increase. When $1.5s[1,1] < s[1,1,1]$, similar operations can be implemented to increase the social welfare. Therefore, we define the surplus of a fractional group to be the maximum total surplus this fractional group of players can obtain by forming 2- or 3-person groups. Numerically, the surplus function is given by:

$$\hat{s}(\hat{G}) = \begin{cases} 1.5 \times 2 = 3, & \text{if } \hat{G} = [1,1,1,2,2,2] \\ 9, & \text{if } \hat{G} = [1,1,2,2,2,2] \\ \max(4, 1.5 \times 3) = 4.5, & \text{if } \hat{G} = [3,3,3,3,3,3] \\ 0, & \text{otherwise} \end{cases}$$

It is worth noting that we define the surplus of the fractional group $\hat{G} = [1,1,1,2,2,2]$ to be the welfare of the group, consisting of 1.5 units of type 1 players and 1.5 units of type 2 players, when all players are assigned into 2-person groups, as no 3-person group corresponds to the fractional group $\hat{G} = [1,1,1,2,2,2]$. However, the maximum welfare of this group of players is at least $2/3 \times 9 = 6$ since it contains $2/3$ mass of 3-person group $[2,3,3]$. Consequently, $\hat{G}$ will not be formed (appear with positive mass in the partition) in any optimal partition of players into fractional groups. Besides, the average surplus of the 2-person group $[3,3]$, which is $3/2$, is larger than the average surplus of the 3-person group $[3,3,3]$, which is 1. Therefore, all type 3 players will be assigned in pairs in any stable assignment.

Given the surplus of fractional groups, the welfare maximization problem is reformulated as

$$\sup_{\hat{\tau}} \sum_{\hat{G} \in \hat{\mathcal{G}}} \hat{s}(\hat{G}) \hat{\tau}(\hat{G})$$

Here, $\hat{\tau} : \hat{\mathcal{G}} \to \mathbb{R}_+$ is the non-negative valued function on the set of fractional groups satisfying the following consistency condition: for any disjoint $i, j, k \in \{1, 2, 3\}$,

$$\begin{aligned}3\tau([i,i,i,i,i,i]) + 2[\tau([i,i,i,i,j,j]) + \tau([i,i,i,i,k,k])] \\ +1.5[\tau([i,i,i,j,j,j]) + \tau([i,i,i,k,k,k])] \\ +[\tau([i,i,j,j,j,j]) + \tau([i,i,k,k,k,k]) + \tau([i,i,k,k,j,j])] = 1\end{aligned}$$

That is, we reformulate the problem in which groups sizes are 2 or 3 to a problem in which the unique group size is $3! = 6$.

2.4.3 Unbounded Group Sizes

When group sizes are unbounded, the existence of a stable assignment has been studied in an approximate manner[28] in Hammond et al. [1989]. In this subsection, we illustrate that an stable assignments may not exist when group sizes are unbounded.

Consider a game with a continuum of homogeneous players. The surplus of any n-person group is given by $s_n = n - 1$. By the equal treatment property of stable assignment, the payoff of any player, in any stable assignment, cannot be higher than or equal to 1. If there is a stable assignment in which all agents have a payoff $u < 1 - 1/m$. Then, m players will form a blocking coalition.

In this example, the surplus function is not uniformly bounded: we have $s_n \to +\infty$ as $n \to +\infty$. When the surplus function is uniformly bounded, the uniform bound will impose a natural upper bound on the group sizes: for formed groups containing too many players, the average payoff of players will goes to zero. Therefore, there is a subset of players in this formed group who have incentive to form a blocking coalition.

2.4.4 An Exchange Economy with Groupwise Externalities

In exchange economies, when there is no consumption externalities, the surplus function is always super-additive. Here, the surplus of a group is defined by the maximum total welfare of the group members who exchange commodities within the group. In contrast, when there are consumption externalities, the surplus function may no longer be super-additive. We give one such example with groupwise envy. This type of groupwise externality differs from the widespread externalities introduced in Hammond et al. [1989].

Now, we describe an economy with groupwise envy. In the economy, agents, with quasi-linear utility functions, either do not trade or trade in pairs. There are one consumption good and two types of agents, type 1 and type 2. Utility functions and initial endowments are given by

$$u_1(m, x, G) = m_1 + \sqrt{x_1} + (x_1 - \max_{j \in G} x_j), \omega_1 = (0, 0)$$

[28] with the approximate feasibility condition.

$$u_2(m, x, G) = m_2 + 100\sqrt{x_2} + (x_2 - \max_{j \in G} x_j), \omega_2 = (0, 100)$$

Here, utility functions are real-valued maps on the monetary good consumption, consumption good consumption and the group the agent is in. The externality term $x_i - \max_{j \in G} x_j$ suggests that an agent would envy his trade partner if his trade partner consumes more consumption goods than him.

By the definition of the surplus function,

$$s([1]) = 0, s([2]) = 1000$$

$$s([1,1]) = 0, s([2,2]) = 2000$$

$$s([1,2]) = \max_{x_1, x_2 \geq 0 : x_1 + x_2 = 100} \sqrt{x_1} + 100\sqrt{x_2} + 100 - 2\max(x_1, x_2) \simeq 900.083$$

where the maximizers are $x_1 \simeq 0.03, x_2 \simeq 99.97$.

That is, the joining of a type 1 agent to the 1-person group consisting of just a type 2 agent will reduce the group surplus due to the existence of groupwise envy. Intuitively, since type 2 agents value the consumption good much more than type 1 agents, in any efficient outcome, almost all consumption good will be consumed by type 2 agents. Consequently, although the type 1 agent consumes a little consumption good by trading with the type 2 agent, his envy makes him much less happy. In addition, the type 2 agent, consuming a little less consumption good, is also less happy. Therefore, with groupwise envy, the sum of payoffs are decreased after the trade.

2.5 Applications

In this part, we apply our results to three problems.

2.5.1 Surplus Sharing Problem

In a market, a continuum of agents of different types form groups in order to share group surpluses. Each agent can participate to form only one group and each group can be formed by at most N agents. The surplus of a formed group only depends on the type distribution of its members.

By our results, there is a stable state in this market. At the stable state, agents are endogenously segmented into small groups and they have a way to share the group surplus such that no subset of agents have incentives to form a new group.

We have three more observations regarding the stable state of the market. Firstly, at the stable state, the same type of agents have the same payoff, which can be understood as the wage level of this type of players. Secondly, at the stable state, the surplus of any formed group is equal to the total payoff of its members. Thirdly, at a stable state, even if an agent can initiate to form a new group and obtain all the group surplus of the new group by paying some other agents an amount no less than their wage levels, no agent has incentive to do so. The reason of this fact is that, at an stable state, the group surplus an agent could obtain by forming a new group minus all the wages he pays to the others cannot be larger than his current wage level.

2.5.2 Production Problem

Given a bag of (finitely many or infinitely many) inputs (Lego bricks), there are numerous ways to combine them into outputs (toys), each of which has a value. The production of any output requires a specific finite combination of inputs (Lego bricks). Production technology specifies a relation between input distributions and output values. Knowing the production technology, there are two natural questions. Firstly, what is the bag's value, defined as the maximum total value of outputs obtained by using the inputs in the bag? Secondly, how to achieve this maximum value?

Figure 3: Production problem[29]

By our results, each type of inputs is endowed with a value defined by the imputation corresponding to a stable assignment. Conceptually, knowing these values, we can answer the first question on the bag's value in the following sense: when the bag contains a continuum of inputs, the sum (integration) of input values in the bag will give the bag's value. When the bag contains only finitely many inputs, the sum of input values in the bag will give an upper bound on the bag's value. When we have

[29]an online picture.

a large number of bags containing the same inputs, the total value of these bags is approximately equal to the total value of all inputs.

Actually, it is easy to see that these two questions can be answered by directly solving a profit maximization problem stated in Equation 5, in which the choice set is the set of all partitions of inputs into small groups. Each small group in the partition will be used to produce an output. When there are a continuum of inputs in the bag, the maximization problem in Equation 5 gives exact answers to both questions. When there are only finitely many inputs, the maximization problem in Equation 5 gives approximate answers to both questions.

However, it is usually computationally infeasible to solve the linear programming problem Equation 5, as the number of unknowns $|\mathcal{G}|$ becomes astronomical even when the maximum group size is very small.[30] For example, when there are $|I| = 1000$ types of inputs and each toy requires a combination of at most 4 pieces of bricks, there are

$$|\mathcal{G}| = \sum_{n=1}^{4} \left(\!\!\binom{1000}{n}\!\!\right) \simeq 4.2 \times 10^{10}$$

types of toys. That is, in order to answer either question, the direct method is to solve a maximization problem with 4.2×10^{10} unknowns.

To deal with the curse of dimensionality, a first thought is that we could apply the duality relation in Theorem 2 to answer the first question: to compute the bag's value, or a upper approximation of the bag's value, we only need to compute the values of inputs. That is, we only need to solve the dual problem in Equation 6 with only $|I|$ unknowns. However, this dual problem is also hard to solve as there are $|\mathcal{G}|$ constraints in it. Moreover, the minimization problem provides no answer to our second question about the optimal way of production.

It appears a better way to deal with the curse of dimensionality is to use Proposition 1 to reformulate the maximization problem as an $N!$-marginal symmetric transport problem. Utilizing the symmetric structure in the symmetric transport problem, Friesecke and Vogler [2018] proved this $N!$-marginal symmetric transport problem can be further reformulated as a combinatorial maximization problem with only $(N!+1)|I|$

[30] $|\mathcal{G}| = \sum_{n=N'}^{N} \left(\!\!\binom{|I|}{n}\!\!\right) = \Theta(|I|^N)$.

unknowns.[31] In the example with 1000 types of inputs and group sizes are up to 4, the reformulated maximization problem has 25000 unknowns. We refer to Khoo and Ying [2019] for a numerical implementation of this dimension reduction trick.

2.5.3 Market Segmentation

Lastly, we study an exchange economy with no centralized market. In the economy, a continuum of agents exchange commodities with each other within small groups of bounded finite sizes. Each type of agent is represented by a continuous quasi-linear utility function and an initial endowment. In this economy, we allow the existence of groupwise externalities: agents' utility functions may depend on the consumption of their trade partners. An example of an economy with groupwise externalities is given in Section 2.4.4.

Our result suggests, with necessary assumptions on the continuity and boundedness of utility functions and initial endowments, there is a stable state in which the market is segmented into small groups and no subset of agents can jointly do better by forming a new group to trade with each other.

In Section 4, we will discuss segmented exchange markets with no externalities in details.

2.6 Related Literature

We review the literature from two perspectives: the conceptual perspective and the methodological perspective.

From the conceptual perspective, the most relevant literature to the material in this section is the f-core literature in Kaneko and Wooders [1982], Kaneko and Wooders [1986], Hammond et al. [1989], Kaneko and Wooders [1996]. With an approximate feasibility condition, this sequence of works proves the existence of an approximately

[31] The number of unknown can be further reduced if not all group sizes are permitted. In particular, we can always replace the number $N!$ by the least common multiple of all group sizes. When N is fixed and $|I|$ is large, $(N!+1)|I| = \Theta(|I|)$. Moreover, we note this reduction technique is not a free lunch. In particular, it transforms a linear programming problem to a combinatorial optimization problem. However, when $|I|$ is large, solving the linear programming problem is impossible due to the large number of unknowns. This reduction trick provides a possibility to solve the optimization problem even though the transformed problem is combinatorial. For instance, Khoo and Ying [2019] used the trick to solve a problem which has 10^{25} unknowns initially.

stable assignment in games with a compact type space.[32]

There are three major differences between the literature mentioned above and the work in this section. Firstly, in this section, we developed a new formulation for the concept f-core by studying a statistical representation of the partition. Secondly, dropping the approximate feasibility assumption, we proved the existence of an stable assignment in a game with possibly a non-compact type space. Thirdly, we showed that, when the surplus function is continuous, similar agents obtain similar payoffs in a stable state.

The framework and results in this section unified the literature on matching problem, roommate problem and f-core. When the unique permitted group size is 2 and the surplus function has a bipartite structure, our work implies results on the matching problem. In particular, Shapley and Shubik [1971] and Roth et al. [1993] studied matching problem with a finite type space. Gretsky et al. [1992], Chiappori et al. [2010] and Chiappori et al. [2016] studied matching problem with a general type space. When the permitted group size is k and the surplus function is assumed to have a k-partite structure, our work implies results on the multi-matching problem such as Carlier and Ekeland [2010] and Pass [2015b]. When there is an upper bound on group sizes, our work implies results on f-core in the transferable cases. In particular, Kaneko and Wooders [1982], Kaneko and Wooders [1986], Wooders [2012] studied f-core with a finite type space. Kaneko and Wooders [1996] studied f-core with a compact atomless type space.

Our model assumes that there are a continuum of players in the game. It is valid to ask, with a *discrete* player set, whether a stable assignment exists or not. To my knowledge, our understanding is complete only when the unique permitted group size is 2: when the surplus function has a bipartite structure, Koopmans and Beckmann [1957] proves the existence of a stable assignment. When the surplus function has no bipartite structure, Chiappori et al. [2014] proves the existence of a stable assignment if the player set is replicated 2 times. Both observations rely on the validity of Birkhoff-von Neumann theorem on matrices. Finding a high dimensional analog of Birkhoff von-Neumann theorem is an ongoing project[33]. Therefore, we need to develop new tools to understand discrete models when the maximum group size is larger than 2.

From the methodological perspective, the literature on f-core such as Kaneko and Wooders [1982], Kaneko and Wooders [1986], Kaneko and Wooders [1996] did not use the optimization method to prove the existence of a stable assignment, as the problem is

[32]This sequence of works are mainly focused on models with non-transferable utility. Here, I am talking about the application of these works to a game with transferable utility. It is worth noting that the method we developed in this paper depending on linear dualities, and thus cannot be applied to nontransferable utility games directly. For a link between duality and games with nontransferable utility, we refer to Nöldeke and Samuelson [2018].

[33]See Linial and Luria [2014] for related literature.

formulated in a non-transferable utility environment. However, in a transferable utility environment, it is most natural to use linear programming to analyze the problem. Indeed, when there are only finitely many types of players in the game, it is well known that linear programming helps to prove the existence of a stable assignment. We refer to Wooders [2012] for details. The idea of using linear programming to prove stability dates back to the ground-breaking works of Bondareva [1963] and Shapley [1967], in which the connection between core and linear programming is established. Several works including Kannai [1969], Schmeidler [1972b], Kannai [1992] extended this connection to infinite type spaces. However, these extensions are not applicable to our model in which group sizes are bounded.

In this section, we connected the literature on f-core to transport problems. This connection helps us to prove the existence of a stable assignment for general type space and significantly reduces the number of unknowns for finding stable assignments in the finite type case.

Actually, there is a long tradition of applying the transportation method to study stability. In this line, most works assume some structure on the surplus function: when the surplus function has a bipartite structure and the unique permitted group size is 2, Koopmans and Beckmann [1957], Shapley and Shubik [1971] studied stable matching with a finite type space. Gretsky et al. [1992], Chiappori et al. [2010], Chiappori et al. [2016] studied stable matching with a general type space. When the surplus function has a k-partite structure and the unique permitted group size is k, Carlier and Ekeland [2010] and Pass [2015b] studied stable multi-matching with general type spaces. When the surplus function has no special structure and the unique permitted group size is 2, Chiappori et al. [2014] studied stable matching in a discrete player model. Their key observation is that any pure matching in the replicated player space corresponds to an assignment. In this section, we propose an identification trick that is inversely related to their observation: an assignment can be identified by some symmetric transport plan in a replicated player space. This identification trick helps us in two ways. Firstly, it helps us to study games with a continuum of players, with small groups containing more than two players and with small groups of different sizes. Secondly, it helps to establish the equivalence relation between the set of stable assignments and the solution of a symmetric optimal transport problem.

Recently, Friesecke and Vogler [2018] introduced a dimension reduction trick to significantly reduce the number of unknowns in a symmetric optimal transport problem. This trick has a huge potential in computations. In particular, Khoo and Ying [2019] used this dimension reduction trick to solve a problem with 10^{25} unknowns. In this section, by relating stable assignments to symmetric transport plans, we enable the possibility of applying this dimension reduction trick to find stable assignments.

For more complete surveys on optimal transport problems, we refer to Villani [2008] for 2-marginal problems, to Pass [2011] and Pass [2015a] for multi-marginal problems.

3 Games with Positive-Size Groups

In Section 2, a small group is defined as a finite subset of players. Therefore, every small group contains at most finitely many types of players and is of "measure zero". In some applications such as Schmeidler [1972a], small groups may contain infinitely many types of players and have small masses. To study this phenomenon, in this section, we extend our previous model and study small groups with small positive masses.

This section is organized as follows. In Section 3.1, we set up the problem. In Section 3.2, we state the results. In Section 3.3, we review the literature. All proofs are postponed to Appendix C.

3.1 Model

We study a cooperative (transferable utility) game represented by the tuple $((I, \mu), s, \varepsilon', \varepsilon)$. Here, the positive numbers $\varepsilon', \varepsilon \in (0, 1]$ are the lower and upper bounds on group sizes respectively.

3.1.1 Type Space

The *type space* of players is summarized by a probability space (I, μ). In the tuple, I is a Polish space representing the set of players' types and $\mu \in \mathcal{P}(I)$ is a probability measure on I representing the distribution of players' types. In particular, μ is not assumed to be atomless.

3.1.2 Groups

Groups are the units in which players interact with each other. Similar to the formulation of sub-population in Che et al. [2019], a group is defined by a non-negative measure $\nu \in \mathcal{M}_+(I)$ on I that is not larger than μ. The size of a group ν is its total measure $\|\nu\| = \nu(I)$ on I. All group sizes are bounded from below by $\varepsilon' \in (0, 1]$[34] and bounded from above by $\varepsilon \in (0, 1]$, where $\varepsilon \geq \varepsilon'$. When $\varepsilon = \varepsilon'$, only one group size is permitted in the game.

[34]Technically, it is important to assume that the group sizes are bounded away from zero. The strictly positive lower bound will imply that the set of assignments defined later is a compact set in weak topology.

Formally, a *group* is a non-negative measure ν on I such that $\nu \leq \mu$ and $\|\nu\| \in [\varepsilon', \varepsilon]$. *The set of groups* $\mathcal{G}$ consists of all groups which sizes are bounded by ε' and ε:

$$\mathcal{G} = \{\nu \in \mathcal{M}_+(I) : \nu \leq \mu, \varepsilon' \leq \nu(I) \leq \varepsilon\}$$

The set of groups $\mathcal{G}$ is endowed with the weak topology[35].

3.1.3 Surplus Function

The surplus function specifies the total amount of surplus group members can share. Formally, a *surplus function* $s : \mathcal{G} \to \mathbb{R}_+$ is a real valued function on the set of groups. There are two assumptions on the surplus function:

(B1) s is upper semi-continuous in the weak topology.

(B2) there is a bounded continuous function $a \in C_b(I)$ such that $s(\nu) \leq \int_I a d\nu$, for all $\nu \in \mathcal{G}$.

Assumption (B1) states the continuity requirement of the surplus function and Assumption (B2) ensures the integrability of the surplus function. We note Assumption (B2) is satisfied if there is a constant $c \geq 0$ such that $s(\nu) \leq c \|\nu\|$ for all $\nu \in \mathcal{G}$. That is, Assumption (B2) is satisfied if the average contribution of a player to any group is bounded from above by a constant $c > 0$.

We say a surplus function is *linear*, if for all $\nu \in \mathcal{G}$ and $\alpha > 0$, we have $s(\alpha\nu) = \alpha s(\nu)$ whenever $\alpha\nu \in \mathcal{G}$. Moreover, we say a surplus function is *super-additive*, if for all $\nu_1, \nu_2 \in \mathcal{G}$, we have $s(\nu_1 + \nu_2) \geq s(\nu_1) + s(\nu_2)$ whenever $\nu_1 + \nu_2 \in \mathcal{G}$. In particular, a surplus function generated from a general equilibrium model with no externalities is both linear and super-additive.

3.1.4 Assignments

An assignment is a partition of a continuum replication of players into groups of permitted sizes. The continuum replication of players is consistent with our study of small groups: a group of players containing 99% of the total population is small in the continuum replicated player set.

[35] By Theorem 8.3.2 in Bogachev [2007], the weak topology on $\mathcal{M}_+(I)$ is metrizable by the Kantorovitch-Rubinstein norm $\|\cdot\|_0$ defined by

$$\|\mu\|_0 = \sup\left\{\int_I f d\mu : f \in Lip_1(I), \|f\|_\infty \leq 1\right\}$$

Formally, an *assignment* is a non-negative measure on the set of groups $\mathcal{G}$ satisfying the consistency condition:

$$\int_{\mathcal{G}} \nu(A) d\gamma(\nu) = \mu(A), \forall \text{measurable } A \subset I$$ [36]

The consistency condition states that all players, whose types are in A, are assigned by the assignment γ. Moreover, we use a set $\Gamma_{\mathcal{G}}$ to denote *the set of assignments*. i.e.

$$\Gamma_{\mathcal{G}} = \left\{ \nu \in \mathcal{M}_+(\mathcal{G}) : \int_{\mathcal{G}} \nu(A) d\gamma(\nu) = \mu(A), \forall \text{measurable } A \subset I \right\}$$

Intuitively, $\gamma(\nu)$ gives the average mass of group ν in the partition of the replicated player set: when the player set is replicated by a large number, the product of $\gamma(\nu)$ and the number of replication specifies the number of groups ν in the partition of the replicated player set determined by assignment γ.

We finish this subsubsection by two remarks.

Firstly, the set of assignments $\Gamma_{\mathcal{G}}$ is non-empty since $\frac{2}{\varepsilon+\varepsilon'} \mathbb{1}_{\frac{\varepsilon+\varepsilon'}{2}\mu} \in \Gamma_{\mathcal{G}}$. That is, all players are assigned to the group $\frac{\varepsilon+\varepsilon'}{2}\mu \in \mathcal{G}$.

Secondly, an assignment is not a probability measure on $\mathcal{G}$ in general. We prove by contradiction. If an assignment is a probability measure on $\mathcal{G}$, then, by taking $A = I$ in the consistency condition, we have that the left hand side of the consistency equation is less than ε while the right hand side of the consistency equation is 1. Contradiction.

3.1.5 Stability

Similar to the case where group sizes are finite, we say an assignment $\gamma \in \Gamma_{\mathcal{G}}$ is *stable* if there is an imputation $u \in L^1(I, \mu)$ such that

1. $\int_I u d\nu \leq s(\nu)$, for $\gamma-$almost all groups $\nu \in \mathcal{G}$[37]
2. $\int_I u d\nu \geq s(\nu)$, for all groups $\nu \in \mathcal{G}$

The first condition is the feasibility condition, which states that, in any formed group (intuitively, a group with a positive mass in the partition), group members can in total share no more than the group surplus. The second condition is the no-blocking condition, which states no group of agents could jointly do better by forming a new group.

[36] For any Borel measurable $A \subset I$, the real valued map $\nu \to \nu(A)$ on $\mathcal{M}_+(I)$ is continuous in the weak topology since

$$|\nu_1(A) - \nu_2(A)| \leq \int_A 1 d|\nu_1 - \nu_2| \leq \int_I 1 d|\nu_1 - \nu_2| \leq \|\nu_1 - \nu_2\|_0$$

Therefore, the map $\nu \to \nu(A)$ is Borel measurable. That is, $\int_{\mathcal{G}} \nu(A) d\gamma(\nu)$ is well-defined.

[37] Here, we need the term "almost everywhere" since we did not prove that there exists a continuous imputation. See Footnote 19 for the related discussions on "formed groups".

We note, when the surplus function is linear, a stable assignment can be interpreted as a stable way to partition of the continuum players into small groups with no lower bound on group sizes[38].

3.2 Results

We prove the existence of a stable assignment by establishing a "continuum"-marginal extension of the Koopmans-Kantorovitch duality theorem. Therefore, we have the following two theorems.

Theorem 3. *For any game $((I, \mu), s, \varepsilon', \varepsilon)$ satisfying Assumption (B1) and Assumption (B2), there is a stable assignment.*

Theorem 4. *For any game $((I, \mu), s, \varepsilon', \varepsilon)$ satisfying Assumption (B1) and Assumption (B2),*

$$\sup_{\gamma \in \Gamma_{\mathcal{G}}} \int_{\mathcal{G}} s d\gamma = \inf_{u \in \mathcal{U}} \int_I u d\mu$$

where $\mathcal{U} = \{u \in L^1(\mu) : \int_I u d\nu \geq s(\nu), \forall \nu \in \mathcal{G}\}$ and the infimum could be attained.

The proofs of these two theorems are postponed to Appendix C.

3.3 Related Literature

Small groups of positive sizes have been studied by Schmeidler [1972a] in exchange economies. In his work, Schmeidler proved that any core allocation cannot be blocked by any small group of epsilon sizes. However, since core allocations are not feasible in general, this result is not enough to imply the existence of a stable assignment.

In this section, we developed a game model with small groups of positive sizes and proved the existence of a stable assignment. In particular, we do not assume the surplus function to be linear or super-additive. Therefore, our model can be used to analyze exchange economies with externalities.

To prove the existence result, we generalized the Koopmans-Kantorovitch duality theorem to a "continuum"-marginal case. Our proof is built on the proof of duality theorem in multi-marginal transport problem in Kellerer [1984] and Dudley [2002].

[38] When the surplus function is linear, the surplus of a group ν is equal to the total surplus of $1/\alpha$ units of groups $\alpha\nu$. Consequently, in any stable assignment γ, for any formed group ν and $\alpha > 0$, $\int_I ud(\alpha\nu) \leq s(\alpha\nu)$ provided the scaled group $\alpha\nu \in \mathcal{G}$. Therefore, in a stable assignment, any group with a positive mass in the partition can be split in an arbitrary uniform way without changing the stability property of the assignment.

4 Economies with Finite-Size Groups

In this section, we study an exchange economy with a continuum of agents who trade only within small groups of bounded finite sizes. In particular, all group sizes are bounded from above by some natural number $N \geq 2$. To apply our results in Section 2, we assume that all agents have quasi-linear utility functions.

We have two findings. First, there is a stable state in any segmented market. (i.e. N-core is non-empty.) Second, every stable state in a segmented market can be supported by some market clearing price. Here, a price is defined to be a cost function on the traded quantities.

This section is organized as follows. In Section 4.1, we develop a general equilibrium model of a segmented market. In Section 4.2 and Section 4.3, we state and prove our results. In Section 4.4, we study three examples. Lastly, in Section 4.5, we review the related literature and propose some directions for future work.

4.1 Model

In this section, we study an economy denoted by the tuple $\mathcal{E} = ((I, \mu), L, (u_i, \omega_i)_{i \in I}, N)$. Compared to the classical general equilibrium models, the only new ingredient is the upper bound $N \in \mathbb{N}$ on group sizes.

There are a unit mass of agents in the economy. The measure space (I, μ) denotes the type space of these agents. In particular, I is a compact metric space[39] representing of agents' types and $\mu \in \mathcal{P}(I)$ is a probability measure on the type space I representing the distribution of agents' types. There are two canonical examples of the type space. Firstly, I is a finite set, and μ is a probability vector with $|I|$ positive entries. Secondly, I is the unit interval $[0, 1]$ and μ is a probability measure on $[0, 1]$ with or without atoms.

There are two types of goods in this economy: one monetary good, money, and $L \in \mathbb{N}$ consumption goods. We will use $m \in \mathbb{R}$ to denote the monetary good consumption and $x \in \mathbb{R}^L_+$ to denote the consumption goods consumption. Therefore, an agent's consumption is summarized by a pair $(m, x) \in \mathbb{R} \times \mathbb{R}^L_+$.

For a type i agent, his utility function on consumption goods is denoted by a function $u_i : \mathbb{R}^L_+ \to \mathbb{R}$. Consequently, his overall utility function $U_i : \mathbb{R} \times \mathbb{R}^L_+ \to \mathbb{R}$, induced by u_i, is

$$U_i(m, x) = m + u_i(x)$$

In addition, for any type i agent, his initial endowment is denoted by a vector $\omega_i \in \mathbb{R} \times \mathbb{R}^L_+ - \{0\}$. We avoid trivial discussions by assuming $\omega_i \neq 0$. Since there are

[39] See Section 4.2.1 for extensions to non-compact type spaces.

two types of goods, we write the $(1+L)$-dimensional vector ω_i as a pair (ω_i^m, ω_i^x), where $\omega_i^m \in \mathbb{R}$ is type i agents' initial endowment of the monetary good, and $\omega_i^x \in \mathbb{R}_+^L$ is type i agents' initial endowment of the consumption goods.

Assumptions on utility functions and initial endowments are as follows:

(C1) for all $i \in I$, u_i is weakly increasing and continuous

(C2) for all $i \in I$, u_i is concave

(C3) ω_i is continuous in I

(C4) for all $x \in \mathbb{R}_+^L$, $u_i(x)$ is continuous in I

Assumption (C1) and Assumption (C2) are the usual assumptions about the monotonicity, continuity and the concavity of utility functions. Assumption (C3) and Assumption (C4) imply that similar types of agents are defined similarly. When there are only finitely many types of agents in the economy, i.e. I is a finite set, Assumption (C3) and Assumption (C4) are automatically satisfied.

4.1.1 Groups and Assignments

In this subsubsection, we repeat our definitions of groups and assignments in Section 2. In the setting of segmented economy discussed in this section, a group is a local market in which agents trade with each other, and an assignment is a market structure which partitions the large market into small local markets.

Formally, the set of groups $\mathcal{G}$ is defined by

$$\mathcal{G} = \bigcup_{n=1}^{N} \mathcal{G}_n$$

where $\mathcal{G}_n$, the set of n-person groups, consists of all multisets of cardinality n on I. Identifying $\mathcal{G}_n$ by the set of equivalence classes $I^n/\sim_n$, we write an n-person group in $\mathcal{G}_n$ as $G = [i_1, ..., i_n]$. We say that a type i agent is in group $G = [i_1, ..., i_n]$, denoted by $i \in G$, if $i = i_k$ for some $1 \le k \le n$. For any function f on I, we define $\sum_{i \in G} f(i) = \sum_{i \in I} |\{k \in \mathbb{N} : i_k = i\}| f(i)$, where $n_i = |\{k \in \mathbb{N} : i_k = i\}|$ is the number of type i agents in group G.

An assignment is a statistical representation of a partition of continuum of agents into small groups, denoted by a tuple of measures $\tau = (\tau_1, ..., \tau_N)$, where τ_n is a non-negative measure on $\mathcal{G}_n$, satisfying the consistency requirement:

$$\sum_{n=1}^{N} \sum_{k=1}^{n} k \tau_n(\mathcal{G}_n(A, k)) = \mu(A), \forall A \subset I$$

where $\mathcal{G}_n(A, k)$ is the set of n-person groups containing exactly k agents whose types are in A. Intuitively, τ_n summarizes the masses of all n-person groups in the economy.

The set of assignments is denoted by T. Given an assignment $\tau \in \mathrm{T}$, we say a group $G \in \mathcal{G}$ is formed, or is a formed group, if $G \in \mathrm{supp}(\tau) = \cup_{n=1}^{N} \mathrm{supp}(\tau_n)$. In addition, in a formed group $G = [i_1, ..., i_m]$, for any $1 \leq q \leq m$, a type i_q agent has a *trading partner* of type i_k for any $k \neq q$.

4.1.2 Feasible Allocations

An allocation specifies agents' consumption. In a segmented economy, an allocation is feasible under a market structure τ if the allocation can be achieved by some redistribution of the initial endowments within each formed group. We note, when market is segmented, two agents of the same type may consume differently if they are in different groups. Therefore, we specify consumption plans group by group.

Formally, a tuple $((m_i^G, x_i^G)_{i \in G})_{G \in \mathrm{supp}(\tau)}$ is a *feasible allocation under assignment* τ, if it satisfies

- for any formed group $G \in \mathrm{supp}(\tau)$ and type i agent in G, $m_i^G \in \mathbb{R}, x_i^G \in \mathbb{R}_+^L$
- for any formed group $G \in \mathrm{supp}(\tau)$, $\sum_{i \in G}(m_i^G, x_i^G) = \sum_{i \in G}(\omega_i^m, \omega_i^x)$

Intuitively, m_i^G and x_i^G are amounts of the monetary and consumption goods consumed by a type i agent in a formed group G.[40] There are two conditions. Firstly, the consumption goods consumption are non-negative. Secondly, in any formed group, the total consumption is equal to the total endowments. That is, in every local market, the market is clear.

Lastly, we compare our definition with the classical notion. In classical general equilibrium models, there is a centralized exchange market. Therefore, an assignment τ represents a partition with only one cell - the grand coalition. As a result, $\mathrm{supp}(\tau) = \{I\}$ and the notion of feasible allocation $((m_i^G, x_i^G)_{i \in G})_{G \in \mathrm{supp}(\tau)}$ defined in segmented markets is reduced to the classical notion $(m_i, x_i)_{i \in I}$.

4.1.3 Stable State and N-core

In a segmented economy, a stable state is a pair of a market structure (assignment) and a feasible allocation such that no group of agents of permitted size can do better by trading with each other. N-core[41] is a set consists of all stable states. In the following,

[40] This notion assumes the same type of agents in the same group consume the same bundle. When all utility functions are concave, we are able to define feasible allocations in this way to obtain a more compact notion.

[41] We use the term N-core to distinguish it from the classical notions core and f-core. Different from core, there is an upper bound on the size of blocking coalitions in the definition of N-core. In addition, different from f-core, there is no approximation in the definition of N-core.

we first define the surplus function on the set of groups. Then, we define the stable states and N-core.

The surplus of a group is defined to be the maximum total payoffs of its members obtained by exchanging commodities with each other in the group. Formally, the *surplus function* $s : \mathcal{G} \to \mathbb{R}$ is defined by

$$s(G) = \max\left\{\sum_{i\in G} U_i(m_i, x_i) : m_i \in \mathbb{R}, x_i \geq 0, \sum_{i\in G}(m_i, x_i) = \sum_{i\in G}(\omega_i^m, \omega_i^x)\right\} \tag{17}$$

for any $G \in \mathcal{G}$. Since utility functions are quasi-linear, we have

$$s(G) = \max\left\{\sum_{i\in G} u_i(x_i) : x_i \geq 0, \sum_{i\in G} x_i = \sum_{i\in G} \omega_i^x\right\} + \sum_{i\in G} \omega_i^m$$

for any $G \in \mathcal{G}$. Therefore, when all utility functions are continuous, the surplus function s is well-defined. Furthermore, as there is no consumption externalities in this section, the surplus function is super-additive[42]. The following lemma states that the surplus function defined by Equation 17 is continuous and bounded from above by some continuous function:

Lemma 9. *For the economy $\mathcal{E}$ satisfying Assumption (C1), Assumption (C3) and Assumption (C4), the surplus function s defined by Equation 17 is continuous on $\mathcal{G}_n$ for all $1 \leq n \leq N$. Moreover, for each every $1 \leq n \leq N$, there is a continuous function $a_n \in L^1(I, \mu)$ such that*

$$s(G) \leq \sum_{i\in G} a_n(i), \forall G \in \mathcal{G}_n$$

Proof. See Appendix D.1. ■

A *stable state* of the economy $\mathcal{E}$ is a pair $(\tau, ((m_i^G, x_i^G)_{i\in G})_{G\in \mathrm{supp}(\tau)})$ where

- τ is an assignment
- $((m_i^G, x_i^G)_{i\in G})_{G\in \mathrm{supp}(\tau)}$ is a feasible allocation under assignment τ
- for any group $G = [i_1, ..., i_n] \in \mathcal{G}$, and any formed group $G_k \in \mathrm{supp}(\tau)$ containing an type i_k agent, we have

$$\sum_{k=1}^{n} U_{i_k}(m_{i_k}^{G_{i_k}}, x_{i_k}^{G_{i_k}}) \geq s(G)$$

and *N-core* of the economy $\mathcal{E}$ is a set containing all stable states.

There are three conditions in the definition of stable state. The first condition clarifies notations. The second condition is the feasibility condition, which implies that

[42] A function s on $\mathcal{G}$ is super-additive if for any n-person group $[i_1, ..., i_n]$ and m-person group $[j_1, ..., j_m]$, if $m + n \leq N$, we have $s([i_1, ..., i_n, j_1, ..., j_m]) \geq s([i_1, .., i_n]) + s([j_1, ..., j_m])$.

the supply and the demand are balanced in all local markets. The third condition is the no-blocking condition, which implies there is no group of agents who can jointly do better by forming a new group.

Every stable state has two properties. First, the same type of agents obtain the same payoff at any stable state. Second, agents in any form group are trading efficiently at any stable state.

Lemma 10. *In an economy $\mathcal{E}$, at any stable state $(\tau, ((m_i^G, x_i^G)_{i\in G})_{G\in supp(\tau)})$, we have*

1. *there is an imputation $v : I \to \mathbb{R}$ such that, for any formed group $G \in supp(\tau)$ containing an type i agent, $U_i(m_i^G, x_i^G) = v_i$.*
2. *for any formed group $G \in supp(\tau)$, we have $\sum_{i\in G} U_i(m_i^G, x_i^G) = s(G)$.*

Proof. For the first property, if two agents of the same type have different payoffs, the total payoff of a group consisting of the lower payoff agent and all trading partners of the higher payoff agent is strictly less than the surplus of this group, thus violating the third condition in the definition of stable state. The second property follows from the feasibility condition and no-blocking condition in the definition of stable state. ■

By the first property in Lemma 10, the no blocking condition in the definition of stable state can be written as: for all group $G \in \mathcal{G}$,

$$\sum_{i\in G} v_i \geq s(G)$$

By Lemma 8 and the second property in Lemma 10, any stable state maximizes the social welfare. However, the converse is not true. For instance, at any stable state, in any formed group, if we reallocate all the monetary resources to one of its members, the resulting new allocation has the same social welfare but is no longer stable.

4.1.4 Price and Competitive Equilibrium

In this subsubsection, we define the competitive equilibrium in a segmented market.

In this paper, by a price, we mean a cost function on the set of traded quantities. Formally, a *price* p is a real-valued, odd[43], increasing and continuous function on the set of net trades $\mathbb{R}^L$.

A net trade $\Delta x = (\Delta x_1, ..., \Delta x_L) \in \mathbb{R}^L$ is a vector of traded quantities of consumption goods. In particular, $\Delta x_l > 0$ corresponds to the purchase of $|\Delta x_l|$ units of good l and $\Delta x_l < 0$ corresponds to the selling of $|\Delta x_l|$ units of good l. The real number $p(\Delta x_1, ..., \Delta x_L)$ represents the units of monetary good needed in order to trade a bundle

[43] p is an odd function if $p(\Delta x) + p(-\Delta x) = 0$ for all $\Delta x \in \mathbb{R}^L$.

$(\Delta x_1, ..., \Delta x_L)$. Intuitively, when all Δx_l are non-negative, agents pay $p(\Delta x_1, ..., \Delta x_L)$ units of money to receive a bundle $(\Delta x_1, ..., \Delta x_L)$.

There are three conditions on the price function p. First, the oddness of price implies that the money payed to buy a bundle is equal to the money received from selling the same bundle. Second, the increasingness of price implies that no agent can pay less for more goods. Last, the continuity of price ensures that similar traded quantities correspond have similar costs.

In classical general equilibrium models, there is a unit price p_ℓ for commodity ℓ. Therefore, the price, or the cost function, defined on traded quantities is $p(\Delta x) = p_1 \Delta x_1 + ... + p_L \Delta x_L$, which is always linear. Thus, there are three major differences between the notion price in this paper and the classical notion. First, the price in this paper is the cost of a bundle, rather than the cost of a unit of some good. Second, the price in this paper may not have the additive separable form, as the traded bundle may contain more than one goods. Third, the price in this paper might be a non-linear function.

Under price p, a type i agent's *budget set* is defined by

$$B_i(p) = \{(m, x) \in \mathbb{R} \times \mathbb{R}^L_+ : m \leq \omega_i^m + p(\omega_i^x - x)\}$$

That is, for any type i agent, his monetary consumption is not larger than the sum of his monetary endowment and the monetary good obtained from selling his consumption goods.

In an economy $\mathcal{E}$, a tuple $(p, \tau, ((m_i^G, x_i^G)_{i \in G})_{G \in \text{supp}(\tau)})$ is a *competitive equilibrium* if

- p is a price
- τ is an assignment
- $((m_i^G, x_i^G)_{i \in G})_{G \in \text{supp}(\tau)})$ is a feasible allocation under assignment τ
- $(m_i^G, x_i^G) \in \arg\max_{(m,x) \in B_i(p)} U_i(m, x)$ for any formed group $G \in \text{supp}(\tau)$ and type $i \in G$

At a competitive equilibrium of the economy $\mathcal{E}$, all agents consume their optimal consumption bundle in their budget sets. In addition, the market is segmented into small local markets according to assignment τ such that the supply and the demand are balanced in every local market.

4.2 Results

In Section 4.1, we defined two solution concepts: stable state and competitive equilibrium. Now, we state results regarding their existence and their relationship.

Firstly, as suggested by Theorem 1, there is a stable state:

Theorem 5. *In an economy $\mathcal{E}$ satisfying Assumption (C1), Assumption (C2)*[44]*, Assumption (C3) and Assumption (C4), there is a stable state $(\tau, ((m_i^G, x_i^G)_{i \in G})_{G \in supp(\tau)})$.*

Proof. See Appendix D.2. ∎

Our main theorem in this section is that all stable states can be supported by some nonlinear price.

Theorem 6. *In an economy $\mathcal{E}$ satisfying Assumption (C1), Assumption (C2), Assumption (C3) and Assumption (C4), for any stable state $(\tau, ((m_i^G, x_i^G)_{i \in G})_{G \in supp(\tau)})$, there is a price p such that the triplet $(p, \tau, ((m_i^G, x_i^G)_{i \in G})_{G \in supp(\tau)})$ is a competitive equilibrium.*

The existence of a competitive equilibrium follows from these two theorems:

Corollary 1. *In an economy $\mathcal{E}$ satisfying Assumption (C1), Assumption (C2), Assumption (C3) and Assumption (C4), a competitive equilibrium exists.*

The existence theorem suggests that there exists a nonlinear market price such that the large market is segmented endogenously into small local markets in a way that the supply and the demand are balanced in every local market.

Lastly, we prove that competitive allocations are stable under pairwise tradings:

Theorem 7. *When $N = 2$, for any competitive equilibrium $(p, \tau, ((m_i^G, x_i^G)_{i \in G})_{G \in supp(\tau)})$, the pair $(\tau, ((m_i^G, x_i^G)_{i \in G})_{G \in supp(\tau)})$ is a stable state.*

Proof. See Appendix D.3. ∎

4.2.1 Extensions

When the type set I is a (possibly noncompact) Polish space, all results in this section still hold if we make more assumptions and vary the definition of "formed groups".

Firstly, when the type set I is not compact, we need to impose the following assumptions:

(C1) for all $i \in I$, u_i is weakly increasing and continuous

(C2) for all $i \in I$, u_i is concave

(C3) ω_i is continuous in I. Moreover, $\omega_i^x \leq M\vec{e}, \forall i \in I$ for some $M > 0$ [45]

[44] Assumption (C2) ensures that there is a stable state such that the same type of agents in the same group consume the same. When Assumption (C2) is dropped, a stable state still exists, but the definition of feasible allocation needs to be changed to allow the possibility that two agents of the same type may consume differently in a group.

[45] $\vec{e} = (1, 1, ..., 1) \in \mathbb{R}^L$.

(C4) for all $x \in \mathbb{R}_+^L$, $u_i(x)$ is bounded and continuous in I

(C5) for any $M > 0$, $\{u_i : i \in I\}$ is uniformly equicontinuous in the domain $[0, M\vec{e}]$ [46]

Compared to the assumptions for the compact type space case in Section 4.1, assumptions (C3) and (C4) are modified to have some boundedness conditions. In addition, there is a new uniform equicontinuity assumption (C5). We note, when I is compact, the additional boundedness conditions in (C3) and (C4) and assumption (C5) are automatically satisfied[47]. Therefore, when the type set is compact, assumptions (C1) - (C5) are degenerated to the assumptions stated in Section 4.1.

It worth noting that Assumption (C5) plays a crucial role in the proof of the continuity of equilibrium price when the type space is not compact. This assumption rules out the collection of utility functions such as $u_n(x) = x^{1/n}, \forall n \in \mathbb{N}$. In Section 4.4.3, we give an example to show that equilibrium price cannot be expected to be continuous without this uniform equicontinuity assumption.

In addition, as explained in Footnote 19, when the type space is not compact, the set of formed groups needs to be redefined as a set $\mathcal{F} \subset \text{supp}(\tau)$ with a full τ-measure, but not necessarily equals to $\text{supp}(\tau)$. Therefore, we need to replace all "$\text{supp}(\tau)$" in the definitions and results by the term "$\mathcal{F}$", where $\mathcal{F} \subset \text{supp}(\tau)$ is a set with a full τ-measure such that, for all groups $G \in \mathcal{F}$, the total payoff of group members in group G is equal to the group surplus.

With these changes on the assumptions and the definition of formed groups, all results in this section hold.

4.3 Proof

In this subsection, we prove Theorem 6. The main part of the proof is the construction of an equilibrium price. We will first introduce our idea of construction, then go to the details of the proof.

4.3.1 Idea of The Proof

For any stable state, we construct an equilibrium price by constructing a budget set for each type of agent. At a given stable state, these budget sets will have the following three properties:

1. all agents' consumption are on the boundary of their budget sets

2. all agents' initial endowments are on the boundary of their budget sets

[46] for any $\varepsilon > 0$, there is a $\delta > 0$ such that $|u_i(x) - u_i(y)| < \varepsilon$, for all $i \in I$, $y \in \mathbb{R}_+^L$ such that $|x - y| < \delta$.

[47] We proved the compactness of the type space, together with continuity assumptions (C3) and (C4), implies (C5) in Lemma 15.

3. the interior of any agent's better than set does not intersect his/her budget set

Here, an agent's better than set is defined to be the set of consumption bundles such that this agent has a higher payoff than his/her payoff at the given stable state.

There are two difficulties in constructing these budget sets. First, since price is the same for all agents, the shapes of all budget sets must be the same. Second, since price is an odd function, the boundary of any agent's budget set must be symmetric with respect to his/her initial endowment.

To deal with the first difficulty, we work in the space of net trades $\mathbb{R}\times\mathbb{R}^L$. Therefore, at any stable state $(\tau, ((m_i^G, x_i^G)_{i\in G})_{G\in \mathrm{supp}(\tau)})$, all agents' initial endowments are translated to the point $(0,0) \in \mathbb{R}\times\mathbb{R}^L$, as consuming the initial endowments corresponds to a net trade $(0,0)$. Moreover, in any formed group $G \in \mathrm{supp}(\tau)$, a type i agent's net trade is given by the vector

$$(\Delta m_i^G, \Delta x_i^G) = (m_i^G, x_i^G) - (\omega_i^m, \omega_i^x)$$

And his/her translated better than set is defined by

$$\begin{aligned}\mathcal{P}_i &= \{(m,x)\in \mathbb{R}\times\mathbb{R}_+^L : U_i(m,x) \geq U_i(m_i^G, x_i^G)\} - (\omega_i^m, \omega_i^x)\\ &= \{(\Delta m, \Delta x) \in \mathbb{R}\times\mathbb{R}^L : U_i(\omega_i^m + \Delta m, \omega_i^x + \Delta x) \geq U_i(m_i^G, x_i^G), \Delta x \geq -\omega_i^x\}\end{aligned}$$

By Lemma 10, the same type of agents have the same payoff at a stable state. Therefore, two agents of the same type have the same translated better than set, even if they consume differently at a stable state.

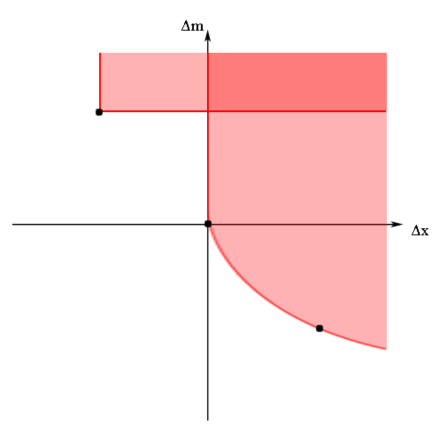

Figure 4: Net Trades and Translated Better Than Sets

In Figure 4, the black dots are net trades of the agents, and the red regions are the translated better than sets. We note there are two black dots on some translated better than set. It corresponds to the case that two agents of the same type consume, or trade, differently at a stable state.

To deal with the second difficulty, when we construct an equilibrium price, or the budget sets, we consider both the translated better than sets and their symmetric images with respect to the origin.

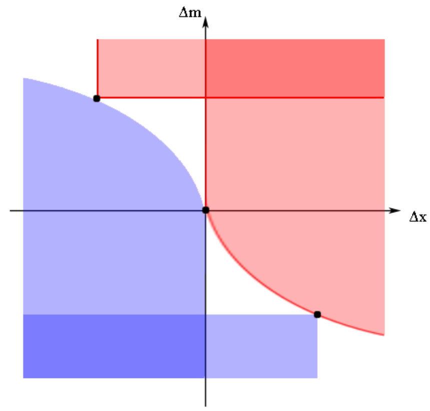

Figure 5: Symmetric Image of Translated Better Than Sets

In Figure 5, the blue regions are the symmetric images of translated better than sets.

The construction of budget sets are proceeded in the following three steps.

First, we set an initial candidate of the budget set $B_0 \subset \mathbb{R}_- \times \mathbb{R}^L$ to be the closure of the intersection of the lower half space $\mathbb{R}_- \times \mathbb{R}^L$ and the union of all symmetric images of translated better than sets.

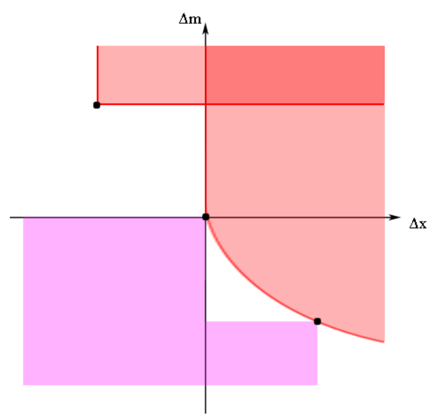

Figure 6: The First Step in The Proof

In Figure 6, the purple region corresponds to the set B_0. Note the set B_0 has the following properties:

1. B_0 is a closed set.
2. B_0 corresponds to a non-decreasing price function.[48]
3. B_0 does not intersect the interior of any translated better than set.
4. The intersection of the lower half space and the symmetric image of any translated better than set is contained in B_0.
5. The symmetric image of any net trade with monetary gains is on the boundary of B_0.

Second, we define $B_- \subset \mathbb{R}_- \times \mathbb{R}^L$ to be the largest subset of the lower half space having these five properties.

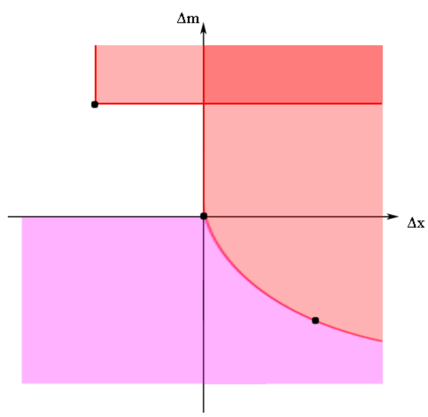

Figure 7: The Second Step in The Proof

In Figure 7, the purple region corresponds to the set B_-. We will show:

1. B_- is unique
2. all net trade with monetary losses are on the boundary of B_-
3. B_- corresponds to a non-decreasing, continuous price

Third, we define a set $B \subset \mathbb{R} \times \mathbb{R}^L$ such that it corresponds to a non-decreasing, continuous and odd price function.

[48]the relation between budget set and price is discussed in the next subsubsection.

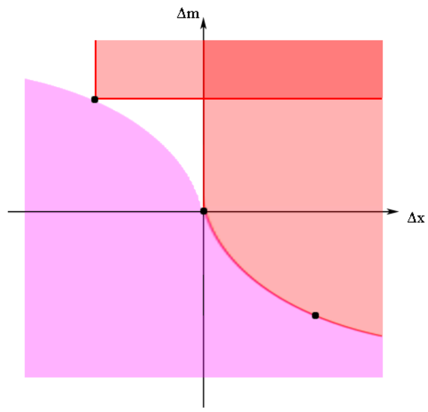

Figure 8: The Third Step in The Proof

In Figure 8, the purple region corresponds to the set B. The boundary of this set correspond to a non-decreasing, continuous and odd price function.

Last, we finish our proof by verifying that this price is an equilibrium price.

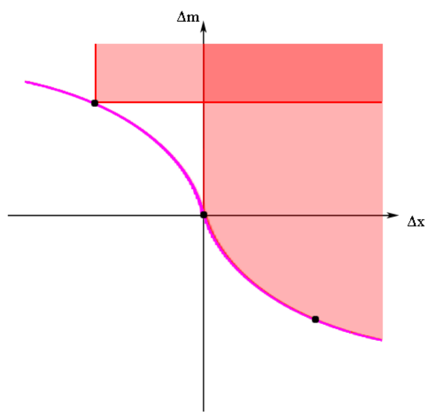

Figure 9: The Last Step in The Proof

In Figure 9, the purple curve correspond to a price. We check it is a market clearing price by checking that

1. the purple curve passes through the origin
2. the purple curve passes through all the net trades
3. all translated better than sets are above the purple curve.

4.3.2 Notations and Lemmas

In this subsubseciton, we introduce a few notations mentioned in the previous subsubsection formally. In addition, we state a few useful lemmas.

To begin with, we fix a stable state $(\tau, ((m_i^G, x_i^G)_{i\in G})_{G\in \mathrm{supp}(\tau)})$. In any formed group $G \in \mathrm{supp}(\tau)$, for a type $i \in G$, *a type i agent's net trade* is

$$(\Delta m_i^G, \Delta x_i^G) = (m_i^G, x_i^G) - (\omega_i^m, \omega_i^x)$$

The set of traded bundles $\Delta \subset \mathbb{R} \times \mathbb{R}^L$ consists of all such vectors:

$$\Delta = \{(\Delta m_i^G, \Delta x_i^G) : G \in \mathrm{supp}(\tau), i \in G\}$$

Moreover, a type i agent's *translated better than set* is defined to be

$$\begin{aligned}\mathcal{P}_i &= \{(m, x) \in \mathbb{R} \times \mathbb{R}_+^L : U_i(m, x) \geq U_i(m_i^G, x_i^G)\} - (\omega_i^m, \omega_i^x) \\ &= \{(\Delta m, \Delta x) \in \mathbb{R} \times \mathbb{R}^L : U_i(\omega_i^m + \Delta m, \omega_i^x + \Delta x) \geq U_i(m_i^G, x_i^G), \Delta x \geq -\omega_i^x\}\end{aligned}$$

Since the same type of agents have the same payoff at a stable state, type i agent's translated better than set does not depend on the group this type i agent is in. That is, if all type i agents have a payoff v_i at a stable state,

$$\mathcal{P}_i = \{(\Delta m, \Delta x) : \Delta x \geq -\omega_i^x, \Delta m \geq v_i - u_i(\omega_i^x + \Delta x)\} \tag{18}$$

The *symmetric image of a type i agent's translated better than set* is $-\mathcal{P}_i$.

All objects defined above are in set of net trade $\mathbb{R} \times \mathbb{R}^L$. We use the term *the lower half space* to denote $\mathbb{R}_- \times \mathbb{R}^L$, which is the set of net trades with monetary losses.

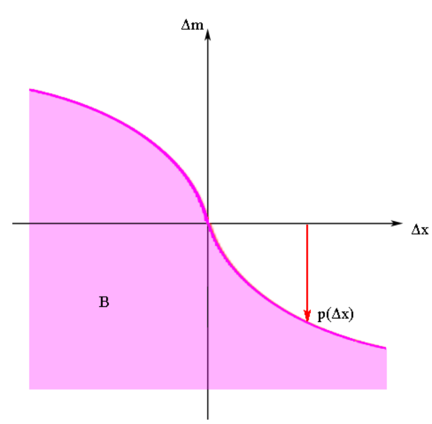

Figure 10: Translated Budget Set and Price Function

In addition, as shown in Figure 10, there is an one to one relationship between translated budget set and price: for any set $B \subset \mathbb{R} \times \mathbb{R}^L$ such that $(\Delta m, \Delta x) \in B$

implies $(\Delta m', \Delta x) \in B$ for all $\Delta m' \leq \Delta m$, we define a function $p_B : \mathbb{R}^L \to \mathbb{R} \cup \{+\infty\}$ by

$$p_B(\Delta x) = - \sup_{(\Delta m, \Delta x) \in B} \Delta m \tag{19}$$

The function p_B has two properties:

Lemma 11. *If $B \subset \mathbb{R} \times \mathbb{R}^L$ is a closed set satisfying $(\Delta m, \Delta x) \in B$ implies $(\Delta m', \Delta x) \in B$ for all $\Delta m' \leq \Delta m$, we have*

$$B = \{(\Delta m, \Delta x) \in \mathbb{R} \times \mathbb{R}^L : \Delta m \leq -p_B(\Delta x)\}$$

Proof. See Appendix D.4. ■

Lemma 12. *p_B is a non-decreasing function on $\mathbb{R}^L$ if and only if the set B has a property that if $(\Delta m, \Delta x) \in B$,*

$$(\Delta m', \Delta x') \leq (\Delta m, \Delta x) \Longrightarrow (\Delta m', \Delta x') \in B$$

Proof. See Appendix D.5. ■

By Lemma 11, when B is closed, we say the price p_B *generates* a budget set B, and equivalently, the budget set B is *generated by* a price p_B.

Lastly, we state four lemmas. Firstly, no net trade is in the interior of any translated better than set.

Lemma 13. *At any stable state $(\tau, ((m_i^G, x_i^G)_{i \in G})_{G \in supp(\tau)})$,*

$$(\Delta m_i^G, \Delta x_i^G) \notin int(\mathcal{P}_j)$$

for any formed group $G \in supp(\tau)$, type $i \in G$ and type $j \in I$.

Proof. For any given formed group G and a type $i \in G$, by the feasibility condition of the stable state, we have

$$\sum_{k \neq i, k \in G} n_k(\Delta m_k^G, \Delta x_k^G) + (n_i - 1)(\Delta m_i^G, \Delta x_i^G) = -(\Delta m_i^G, \Delta x_i^G)$$

where n_i is the number of type i agents in group G. If $(\Delta m_i^G, \Delta x_i^G) \in int(\mathcal{P}_j)$, any type j is strictly better off by consuming $\omega_j + (\Delta m_i^G, \Delta x_i^G)$. Therefore, a type j agent and all members in group G except one type i agent will form a blocking coalition. Contradiction. ■

Secondly, no translated better than set intersects the interior of the symmetric image of any translated better than set.

Lemma 14. *At any stable state* $(\tau, ((m_i^G, x_i^G)_{i\in G})_{G\in supp(\tau)})$,

$$\mathcal{P}_i \cap int(-\mathcal{P}_j) = \emptyset$$

for all $i, j \in I$.

Proof. If $(\Delta m, \Delta x) \in \mathcal{P}_i \cap int(-\mathcal{P}_j)$, any type i agent will be weakly better off by consuming $\omega_i + (\Delta m, \Delta x)$ and any type j agent will be strictly better off by consuming $\omega_j - (\Delta m, \Delta x)$. Therefore, a type i agent and a type j agent will form a blocking coalition. Contradiction. ■

The next two lemmas are for the case when the type set I is infinite. When I is a finite set, both lemmas hold automatically.

The following lemma establishes the equicontinuity property of the class of utility functions in any bounded domain. The key assumption of this lemma is the compactness of the type space. When the type space is not compact, we refer to Section 4.2.1 for further discussions.

Lemma 15. *For any* $M > 0$, *and* $\vec{e} = (1, 1, ..., 1) \in \mathbb{R}^L$, *we have for any* $\varepsilon > 0$, *there is a* $\delta > 0$ *such that, whenever* $|x - y| < \delta$ *for some* $0 \leq x, y \leq M\vec{e}$,

$$|u_i(x) - u_i(y)| < \varepsilon, \forall i \in I$$

Proof. By Assumption (C1), Assumption (C4), and the main theorem in Kruse and Deely [1969], the function $f(i, x) = u_i(x)$ is a continuous function on the compact set $I \times [0, M\vec{e}]$. By Heine-Cantor theorem, f is uniformly continuous. Therefore, for any $\varepsilon > 0$, there is a $\delta > 0$ such that whenever $|(i, x) - (j, y)| < \delta$, we have $|u_i(x) - u_j(y)| < \varepsilon$. Take $i = j$, we finish the proof. ■

The next lemma is crucial for proving the continuity of the market clearing price.

Lemma 16. *At any stable state* $(\tau, ((m_i^G, x_i^G)_{i\in G})_{G\in supp(\tau)})$, *the set of traded bundles* Δ *does not have an accumulation point* $(\Delta m, \Delta x)$ *such that*

$$\Delta m > \Delta m_i^G, \Delta x = \Delta x_i^G$$

for some $G \in supp(\tau)$ *and* $i \in G$.

Proof. See Appendix D.6 for a proof. ■

4.3.3 The Proof

Now, we construct an equilibrium price following the idea of proof we mentioned earlier.

First, at any stable state $(\tau, ((m_i^G, x_i^G)_{i\in G})_{G\in \text{supp}(\tau)})$, we define a collection of candidate translated budget sets in the lower half space:

$$\mathcal{B} = \{B \subset \mathbb{R}_- \times \mathbb{R}^L : B \text{ satisfies } (P1) - (P5)\}$$

where

(P1) B is a closed set, and $(\Delta m, \Delta x) \in B$ implies $(\Delta m', \Delta x) \in B$ whenever $\Delta m' \leq \Delta m$

(P2) the function p_B is non-decreasing

(P3) $B \cap int(\mathcal{P}_i) = \emptyset$ for all $i \in I$

(P4) $-\mathcal{P}_i \cap (\mathbb{R}_- \times \mathbb{R}^L) \subset B$ for all $i \in I$

(P5) for any $(\Delta m_i^G, \Delta x_i^G)$ such that $\Delta m_i^G \geq 0$, $-(\Delta m_i^G, \Delta x_i^G) \in \partial B$

By (P1), any set $B \in \mathcal{B}$ is generated by some price function p_B. By (P2), any set $B \in \mathcal{B}$ has a "downward sloping" boundary. By (P3), any set $B \in \mathcal{B}$ does not intersect the interior of any translated better than set. By (P4), for any set $B \in \mathcal{B}$, the part of symmetric image of any translated better than sets that is in the lower half space is in set B. By (P5), for any set $B \in \mathcal{B}$, the symmetric images of all net trades without monetary losses are on the boundary of set B.

To begin with, we prove $\mathcal{B}$ is non-empty by defining a set $B_0 \subset \mathbb{R}_- \times \mathbb{R}^L$ where

$$B_0 = cl\left[\left(\bigcup_{i\in I} -\mathcal{P}_i\right)\bigcap (\mathbb{R}_- \times \mathbb{R}^L)\right]$$

Lemma 17. *$B_0 \in \mathcal{B}$.*

Proof. See Appendix D.7 for a proof. ■

Next, we define B_- to be the largest element in $\mathcal{B}$:

$$B_- = cl\left(\bigcup_{B\in\mathcal{B}} B\right)$$

We prove B_- has the following properties in order:

1. $B_- \in \mathcal{B}$.
2. $p_{B_-}(\Delta x) > 0$ implies $p_{B_-}(-\Delta x) = 0$ for all $\Delta x \in \mathbb{R}^L$.
3. p_{B_-} is real-valued and continuous.
4. for any $(\Delta m_i^G, \Delta x_i^G)$ such that $\Delta m_i^G < 0$, $-p_{B_-}(\Delta x_i^G) = \Delta m_i^G$.

Lemma 18. *$B_- \in \mathcal{B}$.*

Proof. See Appendix D.8 for a proof. ■

Lemma 19. *$p_{B_-}(\Delta x) > 0$ implies $p_{B_-}(-\Delta x) = 0$ for all $\Delta x \in \mathbb{R}^L$.*

Proof. See Appendix D.9 for a proof. ■

Lemma 20. *p_{B_-} is real-valued and continuous.*

Proof. See Appendix D.10 for a proof. ■

Lemma 21. *For any $(\Delta m_i^G, \Delta x_i^G) \in \Delta$ such that $\Delta m_i^G < 0$, $-p_{B_-}(\Delta x_i^G) = \Delta m_i^G$*

Proof. See Appendix D.11 for a proof. ■

Last, we define an odd, continuous, non-decreasing price function and prove it is an equilibrium price at the given stable state. The function $p : \mathbb{R}^L \to \mathbb{R}$ is defined by

$$p(\Delta x) = \begin{cases} -p_{B_-}(-\Delta x), & \text{if } p_{B_-}(-\Delta x) > 0 \\ p_{B_-}(\Delta x), & \text{otherwise} \end{cases} \tag{20}$$

We finish our proof by proving that p is continuous, odd and non-decreasing and is an equilibrium price.

Lemma 22. *The function p defined in Equation 20 is continuous, odd and non-decreasing in $\mathbb{R}^L$.*

Proof. We start by proving the function p is continuous pointwisely. For any point $\Delta x \in \mathbb{R}^L$ such that $p(\Delta x) > 0$, in an open neighborhood of Δx, $p = p_{B_-}$, thus p is continuous at this point as p_{B_-} is continuous by Lemma 20. In contrast, for any point $\Delta x \in \mathbb{R}^L$ such that $p(\Delta x) < 0$, in an open neighborhood of Δx, $p = -p_{B_-}(-\cdot)$, thus p is continuous at this point as p_{B_-} is continuous by Lemma 20. Furthermore, for any point $\Delta x \in \mathbb{R}^L$ such that $p(\Delta x) = 0$, we must have $p_{B_-}(\Delta x) = p_{B_-}(-\Delta x) = 0$ by Lemma 19. Therefore, for any small $\varepsilon > 0$, we can find an open neighborhood of Δx such that the absolute values of both p_{B_-} and $-p_{B_-}(-\cdot)$ are smaller than ε. That is, p is continuous at this point.

The oddness of the function p follows from the definition. To see p is non-decreasing, we take any $\Delta x' \geq \Delta x$ in $\mathbb{R}^L$. When $p(\Delta x) \geq 0$, $0 \leq p(\Delta x) = p_{B_-}(\Delta x) \leq p_{B_-}(\Delta x')$ since, by (P2), p_{B_-} is non-decreasing. Therefore, $p(\Delta x') = p_{B_-}(\Delta x') \geq p(\Delta x)$. When $p(\Delta x) < 0$, we prove by contradiction and suppose $p(\Delta x') < p(\Delta x) < 0$. Then we have $p_{B_-}(-\Delta x') > 0$. Since $-\Delta x \geq -\Delta x'$, $p_{B_-}(-\Delta x) \geq p_{B_-}(-\Delta x')$ as p_{B_-} is non-decreasing. Therefore, $p(\Delta x) = -p_{B_-}(-\Delta x) \leq -p_{B_-}(-\Delta x') = p(\Delta x')$. Contradiction. Hence, p is a non-decreasing function. ■

Lemma 23. *If p is defined in Equation 20, $(p, \tau, ((m_i^G, x_i^G)_{i \in G})_{G \in supp(\tau)})$ is a competitive equilibrium.*

Proof. In Lemma 22, we proved p is a price. Since $(\tau, ((m_i^G, x_i^G)_{i\in G})_{G\in \text{supp}(\tau)})$ is a stable state, by definition, τ is an assignment and $((m_i^G, x_i^G)_{i\in G})_{G\in \text{supp}(\tau)}$ is a feasible allocation under the assignment τ. It remains to show, for any formed group $G \in \text{supp}(\tau)$ and any type i agent in G, we have

- $(m_i^G, x_i^G) \in B_i(p)$
- $U_i(m_i^G, x_i^G) = \max_{(m_i,x_i)\in B_i(p)} U_i(m_i, x_i)$

Firstly, to prove the consumption bundle (m_i^G, x_i^G) is affordable to a type i agent under price p, we need to show $m_i^G \le \omega_i^m + p(\omega_i^x - x_i^G)$, which is equivalent to $\Delta m_i^G \le -p(\Delta x_i^G)$. When $\Delta m_i^G \ge 0$, as B_- satisfies (P5), $-\Delta m_i^G = p_{B_-}(-\Delta x_i^G) = p(\Delta x_i^G)$. When $\Delta m_i^G < 0$, by Lemma 21, we have $\Delta m_i^G = -p(\Delta x_i^G)$. That is, $(m_i^G, x_i^G) \in B_i(p)$.

Secondly, to see the consumption bundle (m_i^G, x_i^G) is an optimal consumption for a type i agent, we prove the interior of his better than set $\mathcal{P}_i + \omega_i$ has no intersection with his budget set $B_i(p)$. Suppose the contrary that $(m, x) \in B_i(p) \cap (int(\mathcal{P}_i) + \omega_i)$, we have some net trade bundle $(\Delta m, \Delta x) \in int(\mathcal{P}_i)$ and $\Delta m \le -p(\Delta x)$. If $p(\Delta x) \ge 0$, $\Delta m \le -p(\Delta x) = -p_{B_-}(\Delta x)$. Therefore, $(\Delta m, \Delta x) \in B_-$, contradicts (P3). If $p(\Delta x) < 0$, $\Delta m \le -p(\Delta x) = p_{B_-}(-\Delta x)$. That is, $-\Delta m \ge p_{B_-}(-\Delta x)$. For small enough $\varepsilon > 0$, $-\Delta m + \varepsilon > p_{B_-}(-\Delta x)$. That is, $(-\Delta m + \varepsilon, -\Delta x) \notin B_-$. But since $(\Delta m, \Delta x) \in int(\mathcal{P}_i)$, for small enough $\varepsilon > 0$, $(\Delta m - \varepsilon, \Delta x) \in \mathcal{P}_i$. Therefore, $(-\Delta m + \varepsilon, -\Delta x) \in -\mathcal{P}_i$. If $\Delta m > 0$, $(-\Delta m + \varepsilon, -\Delta x) \in -\mathcal{P}_i \cap (\mathbb{R}_- \times \mathbb{R}^L)$, contradicts (P4). If $\Delta m \le 0$, $(0, -\Delta x) \in -\mathcal{P}_i \cap (\mathbb{R}_- \times \mathbb{R}^L)$. But since $p_{B_-}(-\Delta x) = -p(\Delta x) > 0$, $(0, -\Delta x) \notin B_-$, which contradicts (P4). ∎

4.4 Examples

In this subsection, we give two examples. In the first example, we explain why linear prices are incompatible with the models with a segmented market structure. In the second example, we illustrate that a nonlinear equilibrium price exists in a segmented market in an Edgeworth box. In the third example, we illustrate the continuity of equilibrium price may fail if the type space is not compact and utility functions are not uniformly equicontinuous.

4.4.1 Non-existence of a Linear Market Clearing Price

First, in a segmented market, feasible allocations are usually not Pareto efficient. On the other hand, by the first welfare theorem, any allocation supported by a linear market clearing price must be Pareto efficient. Therefore, there is usually no linear market clearing price in a segmented market.

More concretely, we consider an economy with I types of agents and $L = I$ types of consumption goods, where $I \ge 2$ is some natural number. Each type of agent has a mass $1/I$. For any $i \in I$, type i agents' utility function and initial endowment are given by

$$u_i(m, x_1, ..., x_I) = m + \min(x_1, ..., x_I)$$

$$\omega_i = (0, e_i) \in \mathbb{R} \times \mathbb{R}_+^I$$

where $e_i = (0, ..., 0, 1, 0, ..., 0) \in \mathbb{R}_+^I$ is a unit vector having an entry 1 at the i-th entry. In this economy, agents can trade in groups of size up to N, where $2 \leq N \leq I - 1$.

Since no trading group can contain more than I agents, no agent can consume all varieties of consumption goods after any trade. Therefore, any assignment and any feasible allocation under this assignment will form a stable state of the economy. However, no feasible allocation is Pareto efficient, as every agent will be better off by consuming $(0, 1/I, ..., 1/I)$. Consequently, by the first welfare theorem, there is no linear market clearing price in this segmented market.

To see that any stable state can be supported by some nonlinear market clearing price, we give the solution for the case when $I = 3$ and $N = 2$ here, and postpone the discussion on general case to Appendix D.12. When $I = 3$ and $N = 2$, an market clearing price is given by a function $p : \mathbb{R}^3 \to \mathbb{R}$, where $p(\Delta x_1, \Delta x_2, \Delta x_3)$ is the second smallest element in the set $\{\Delta x_1, \Delta x_2, \Delta x_3\}$.

Next, we first verify the price defined above is a market clearing price for the no-trade allocation, then verify the price is a market clearing price for any feasible allocation. The continuity and increasingness of p are straightforward. To see p is an odd function, we note if Δx_k is the the second smallest element in the set $\{\Delta x_1, \Delta x_2, \Delta x_3\}$, then $-\Delta x_k$ is the second smallest element in the set $\{-\Delta x_1, -\Delta x_2, -\Delta x_3\}$. Lastly, to see the individual optimality, by the symmetry among agents, we only prove the individual optimality of some type 1 agent. There are three cases:

1. When he tries to buy everything, i.e. his net trade $(\Delta x_1, \Delta x_2, \Delta x_3)$ is non-negative, his payoff increment is the smallest entry in the set $\{\Delta x_1, \Delta x_2, \Delta x_3\}$. But under the price system specified above, he needs to pay the second smallest entry in the set $\{\Delta x_1, \Delta x_2, \Delta x_3\}$ to trade the bundle $(\Delta x_1, \Delta x_2, \Delta x_3)$. Therefore, he never has an strict incentive to trade in this case.

2. When he tries to sell his endowment and buy some other goods, i.e. his net trade $(\Delta x_1, \Delta x_2, \Delta x_3)$ has the property that $\Delta x_1 < 0$, $\Delta x_2, \Delta x_3 \geq 0$, his payoff increment is at most the smaller entry in the set $\{\Delta x_2, \Delta x_3\}$. On the other hand, he needs to pay the smaller entry in the set $\{\Delta x_2, \Delta x_3\}$ to trade the bundle $(\Delta x_1, \Delta x_2, \Delta x_3)$. Therefore, he never has an strict incentive to trade in this case.

3. For all other possibilities, i.e. he tries to sell more than one types of consumption goods, he does not have sufficient endowment to support this trade.

Therefore, p is a market clearing price. Moreover, at any stable state, any agent's net trade bundle $\{\Delta x_1, \Delta x_2, \Delta x_3\}$ must have a zero entry. Therefore, the cost to trade this bundle is zero. Therefore, any stable state can be supported by the price p we defined above.

4.4.2 Nonlinear Market Clearing Price

Next, we use an example, with two types of agents and one consumption commodity, to illustrate the existence of a nonlinear market clearing price in an Edgeworth box.

The example is as follows. There are two types of consumers in the economy: Travelers (Buyers) and Hotels (Sellers). The only consumption good in this economy is hotel time. In this economy, no trading group of size larger than 2 is allowed. For instance, a hotel is not allowed to accommodate more than one traveler in one business day. (Perhaps because it has only one room.) Each hotel has a unit mass of hotel time, and the hotel can sell the whole unit or a part of its hotel time to some traveler. Each traveler has 0.5 unit of money initially. Moreover, there are more travelers than hotels in the market. Therefore, some travelers will not have a hotel to trade with. In other words, they must sleep on the street.

Formally, the mass of hotels is given by $\mu(H) > 0$ and the mass of the travelers is given by $\mu(T) > 0$. We assume $\mu(T) > \mu(H)$. The utility function and initial endowment of hotels are given by:

$$u_H(m, x) = m, \omega_H = (0, 1)$$

In contrast, the utility function and initial endowment of travelers are given by:

$$u_T(m, x) = m + \sqrt{x}, \omega_T = (0.5, 0)$$

Therefore, hotels only care about their monetary revenue and travelers wish to pay an infinite amount of money to buy a small amount of hotel time initially as the marginal utility of hotel time is infinite at zero.

In this economy, $N' = N = 2$. That is, agents can only trade in pairs. Therefore, there are three types of groups:

$$\mathcal{G} = \{[H, H], [T, T], [H, T]\}$$

In this case, an assignment τ is a nonnegative-valued function on $\mathcal{G}$ such that

$$2\tau([H, H]) + \tau([H, T]) = \mu(H)$$

$$2\tau([T, T]) + \tau([H, T]) = \mu(T)$$

The surplus function can be computed easily as follows:

$$s([T,T]) = 1, s([H,T]) = 1.5, s([H,H]) = 0$$

To maximize the social welfare, all hotels must trade with some travelers. Therefore, the unique stable assignment is given by

$$\tau([H,T]) = \mu(H), \tau([T,T]) = \frac{\mu(T) - \mu(H)}{2}, \tau([H,H]) = 0$$

In particular, $\tau([T,T]) > 0$. By the equal treatment property of the stable state (Lemma **??**), any feasible allocation at a stable state yields a payoff $v(T) = 0.5$ for travelers and a payoff $v(H) = 1$ for hotels. Intuitively, it means that no traveler would benefit from the trade - if there is a traveler who benefits from the trade with some hotel, any traveler who has no hotel to stay at has an incentive to pay more than this traveler to trade with this hotel.[49]

Knowing the equilibrium payoffs, we can compute consumption. In particular, there are three types of agents in the segmented market under assignment τ: hotels, travelers who trade with a hotel, and travelers who do not trade with a hotel. Their consumption are given by

$$\begin{aligned} &(m_H^G, x_H^G) = (1, 0), && G = [H,T] \\ &(m_T^G, x_T^G) = (-0.5, 1), && G = [H,T] \\ &(m_T^G, x_T^G) = (0.5, 0), && G = [T,T] \end{aligned}$$

That is, all hotels will sell all their available hotel time to obtain 1 unit of money. For the travelers who trade with some hotel, they pay 1 unit of money to obtain 1 unit of hotel time. For the travelers who do not trade with any hotel, they consume their initial endowment.

Next, we find an equilibrium price for this allocation. The budget sets of the hotels and the travelers are,

$$B_H(p) = \{(m,x) \in \mathbb{R} \times \mathbb{R}_+ : -m + p(1-x) \geq 0\}$$

$$B_T(p) = \{(m,x) \in \mathbb{R} \times \mathbb{R}_+ : 0.5 - m + p(-x) \geq 0\}$$

For hotels, the bundle $(1,0)$ is affordable and optimal. Therefore,

$$\begin{cases} (1,0) \in B_H(p) : -1 + p(1-0) \geq 0 \\ 1 \geq m, \forall (m,x) \in \mathbb{R}^2 : -m + p(1-x) \geq 0, x \geq 0 \end{cases}$$

[49]That is, we observed that the trade surplus might go to one of the trading parties in a segmented market. In such cases, it is clear no linear equilibrium price exists by the first order condition of the individual optimization problem.

By the fact p is increasing, these two inequalities implies $p(1) = 1$.

For travelers, the bundles $(-0.5, 1)$ and $(0, 5, 0)$ are affordable and optimal:

$$\begin{cases} (-0.5, 1) \in B_T(p) : 1 + p(-1) \geq 0 \\ (0, 5, 0) \in B_T(p) : 0 + p(0) \geq 0 \\ 0.5 \geq m + \sqrt{x}, \forall (m, x) \in \mathbb{R}^2 : 0.5 - m + p(-x) \geq 0, x \geq 0 \end{cases}$$

By the fact p is odd and increasing, these three inequalities implies $p(x) \geq \sqrt{x}, \forall x \geq 0$ and $p(1) = 1$.

In sum, $(p, \tau, ((m_i^G, x_i^G)_{i \in G})_{G \in \text{supp}(\tau)})$ is a competitive equilibrium if $p : \mathbb{R} \to \mathbb{R}$ is an odd, increasing, continuous function satisfying

$$p(x) \geq \sqrt{x}, \forall x \geq 0 \text{ and } p(1) = 1$$

There are infinitely many equilibrium prices. A class of equilibrium price is given by $p(x) = sgn(x)x^\alpha, \forall \alpha \geq 2$. Among all equilibrium prices, the most "natural" one is $p(x) = sgn(x)\sqrt{x}$.

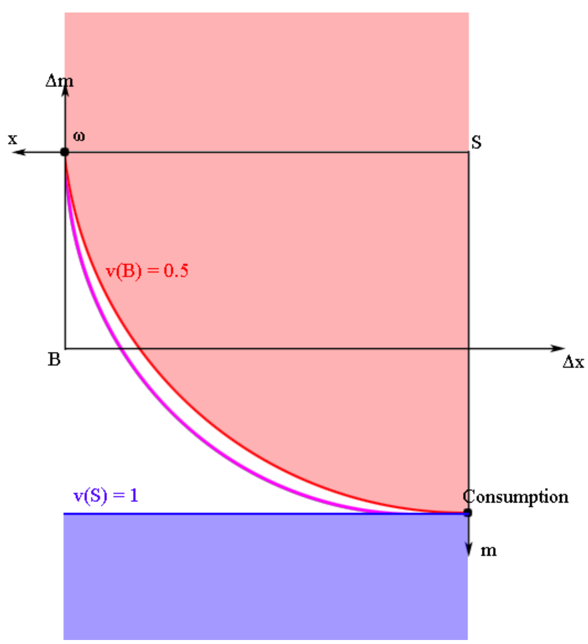

Figure 11: Edgeworth Box when $\mu(T) > \mu(H)$

Next, we illustrate our work in an Edgeworth box in Figure 11.

Given the feasible allocation at the stable state, the payoffs of travelers and hotels are $v(T) = 0.5$ and $v(H) = 1$ respectively. Therefore, we could draw the indifference curves and better than sets of hotels and travelers. The indifference curve and better than sets of travelers are drawn in red and the indifference curves and better than sets of hotels are drawn in blue.

A classical exercise for finding an equilibrium price is to draw a line which separates the blue region from the red region while the line passes both the initial endowment and the consumption. It is easy to see no such line exists. However, a nonlinear curve in purple will do the job. Any such purple curve will correspond to a nonlinear equilibrium price.

4.4.3 Infinite Types, Pairwise Tradings

In this subsubsection, we give an example to show that, when the type set is not compact, the equilibrium "price" might be discontinuous when we drop the uniform equicontinuity assumption Assumption (C5).[50]

The example is a modification of the previous example. Let the type space be $I = \{2\} \cup (2, +\infty)$. Type 2 corresponds to a set of homogeneous hotels H, and type α, for any $\alpha > 2$, corresponds to a set of type α travelers.

Utility functions and initial endowments of hotels and type α travelers are given by

$$u_H(m, x) = m, \omega_H = (0, 1)$$

$$u_\alpha(m, x) = m + \sqrt[\alpha]{x}, \omega_\alpha = (0.5, 0)$$

We take $N = 2$ as before. That is, agents must trade in pairs. Moreover, in the market, we assume that there are 1 unit mass of hotels and the density of travelers is given by the function $4/\alpha^2$. That is, the total mass of travelers is 2. Consequently, same as before, some travelers cannot trade with any hotel.

Now, we are able to describe the consumption at the stable state. We note, the gain from trade is always 1 between a hotel and a traveler, and is always 0 otherwise. Therefore, an assignment assigning a mass $2/\alpha^2$ to the group consisting of a hotel and a type α traveler and assigning the remaining travelers pairwisely is stable. Under such assignment, all hotels consumes $(1, 0)$, all travelers traded with a hotel consume $(-0, 5, 1)$ and all other travelers consume $(0.5, 0)$.

By studying the Edgeworth box, all equilibrium "price" must satisfy the following relation:

$$p(0) = 0, p(x) = 1, \forall x \in (0, 1]$$

Therefore, there is no continuous equilibrium "price" in this economy. To see there

[50] In this subsubsection, "price" is a function on the set of net trades $\mathbb{R}_+^L$ that is increasing, odd, but may not be continuous. However, the equilibrium "price" p will still have some semi-continuity property based on our proof: $p_- = \min(p, 0)$ will always be lower semi-continuous.

exists a discontinuous equilibrium "price", we give one example:

$$p(x) = sgn(x) \cdot \begin{cases} 0, & x = 0 \\ 1, & 0 < |x| \leq 1 \\ \sqrt{|x|}, & |x| \geq 1 \end{cases}$$

4.5 Related Literature and Future Work

Exchange economies with a segmented market structure have been studied by Schmeidler [1972a] and Hammond et al. [1989]. When group sizes are bounded by epsilon percentiles, Schmeidler [1972a], together with the classical existence and core equivalence result in Aumann [1966] and Aumann [1964], proved that competitive allocations cannot be blocked by any small group of epsilon sizes. When group sizes are finite but unbounded, Hammond et al. [1989] proved the existence of a competitive equilibrium in an approximate manner[51]. In both works, competitive allocations cannot be achieved by exchanges in small groups. Moreover, when group sizes are finite and bounded, to my knowledge, there is no relevant material.

In this section, we fill in the gaps in the literature by studying economies with groups of bounded finite sizes. In particular, we propose a model connecting general equilibrium theory and matching theory. By introducing nonlinear prices to general equilibrium models, we prove the existence of a competitive equilibrium with a potentially nonlinear equilibrium price.

Actually, nonlinear prices have been introduced to general equilibrium models before. For example, the groundbreaking work Aliprantis et al. [2001] and its continuation work Basile and Graziano [2013] studied nonlinear market price generated from personalized linear prices in complete markets. However, as the example in Section 4.4.2 shows, personalized linear price may fail to exist in a segmented market. Therefore, our work provides a different source of price nonlinearity to general equilibrium models.

Lastly, we propose three directions for further explorations.

Firstly, it might be possible to refine the definition of price in this model. Currently, a price is a function on the net trade bundles. It might be possible to have a nonlinear price function for the net trade of each consumption good. A refined definition of price might lead to the determinacy of the equilibrium.

Secondly, suggested by the work of Hammond et al. [1989], it is reasonable to conjecture that, as the maximum group size becomes very large but remains to be finite, there is a competitive equilibrium with an almost linear market price.

Thirdly, the current model rules out the possibility of retrade. One way to incorporate the possibility of retrade is to study a multi-period extension of the current model.

[51] The competitive allocation is approximately feasible.

My conjecture is when the number of periods goes is large enough, there is an almost linear market clearing price.

5 Conclusions

In this paper, we study games and exchange economies with transferable utility and a continuum of agents, who may be of different types and can interact only in small groups.

Firstly, we study a game with a continuum of agents who form small groups in order to share group surpluses. Group sizes are exogenously bounded by natural numbers or percentiles. We prove that there exists a stable assignment, where no group of agents can jointly do better. Conceptually, our work provides the only existence result to this problem on our level of generality, as well as a uniform way to understand diverse solution concepts, such as stable matching, fractional core, f-core, and epsilon-sized core. Computationally, when there are finitely many types of players and group sizes are finite, we reduce the number of unknowns in the problem of finding stable assignments from about $|I|^N$ to about $|I|$, where $|I|$ is the number of player types, N is the maximum group size and $|I|$ is much larger than N. We achieve this reduction by reformulating the welfare maximization problem as a symmetric transport problem.

Secondly, we study an exchange economy with finitely many goods and a continuum of agents who can exchange commodities only within small groups of some bounded finite sizes. By introducing the idea of a nonlinear price in which expenditures on traded quantities are defined by the same nonlinear function in every group, we prove the existence of a competitive equilibrium with a potentially nonlinear market price, provided that agents have quasi-linear utility functions. It appears that only nonlinear market prices are compatible with models in which all trade surplus might go to one of the trading parties. Therefore, our result suggests that market segmentation might lead to price nonlinearity. This work fills in gaps in the work of Hammond et al. [1989] on economies with small groups of arbitrary finite sizes.

References

C. D. Aliprantis and K. C. Border. *Infinite Dimensional Analysis: A Hitchhiker's Guide*. Springer Science & Business Media, 2007.

C. D. Aliprantis, R. Tourky, and N. C. Yannelis. A theory of value with non-linear prices: equilibrium analysis beyond vector lattices. *Journal of Economic Theory*, 100(1):22–72, 2001.

R. J. Aumann. Markets with a continuum of traders. *Econometrica: Journal of the Econometric Society*, pages 39–50, 1964.

R. J. Aumann. Existence of competitive equilibria in markets with a continuum of traders. *Econometrica: Journal of the Econometric Society*, pages 1–17, 1966.

A. Basile and M. G. Graziano. Core equivalences for equilibria supported by non-linear prices. *Positivity*, 17(3):621–653, 2013.

D. P. Bertsekas. Nonlinear programming. athena scientific belmont. *Massachusets, USA*, 1999.

V. I. Bogachev. *Measure theory (Vol. 2)*. Springer Science & Business Media, 2007.

O. N. Bondareva. Some applications of linear programming methods to the theory of cooperative games. *Problemy kibernetiki*, 10:119–139, 1963.

G. Carlier and I. Ekeland. Matching for teams. *Economic theory*, 42(2):397–418, 2010.

G. Carlier and B. Nazaret. Optimal transportation for the determinant. *ESAIM: Control, Optimisation and Calculus of Variations*, 14(4):678–698, 2008.

Y.-K. Che, J. Kim, and F. Kojima. Stable matching in large economies. *Econometrica*, 87(1):65–110, 2019.

P.-A. Chiappori, R. J. McCann, and L. P. Nesheim. Hedonic price equilibria, stable matching, and optimal transport: equivalence, topology, and uniqueness. *Economic Theory*, 42(2):317–354, 2010.

P.-A. Chiappori, A. Galichon, and B. Salanié. The roommate problem is more stable than you think. 2014.

P.-A. Chiappori, R. McCann, and B. Pass. Multidimensional matching. *arXiv preprint arXiv:1604.05771*, 2016.

J. Diestel and J. Uhl Jr. Vector measures. *Bull. Amer. Math. Soc*, 84:681–685, 1978.

R. M. Dudley. *Real analysis and probability*, volume 74. Cambridge University Press, 2002.

G. Friesecke and D. Vogler. Breaking the curse of dimension in multi-marginal kantorovich optimal transport on finite state spaces. *SIAM Journal on Mathematical Analysis*, 50(4):3996–4019, 2018.

W. Gangbo and A. Świech. Optimal maps for the multidimensional monge-kantorovich problem. *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences*, 51(1):23–45, 1998.

N. E. Gretsky, J. M. Ostroy, and W. R. Zame. The nonatomic assignment model. *Economic Theory*, 2(1):103–127, 1992.

P. J. Hammond, M. Kaneko, and M. H. Wooders. Continuum economies with finite coalitions: Core, equilibria, and widespread externalities. *Journal of Economic Theory*, 49(1):113–134, 1989.

M. Kaneko and M. H. Wooders. Cores of partitioning games. *Mathematical Social Sciences*, 3(4):313–327, 1982.

M. Kaneko and M. H. Wooders. The core of a game with a continuum of players and finite coalitions: The model and some results. *Mathematical social sciences*, 12(2): 105–137, 1986.

M. Kaneko and M. H. Wooders. The nonemptiness of thef-core of a game without side payments. *International Journal of Game Theory*, 25(2):245–258, 1996.

Y. Kannai. Countably additive measures in cores of games. *Journal of Mathematical Analysis and Applications*, 27(2):227–240, 1969.

Y. Kannai. The core and balancedness. *Handbook of game theory with economic applications*, 1:355–395, 1992.

H. G. Kellerer. Duality theorems for marginal problems. *Zeitschrift fr Wahrscheinlichkeitstheorie und verwandte Gebiete*, 67(4):399–432, 1984.

Y. Khoo and L. Ying. Convex relaxation approaches for strictly correlated density functional theory. *SIAM Journal on Scientific Computing*, 41(4):B773–B795, 2019.

T. C. Koopmans and M. Beckmann. Assignment problems and the location of economic activities. *Econometrica: journal of the Econometric Society*, pages 53–76, 1957.

R. Kruse and J. Deely. Joint continuity of monotonic functions. *The American Mathematical Monthly*, 76(1):74–76, 1969.

N. Linial and Z. Luria. On the vertices of the d-dimensional birkhoff polytope. *Discrete & Computational Geometry*, 51(1):161–170, 2014.

G. Nöldeke and L. Samuelson. The implementation duality. *Econometrica*, 86(4):1283–1324, 2018.

B. Pass. *Structural results on optimal transportation plans.* Citeseer, 2011.

B. Pass. Multi-marginal optimal transport: theory and applications. *ESAIM: Mathematical Modelling and Numerical Analysis*, 49(6):1771–1790, 2015a.

B. Pass. Multi-marginal optimal transport: theory and applications. *ESAIM: Mathematical Modelling and Numerical Analysis*, 49(6):1771–1790, 2015b.

A. E. Roth, U. G. Rothblum, and J. H. Vande Vate. Stable matchings, optimal assignments, and linear programming. *Mathematics of operations research*, 18(4):803–828, 1993.

D. Schmeidler. A remark on the core of an atomless economy. *Econometrica (pre-1986)*, 40(3):579, 1972a.

D. Schmeidler. Cores of exact games, i. *Journal of Mathematical Analysis and applications*, 40(1):214–225, 1972b.

L. S. Shapley. On balanced sets and cores. *Naval Research Logistics (NRL)*, 14(4): 453–460, 1967.

L. S. Shapley and M. Shubik. The assignment game i: The core. *International Journal of game theory*, 1(1):111–130, 1971.

C. Villani. *Optimal transport: old and new.* Vol. 338. Springer Science & Business Media, 2008.

M. Wooders. The theory of clubs and competitive coalitions. *Annu. Rev. Econ.*, 4(1): 595–626, 2012.

A Notations

In this section, we clarify notations we used in this paper. In the following definitions, I is always assumed to be a Polish (complete separable metric) space.

- $\Theta(x^k)$: the growth rate of a function on $\mathbb{N}$ is exactly x^k (See Remark 1)
- $\mathcal{M}_+(I)$: the space of non-negative Borel measures on I
- $\mathcal{P}(I)$: the space of probability measures on I
- S_n: the set of bijective maps from $\{1, 2, ..., n\}$ to itself
- $USC(I)$: the set of upper semi-continuous continuous functions on I
- $LSC(I)$: the set of lower semi-continuous functions on I

- $C(I)$: the set of continuous functions on I
- $C_b(I)$: the set of bounded continuous functions on I
- $Lip_1(I)$: the set of 1-Lipchitz continuous functions on I
- $L^1(I, \mu)$: the set of integrable functions on (I, μ)
- $\|f\|_\infty$: the essential suppremum of the function f
- $\|\nu\|$: the total measure of a nonnegative measure ν on I
- $\mu_1 \leq \mu_2$: μ_1 is less than equal to μ_2 on all measurable sets
- $\text{supp}(\mu)$: the support of measure μ, see (Remark 2)
- $T_\# \mu$: the push-forward measure of μ under the map T (See Remark 3)
- Symmetric measure: a measure γ on I^n is symmetric if

$$\gamma(A_1, ..., A_n) = \gamma(A_{\sigma(1)}, ..., A_{\sigma(n)}), \forall A_1, ..., A_n \subset I, \sigma \in S_n$$

Remark 1: For a function $f : \mathbb{N} \to \mathbb{R}$, $f(x) = \Theta(x^k)$ if $cx^k \leq f(x) \leq Cx^k$ for some constants $c, C > 0$. Since big Θ notation characterizes functions according to their growth rates, different functions with the same growth rate may be represented using the same big Θ notation.
Remark 2: For a measure μ on I, $\text{supp}(\mu)$ is a closed set satisfying: 1. $\mu((\text{supp}\mu)^c) = 0$. 2. for every open G intersecting $\text{supp}(\mu)$, $\mu(G \cap \text{supp}(\mu)) > 0$. When I is a Polish space, μ has a unique support.[52]
Remark 3: For Polish spaces I, J, a measure μ on I and a measurable function T from I to J, $T_\# \mu$ is a measure on J such that for any measurable $A \subset J$, $T_\# \mu(A) = \mu(T^{-1}(A))$.[53]

B Omitted Proofs in Section 2

B.1 Metrizability of The Quotient space $I^n / \sim_n$

In this subsection, we show the quotient space $I^n / \sim_n$, defined in Section 2.1.2, is metrizable. In this proof, we omit the subscript of the equivalence relation and write it as $\sim$.

To start with, it is well known that the quotient space of a metric space (I, d_0) is endowed with a pseudo-metric defined by

$$d([x], [y]) = \inf \sum_{k=1}^{n} d_0(p_k, q_k)$$

[52]Definitions and results are taken from Aliprantis and Border [2007].
[53]This definition is taken from Villani [2008].

where the choice set is given by the set of all finite sequence $(p_k, q_k)_{k=1}^n$ such that $p_k, q_k \in I$, $p_1 = x$, $q_n = y$ and $q_k \sim p_{k+1}$ for all $k \in \mathbb{N}$. Now, we prove the pseudo-metric d defined above is a metric on $I^n / \sim_n$.

Lemma 24.

$$d([x],[y]) = \min_{y' \sim y} d_0(x, y')$$

This lemma will imply that d is a metric on the quotient space: for any $y \in I^n$, there are only finite many $y' \in I^n$ such that $y \sim y'$. Therefore, $d([x],[y]) = 0$ implies $d_0(x, y') = 0$ for some $y' \sim y$. Hence, $x \sim y$.

Proof. Firstly, by definition, we have $d([x],[y]) = d([x],[y']) \le d_0(x, y')$ for all $y' \sim y$. Therefore, $d([x],[y]) \le \min_{y' \sim y} d_0(x, y')$. Now we prove the converse $d([x],[y]) \ge \min_{y' \sim y} d_0(x, y')$. To prove the converse, we use the distance preserving property of the permutation: the distance of two points in I^n are not changed if their indices are permuted by the same permutation. Consequently, for any sequence $(p_k, q_k)_{k=1}^n$ in the choice set, there is a sequence of $(r_k)_{k=2}^n$ in I^n, defined inductively, such that

$$r_1 = q_1$$

$$d_0(p_k, q_k) = d_0(r_{k-1}, r_k), \forall k \ge 2$$

$$r_k \sim q_k$$

Therefore,

$$\sum_{k=1}^n d_0(p_k, q_k) = d_0(x, r_1) + \sum_{k=2}^n d_0(r_{k-1}, r_k) \ge d(x, r_k)$$

Since $r_n \sim q_n = y$,

$$\sum_{k=1}^n d_0(p_k, q_k) \ge \min_{y' \sim y} d_0(x, y')$$

As the choice of $(p_k, q_k)_{k=1}^n$ in the choice set is arbitrary, the lemma is proved. ∎

B.2 Measurability of $\mathcal{G}_n(A, k)$

In this subsection, we show the $\mathcal{G}_n(A, k)$, defined in Section 2.1.4, is measurable in $I^n / \sim_n$.

Firstly, we prove for any Borel measurable $A_1, ..., A_n \subset I$,

$$Z = \{[i_1, ..., i_n] : i_k \in A_k, \forall 1 \le k \in n\}$$

is Borel measurable in $I^n / \sim_n$. By definition of Borel measurability, we only need to prove the cases that $A_1, ..., A_n$ are open: fix $i_1, ..., i_n \in A_1 \times A_2 \times ... \times A_n$, for any

$\delta > 0$ and $[j_1, ..., j_n] \in I^n/\sim_n$, if $d([i_1, ..., i_n], [j_1, ..., j_n]) < \delta$, by Lemma 24, there is a permutation $\sigma \in S_n$ such that

$$d_0((i_1, ..., i_n), (j_{\sigma(1)}, ..., j_{\sigma(n)})) < \delta$$

Therefore, $d_I(i_m, j_{\sigma(m)}) < \delta$ for all $1 \le m \le n$. Here, d_I is the metric in I. On the other hand, all A_m are open. By taking δ small enough, we have $j_{\sigma(m)} \in A_m$. Therefore, $[j_1, ..., j_n] \in Z$. Consequently, Z is open, thus Borel measurable.

Next, for any $1 \le k \le n$ and measurable $A \subset I$, we take $A_1 = ... = A_k = A$ and $A_{k+1} = ... = A_n = A^c$, we have

$$\mathcal{G}_n(A, k) = \{[i_1, ..., i_n] : i_k \in A_k, \forall 1 \le k \in n\}$$

Therefore, $\mathcal{G}_n(A, k)$ is Borel measurable.

B.3 Measurability of The Function c

In this subsection, we show the function $c : I^n \to \mathbb{R}$ defined in Section 2.3.2 is Borel measurable in I^n. Recall,

$$c(i_1, ..., i_n) = \frac{1}{(n-1)!} \prod_{i \in I} n_i!$$

where $n_i = |\{k : i_k = i\}|$.

Since there are finitely many sequence $(n_i)_{i \in I}$ such that, for all $i \in I$, $n_i \in \mathbb{N}$ and $\sum_{i \in I} n_i = n$, the range of the function c is a finite set in $\mathbb{R}$. Consequently, to prove c is Borel measurable, it is sufficient to show for any sequence of positive integers $m_1 \ge m_2 \ge ... \ge m_k \ge 1$ such that $m_1 + ... + m_k = n$, the set

$$S_0(m_1, ...m_k) = \{(\underbrace{i_1, ..., i_1}_{m_1 \text{ many}}, \underbrace{i_2, ..., i_2}_{m_2 \text{ many}}, ..., \underbrace{i_k, ..., i_k}_{m_k \text{ many}}) \in I^n : i_1, ..., i_k \text{ are disjoint}\}$$

is Borel measurable.

To see the sufficiency, we first define a set $S(m_1, ...m_k)$ to be a subset of I^n consisting of all permuted elements in the set $S_0(m_1, ...m_k)$. Clearly, the cardinality of S depends on the sequence $(m_1, ...m_k)$. If $S_0(m_1, ...m_k)$ is measurable, we have $S(m_1, ...m_k)$, as a finite union of measurable sets, is measurable. Moreover, there are at most finitely many decreasing sequences of positive integers $(m'_1, ..., m'_l)$ such that $m'_1!...m'_l! = m_1!...m_k!$ and $m'_1 + ... + m'_l = n$. Therefore, the set $c^{-1}(\frac{m_1!...m_k!}{(n-1)!}) \subset I^n$ is at most a finite union of the set $S(m'_1, ..., m'_l)$ over all positive decreasing sequences $(m'_1, ..., m'_l)$ such that $m'_1!...m'_l! = m_1!...m_k!$, thus is measurable.

Lastly, we prove, for any decreasing sequence $m_1 \ge m_2 \ge ... \ge m_k \ge 1$ such that $m_1 + ... + m_k = n$, $S_0(m_1, ..., m_k)$ is measurable. Firstly, the set

$$\{(\underbrace{i_1, ..., i_1}_{m_1 \text{ many}}, \underbrace{i_2, ..., i_2}_{m_2 \text{ many}}, ..., \underbrace{i_k, ..., i_k}_{m_k \text{ many}}) \in I^n : i_1, ..., i_k \in I\}$$

is Borel measurable as it is closed. In addition, $S_0(m_1, ..., m_k)$ is obtained by subtracting a finite union of degenerate cases from this set, all of which are closed. Therefore, $S_0(m_1, ..., m_k)$ is Borel measurable.

B.4 Proof of Lemma 1

In this subsection, we prove Lemma 1.

Firstly, it is easy to prove that the set K_n is closed in $I^{N!}/\sim_{N!}$ as it is a finite union of closed set.

Moreover, for any permitted group size n, $\hat{s}_n$ is upper semi-continuous. We prove by contradiction. Suppose there is a sequence such that $\hat{G}_k \to \hat{G}$ in $I^{N!}/\sim_{N!}$ such that $\limsup_{\hat{G}_k \to \hat{G}} \hat{s}_n(\hat{G}_k) > \hat{s}_n(\hat{G})$. Firstly, when $\hat{G} \notin K_n$, we have $\hat{G}_k \notin K_n$ for all large enough k. Therefore, $\limsup_{\hat{G}_k \to \hat{G}} \hat{s}_n(\hat{G}_k) = \hat{s}_n(\hat{G}) = 0$, which yields contradiction. Secondly, $\hat{G} \in K_n$. Then $\hat{G}_k \in K_n$ for infinitely many large k, since otherwise the limsup term will be zero. But $\hat{s}_n \circ P_n = s_n$ in K_n. By the upper semi-continuity of s_n, we have

$$\limsup_{\hat{G}_k \to \hat{G}} \hat{s}_n(\hat{G}_k) \leq \hat{s}_n(\hat{G})$$

Contradiction. Recall that $\hat{s}$ is defined by

$$\hat{s} = N \max\left(\frac{1}{N'}\hat{s}_{N'}, ..., \frac{1}{N}\hat{s}_N\right)$$

Therefore, $\hat{s}$ is upper semi-continuous.

On the other hand, by Assumption (A2), there is a lower semi-continuous function $a \in L^1(I, \mu)$ such that

$$s_n([i_1, ..., i_n]) \leq a(i_1) + ... + a(i_n)$$

for all $N' \leq n \leq N$, $i_1, ..., i_n \in I$. Consequently,

$$\hat{s}_n(P_n^{-1}([i_1, ..., i_n])) \leq a(i_1) + ... + a(i_n)$$

Moreover, since s_n is non-negative, a is non-negative everywhere. Therefore, for any $[i_1, ..., i_{N!}] \in R_0$,

$$\hat{s}([i_1, ..., i_{N!}]) = 0 \leq \sum_{k=1}^{N!} a(i_k)$$

For any $[\underbrace{i_1, ..., i_1}_{N!/n \text{ many}}, ..., \underbrace{i_n, ..., i_n}_{N!/n \text{ many}}] \in R_n \subset K_n$, by the property of R_n, we have

$$\hat{s}([i_1, ..., i_1, ..., i_n, ..., i_n]) = \frac{N}{n}\hat{s}_n([i_1, ..., i_1, ..., i_n, ..., i_n]) = \frac{N}{n}s_n([i_1, ..., i_n]) \leq \frac{N}{n}\sum_{k=1}^{n} a(i_k)$$

Therefore, we define $\hat{a} = a/(N-1)!$, we have,

$$\hat{s}([i_1, ..., i_{N!}]) \leq \sum_{k=1}^{N!} \hat{a}(i_k)$$

B.5 Proof of Lemma 3

In this subsection, we prove Lemma 3. For any $S \subset I^n / \sim_n$,

$$\begin{aligned}
(Q_n)_\# \gamma_n(S) &= \gamma_n(Q_n^{-1}(S)) \\
&= \int_{Q_n^{-1}(S)} c d(Q_n^\# \tau_n) \\
&= \sum_{\beta \in \mathcal{A}} \int_{Q_n^{-1}(S) \cap J_\beta} c d(Q_n^\# \tau_n)
\end{aligned}$$

For each fixed β, by definition, $J_\beta = J_{m,\sigma}$ for some $m \in \mathcal{M}$ and $\sigma \in S_n$. Therefore, $c = \frac{m_1!...m_k!}{(n-1)!}$ in J_β and we have

$$\begin{aligned}
\int_{Q_n^{-1}(S) \cap J_\beta} c d(Q_n^\# \tau_n) &= \frac{m_1!...m_k!}{(n-1)!} Q_n^\# \tau_n(Q_n^{-1}(S) \cap J_\beta) \\
&= \frac{m_1!...m_k!}{(n-1)!} \tau_n(Q_n(Q_n^{-1}(S) \cap J_\beta))
\end{aligned}$$

On the other hand, there are $\frac{n!}{m_1!...m_k!}$ many $\alpha \in$ A such that $Q_n(Q_n^{-1}(S) \cap J_\beta) = Q_n(Q_n^{-1}(S) \cap J_\alpha)$. We use $[\beta] \subset \mathcal{A}$ to denote this collection of such indices. By definition, A can be partitioned into $|\mathcal{M}|$ components $\{[\beta_m]\}_{m \in \mathcal{M}}$ and

$$\bigcup_{m \in \mathcal{M}} Q_n(Q_n^{-1}(S) \cap J_{\beta_m}) = S$$

where β_m is a element in $[\beta_m]$. Therefore,

$$\begin{aligned}
(Q_n)_\# \gamma_n(S) &= \sum_{\beta \in \mathcal{A}} \int_{Q_n^{-1}(S) \cap J_\beta} c d(Q_n^\# \tau_n) \\
&= \sum_{m \in \mathcal{M}} \sum_{\beta \in [\beta_m]} \int_{Q_n^{-1}(S) \cap J_\beta} c d(Q_n^\# \tau_n) \\
&= \sum_{m \in \mathcal{M}} \frac{n!}{m_1!...m_k!} \cdot \frac{m_1!...m_k!}{(n-1)!} \tau_n(Q_n(Q_n^{-1}(S) \cap J_{\beta_m})) \\
&= \sum_{m \in \mathcal{M}} n \tau_n(Q_n(Q_n^{-1}(S) \cap J_{\beta_m})) \\
&= n \tau_n(S)
\end{aligned}$$

B.6 Proof of Lemma 4

In this subsection, we prove Lemma 4.

Firstly, we prove the measure γ_n is symmetric. For any permutation $\sigma \in S_n$, and any measurable sets $A_1, ..., A_n \subset I$, we have

$$\begin{aligned}
\gamma_n(A_{\sigma(1)} \times ... \times A_{\sigma(n)}) &= \int_{A_{\sigma(1)} \times ... \times A_{\sigma(n)}} cd(Q_n^{\#}\tau_n) \\
&= \sum_{\alpha \in \mathrm{A}} c(\alpha) Q_n^{\#}\tau_n(A_{\sigma(1)} \times ... \times A_{\sigma(n)} \cap J_\alpha) \\
&= \sum_{\alpha \in \mathrm{A}} c(\alpha) \tau_n(Q_n(A_{\sigma(1)} \times ... \times A_{\sigma(n)} \cap J_\alpha)) \\
&= \sum_{\alpha \in \mathrm{A}} c(\alpha) \tau_n(Q_n(A_1 \times ... \times A_n \cap J_\alpha)) \\
&= \sum_{\alpha \in \mathrm{A}} c(\alpha) Q_n^{\#}\tau_n(A_1 \times ... \times A_n \cap J_\alpha) \\
&= \gamma_n(A_1 \times ... \times A_n)
\end{aligned}$$

where $\{c(\alpha)\}_{\alpha \in \mathrm{A}}$ are a collection of constant determined by the function c on the set J_α.

The second property will be proved by using the symmetric property of γ_n and Lemma 3. By Lemma 3, we have

$$\sum_{k=0}^{n} k\tau_n(\mathcal{G}_n(A,k)) = \sum_{k=0}^{n} \frac{k}{n}(Q_n)_{\#}\gamma_n(\mathcal{G}_n(A,k)) = \sum_{k=0}^{n} \frac{k}{n}\gamma_n(Q_n^{-1}(\mathcal{G}_n(A,k)))$$

In particular, $Q_n^{-1}(\mathcal{G}_n(A,k))$ is the union of Cartesian products of k sets A and $n-k$ sets A^c. There are $\frac{n!}{k!(n-k)!}$ many such Cartesian products. Therefore, since γ_n is symmetric,

$$\gamma_n(Q_n^{-1}(\mathcal{G}_n(A,k))) = \frac{n!}{k!(n-k)!}\gamma_n(\underbrace{A \times ... \times A}_{k \text{ many}} \times \underbrace{A^c \times ... \times A^c}_{n-k \text{ many}})$$

On the other hand, there are $\frac{(n-1)!}{(k-1)!(n-k)!}$ Cartesian products of k sets A and $n-k$ sets A^c in the form $A \times S_2 \times ... \times S_n$ where $S_j \in \{A, A^c\}$. We use $\mathcal{S}_k$ to denote the collection of such Cartesian products. i.e.

$$\mathcal{S}_k = \{A \times S_2 \times ... \times S_n : S_j \in \{A, A^c\}, |\{j : S_j = A\}| = k-1\}$$

Therefore, again by the symmetry of the measure γ_n,

$$\gamma_n(\cup_{S \in \mathcal{S}_k} S) = \frac{(n-1)!}{(k-1)!(n-k)!}\gamma_n(\underbrace{A \times ... \times A}_{k \text{ many}} \times \underbrace{A^c \times ... \times A^c}_{n-k \text{ many}})$$

Consequently,

$$\sum_{k=0}^{n} k\tau_n(\mathcal{G}_n(A,k)) = \sum_{k=0}^{n} \frac{k}{n}\gamma_n(Q_n^{-1}(\mathcal{G}_n(A,k))) = \sum_{k=1}^{n} \gamma_n(\cup_{S \in \mathcal{S}_k} S) = \gamma_n(\cup_{S \in \cup_{k=1}^{n} \mathcal{S}_k} S)$$

Note that,

$$\cup_{k=1}^{n}\mathcal{S}_k = \{A \times S_2 \times ... \times S_n : S_j \in \{A, A^c\}\}$$

Thus,

$$\cup_{S\in\cup_{k=1}^{n}\mathcal{S}_k} S = A \times I \times ... \times I$$

In conclusion,

$$\sum_{k=0}^{n} k\tau_n(\mathcal{G}_n(A,k)) = \gamma_n(A \times I \times ... \times I)$$

B.7 Proof of Lemma 5

In this subsection, we prove Lemma 5. We start by stating and proving the following lemma:

Lemma 25. *If S is a symmetric function,*

$$\sup_{\hat{\gamma}\in\hat{\Gamma}_{sym}} \int_{I^{N!}} Sd\hat{\gamma} = \sup_{\hat{\gamma}\in\hat{\Gamma}} \int_{I^{N!}} Sd\hat{\gamma}$$

Proof. Since $\hat{\Gamma}_{sym} \subset \hat{\Gamma}$, we have

$$\sup_{\hat{\gamma}\in\hat{\Gamma}_{sym}} \int_{I^{N!}} Sd\hat{\gamma} \leq \sup_{\hat{\gamma}\in\hat{\Gamma}} \int_{I^{N!}} Sd\hat{\gamma}$$

Conversely, for any $\gamma_0 \in \hat{\Gamma}$, we define

$$\hat{\gamma} = \frac{1}{(N!)!} \sum_{\sigma\in S_{N!}} (f_\sigma)_{\#}\gamma_0$$

where $f_\sigma : I^{N!} \to I^{N!}$ is defined by $f_\sigma(i_1, ..., i_{N!}) = (i_{\sigma(1)}, ..., i_{\sigma(N!)})$.

We claim $\hat{\gamma}$ is in $\hat{\Gamma}_{sym}$ and

$$\int_{I^{N!}} Sd\hat{\gamma} = \int_{I^{N!}} Sd\gamma_0$$

We omit the routine work to check $\hat{\gamma}$ is in $\hat{\Gamma}_{sym}$. Since S is symmetric, γ induces the same total welfare as γ:

$$\begin{aligned}\int_{I^{N!}} Sd\hat{\gamma} =& \frac{1}{(N!)!} \sum_{\sigma\in S_{N!}} \int_{I^{N!}} Sd(f_\sigma)_{\#}\gamma_0 = \frac{1}{(N!)!} \sum_{\sigma\in S_{N!}} \int_{I^{N!}} s \circ f_\sigma d\gamma_0 \\ =& \frac{1}{(N!)!} \sum_{\sigma\in S_{N!}} \int_{I^{N!}} Sd\gamma_0 = \int_{I^{N!}} Sd\gamma_0\end{aligned}$$

Since $\gamma_0 \in \hat{\Gamma}$ is arbitrary,

$$\sup_{\hat{\gamma}\in\hat{\Gamma}_{sym}} \int_{I^{N!}} Sd\hat{\gamma} \geq \sup_{\hat{\gamma}\in\hat{\Gamma}} \int_{I^{N!}} Sd\hat{\gamma}$$

■

By the duality theorem for multi-marginal transport problem Kellerer [1984], we have

$$\sup_{\hat{\gamma}\in\hat{\Gamma}} \int_{I^{N!}} \frac{S}{N} d\hat{\gamma} = \inf_{(u_j)_{j=1}^{N!}\in\tilde{\mathcal{U}}} \sum_{j=1}^{N!} \int_I u_j d\mu$$

where

$$\tilde{\mathcal{U}} = \left\{ (u_1, ..., u_{N!}) \in (L^1(I,\mu))^{N!} : \sum_{j=1}^{N!} u_j(i_j) \geq \frac{S(i_1, ..., i_{N!})}{N}, \forall i_1, i_2, ..., i_{N!} \in I \right\}$$

and the infimum could be achieved. Moreover, by Lemma 25,

$$\sup_{\hat{\gamma}\in\hat{\Gamma}_{sym}} \int_{I^{N!}} \frac{S}{N} d\hat{\gamma} = \sup_{\hat{\gamma}\in\hat{\Gamma}} \int_{I^{N!}} \frac{S}{N} d\hat{\gamma}$$

So it remains to show

$$\inf_{(u_j)_{j=1}^{N!}\in\tilde{\mathcal{U}}} \sum_{j=1}^{N!} \int_I u_j d\mu = \inf_{u\in\hat{\mathcal{U}}} \int_I u d\mu \tag{21}$$

and the infimum on the right hand side could be achieved.

Firstly, take any $u \in \hat{\mathcal{U}}$, we have $(u, u, ..., u) \in \tilde{\mathcal{U}}$. Therefore,

$$\inf_{(u_j)_{j=1}^{N!}\in\tilde{\mathcal{U}}} \sum_{j=1}^{N!} \int_I u_j d\mu \leq \inf_{u\in\hat{\mathcal{U}}} \int_I u d\mu$$

Conversely, for any minimizer $(u_j^*)_{j=1}^{N!} \in \tilde{\mathcal{U}}$ solving the left hand side of Equation 21, we define

$$u = \sum_{n=1}^{N!} u_n^* \tag{22}$$

Then, for any $i_1, ..., i_{N!} \in I$, we have,

$$\begin{aligned}\sum_{j=1}^{N!} u(i_j) &= \sum_{j=1}^{N!}\sum_{n=1}^{N!} u_n^*(i_j) = \sum_{j=1}^{N!}\left[\sum_{k=0}^{N!-1} u_j^*(i_{j+k})\right] \\ &\geq \frac{1}{N}\left[S(i_1, ..., i_{N!}) + S(i_2, ..., i_{N!}, i_1) + ... + S(i_{N!}, i_1, ..., i_{N!-1})\right] \\ &= (N-1)! S(i_1, ..., i_{N!})\end{aligned}$$

where $i_{N!+k}$ is defined to be i_k for $1 \leq k \leq N!$ and the last equality is implied by the symmetry of S. Moreover, as a finite sum of upper semi-continous integrable functions, $u \in L^1(I, \mu)$. Thus, $u \in \mathcal{U}$, and we have

$$\inf_{(u_j)_{j=1}^{N!}\in\tilde{\mathcal{U}}} \sum_{j=1}^{N!} \int_I u_j d\mu \geq \inf_{u\in\hat{\mathcal{U}}} \int_I u d\mu$$

In conclusion,

$$\inf_{(u_j)_{j=1}^{N!}\in\tilde{\mathcal{U}}}\sum_{j=1}^{N!}\int_I u_j d\mu = \inf_{u\in\hat{\mathcal{U}}}\int_I u d\mu$$

and $u=\sum_{n=1}^{N!}u_n^*$ solves the minimization problem on the right hand side of Equation 21.

B.8 Proof of Lemma 6

In this subsection, we prove Lemma 6. Firstly, take any $u\in\hat{\mathcal{U}}$ and any $[i_1,...,i_1,i_2,...,i_2,...,i_n,...,i_n]\in K_n$, We have

$$\begin{aligned}\sum_{k=1}^{n}\frac{N!}{n}u(i_k)\geq&(N-1)!\hat{s}([i_1,...,i_1,i_2,...,i_2,...,i_n,...,i_n])\\ \geq&\frac{N!}{n}\hat{s}_n([i_1,...,i_1,i_2,...,i_2,...,i_n,...,i_n])\\ =&\frac{N!}{n}s_n([i_1,...,i_n])\end{aligned}$$

Therefore, $u\in\mathcal{U}_n$. Since n is arbitrary, $u\in\mathcal{U}=\cap_{n=N'}^{N}\mathcal{U}_n$. Conversely, take any $u\in\mathcal{U}$, we check $u\in\hat{\mathcal{U}}$ region by region. Firstly, take $[i_1,...,i_{N!}]\in R_0$, it is clear that

$$\sum_{k=1}^{N!}u(i_k)\geq 0=(N-1)!\hat{s}([i_1,...,i_{N!}])$$

Moreover, for any permitted group size $N'\leq n\leq N$, we take $[i_1,...,i_1,i_2,...,i_2,...,i_n,...,i_n]\in R_n$,

$$\begin{aligned}\sum_{k=1}^{n}\frac{N!}{n}u(i_k)=&\frac{N!}{n}\sum_{k=1}^{n}u(i_k)\\ \geq&\frac{N!}{n}s_n([i_1,...,i_n])\\ =&\frac{N!}{n}\hat{s}_n([i_1,...,i_1,i_2,...,i_2,...,i_n,...,i_n])\\ =&(N-1)!\hat{s}([i_1,...,i_1,i_2,...,i_2,...,i_n,...,i_n])\end{aligned}$$

Since $\{R_0,R_{N'},...,R_N\}$ is a partition of $I^{N!}/\sim_{N!}$, we have $u\in\hat{\mathcal{U}}$.

B.9 Proof of Theorem 2

Now, we finish the proof of duality theorem by proving there is a continuous minimizer for the welfare minimization problem

$$\inf_{u\in\mathcal{U}}\int_I u d\mu \tag{23}$$

where $\mathcal{U} = \{u \in L^1(I, \mu) : \sum_{i \in G} u(i) \geq s(G), \forall G \in \mathcal{G}\}$.

Firstly, by Lemma 5 and Lemma 6, there is a minimizer $u^* \in L^1(I, \mu)$ minimizing Equation 23. On the other hand, by Lemma 7, there is a maximizer τ^* for the welfare maximization problem. Define $\mu_n = \tau^*(\mathcal{G}_n)$. That is, μ_n is the type distribution of players who are assigned to some n-person group under assignment τ^*. Clearly, $\sum_{n=N'}^{N} \mu_n = \mu$.

We claim that, for any permitted group size n, there is a continuous function $v_n \in C(I) \cap \mathcal{U}_n$ such that $v_n \leq u^*$ on I and $\int_I u^* d\mu_n = \int_I v_n d\mu_n$. We note, if the claim is proved to be true, we can take $v = \max(v_{N'}, ..., v_N)$. It is clear, as the maximum of finitely many continuous functions, v is continuous. Moreover, as $v_n \in \mathcal{U}_n$, v is in $\mathcal{U}$. By $v \leq u^*$, v is also a minimizer of Equation 23.

Now, we fixed a permitted group size n and prove the claim. Our proof is based on the trick used in the proof of Theorem 4.1.1 in Pass [2011]. This trick can also be found in Gangbo and Święch [1998] and Carlier and Nazaret [2008].

We define γ_n^* by τ_n^* using the change of variable trick in Equation 9. By Lemma 3,

$$\int_{I^n} \frac{\tilde{s}_n}{n} d\gamma_n^* = \int_{\mathcal{G}_n} s_n d\tau_n^*$$

where $\tilde{s}_n = s \circ Q_n$ is a continuous function on I^n. Moreover, by the proof of equivalence lemma in Appendix B.11, $\sum_{i \in G} u^*(i) = s_n(G)$ for τ_n^*-almost all $G \in \mathcal{G}_n$. Therefore,

$$\int_{I^n} \tilde{s}_n d\gamma_n^* = n \int_I u^* d\mu_n$$

We define n functions $(w_1, ..., w_n)$ as follows:

$$w_1(i_1) = \sup_{i_k \in I, k \geq 2} \left(\tilde{s}_n(i_1, ..., i_k) - \sum_{k=2}^{n} u^*(i_k) \right)$$

Inductively, for $m \geq 2$,

$$w_m(i_m) = \sup_{i_k \in I, k \neq m} \left(\tilde{s}_n(i_1, ..., i_k) - \sum_{k=1}^{m-1} w_k(i_k) - \sum_{k=m+1}^{n} u^*(i_k) \right)$$

Inductively, by $\sum_{k=1}^{n} u^*(i_k) \geq \tilde{s}_n(i_1, ..., i_k)$ for all $i_1, ..., i_k \in I$, we have

$$u^*(x_k) \geq w_k(x_k) \tag{24}$$

for all $1 \leq k \leq n$, $x_k \in I$. Moreover, recall,

$$w_n(i_n) = \sup_{i_k \in I, k \neq n} \left(\tilde{s}_n(i_1, ..., i_n) - \sum_{k=1}^{n-1} w_k(i_k) \right)$$

we have for all $1 \leq m \leq n$,

$$w_m(i_m) \geq \sup_{i_k \in I, k \neq m} \left(\tilde{s}_n(i_1, ..., i_n) - \sum_{k \neq m} w_k(i_k) \right)$$

The definition of w_{m-1} implies for all $i_1, ..., i_n \in I$,

$$u^*(i_m) \geq s_n(i_1, ..., i_n) - \sum_{k=1}^{m-1} u^*(i_k) - \sum_{k=m+1}^{n} w_k(i_k)$$

Therefore,

$$\begin{aligned} w_m(x_m) &= \sup_{i_k \in I, k \neq m} \left(\tilde{s}_n(i_1, ..., i_k) - \sum_{k=1}^{m-1} w_k(i_k) - \sum_{k=m+1}^{n} u^*(i_k) \right) \\ &\leq \sup_{i_k \in I, k \neq m} \left(\tilde{s}_n(i_1, ..., i_n) - \sum_{k \neq m} w_k(i_k) \right) \end{aligned}$$

Consequently, for all $1 \leq m \leq n$ and all $i_1, ..., i_k \in I$,

$$w_m(i_m) = \sup_{i_k \in I, k \neq m} \left(\tilde{s}_n(i_1, ..., i_n) - \sum_{k \neq m} w_k(i_k) \right)$$

That is, $(w_1, ..., w_n)$ are $\tilde{s}_n$-conjugate. Since $\tilde{s}_n$ is continuous and I is compact, $w_1, ..., w_n$ are continuous. Define $v_n = \frac{1}{n}(w_1 + ... + w_n) \in C(I)$. By Equation 24, $v_n \leq u^*$. By the symmetry of $\tilde{s}_n$, $v_n \in \mathcal{U}_n$. Moreover, as all marginals of γ_n^* are μ_n,

$$n \int_I u^* d\mu_n \geq \sum_{k=1}^{n} \int_I w_k d\mu_n \geq \int_{I^n} \tilde{s}_n d\gamma_n^*$$

But $\int_{I^n} \tilde{s}_n d\gamma_n^* = n \int_I u^* d\mu_n$. Thus, $\int_I u^* d\mu_n = \int_I v_n d\mu_n$. Therefore, the claim is proved.

B.10 Proof of Lemma 7

In this subsection, we prove Lemma 7. By Proposition 1 and Lemma 25, we have

$$\sup_{\tau \in \mathrm{T}} \sum_{n=N'}^{N} \int_{\mathcal{G}_n} s_n d\tau_n = \sup_{\hat{\gamma} \in \hat{\Gamma}_{sym}} \int_{I^{N!}} \frac{S}{N} d\hat{\gamma} = \sup_{\hat{\gamma} \in \hat{\Gamma}} \int_{I^{N!}} \frac{S}{N} d\hat{\gamma}$$

We first show there is a maximizer $\hat{\gamma}$ solving the right maximization problem, then we construct a maximizer τ for the left maximization problem.

To show the existence of a maximizer $\hat{\gamma}$, we show $\hat{\Gamma}$ is compact and the map $\hat{\gamma} \to \int_{I^{N!}} S d\hat{\gamma}$ is upper semi-continuous.

Lemma 26. *$\hat{\Gamma}$ is compact in the weak-* topology.*

Proof. Since I is Polish space, by Ulam's theorem, $\{\mu / \|\mu\|\}$ is tight in I. i.e. for any $\varepsilon > 0$, there exists a compact set K_ε, such that $\mu(I - K_\varepsilon) < \varepsilon$. Hence, $\hat{\Gamma}$ is uniformly tight as, for any $\hat{\gamma} \in \hat{\Gamma}$,

$$\hat{\gamma}(I^{N!} - K_\epsilon^{N!}) \leq N! \mu(I - K_\varepsilon) \leq N! \varepsilon$$

Moreover, $\hat{\Gamma}$ is uniformly bounded as, for any $\hat{\gamma} \in \hat{\Gamma}$, $\|\hat{\gamma}\| = \|\mu\|$. By Prokhorov's theorem (Theorem 8.6.2 in Bogachev [2007]), $\hat{\Gamma}$ has compact closure in weak* topology.

Thus, to show $\hat{\Gamma}$ is compact, it is sufficient to show it is closed: for any convergent sequence $(\hat{\gamma}_k)_{k\in\mathbb{N}}$ in $\hat{\Gamma}$ such that $\hat{\gamma}_k \to \hat{\gamma}$, we have, for all measurable $S \subset I^{N!}$,

$$\hat{\gamma}(S) = \lim_{k\to\infty} \hat{\gamma}_k(S)$$

For any measurable set $A \subset I$, by taking S to be the Cartesian product of one set A and $N! - 1$ sets I, we can show all the marginals of $\hat{\gamma}$ is μ. Therefore, $\hat{\gamma} \in \hat{\Gamma}$. i.e. $\hat{\Gamma}$ is closed. ∎

Lemma 27. *The functional $\hat{\gamma} \to \int_{I^{N!}} Sd\hat{\gamma}$ is upper semi-continuous in the weak-* topology.*

Proof. Firstly, we claim it is without loss of generality to assume S is non-positive: by Equation 14, S is bounded from above by the function lower-semi continuous and integrable $A : I^{N!} \to \mathbb{R}$ where

$$A(i_1, ..., i_{N!}) = \sum_{j=1}^{N!} \hat{a}(i_j)$$

We can study $s - A$, which is a non-positive upper semi-continuous and integrable function. By the non-positive upper semi-continuity, there is a decreasing sequence of S_l converging to S pointwisely, where S_l is a continuous bounded function on $I^{N!}$. Taking $\hat{\gamma}_k \to \hat{\gamma}$, by monotone convergence theorem, we have

$$\int_{I^{N!}} Sd\hat{\gamma} = \lim_{l\to\infty} \int_{I^{N!}} S_l d\hat{\gamma} = \lim_{l\to\infty} \lim_{k\to\infty} \int_{I^{N!}} S_l d\hat{\gamma}_k \geq \limsup_{k\to\infty} \int_{I^{N!}} Sd\hat{\gamma}_k$$

∎

Therefore, by Lemma 26 and Lemma 27, there is a $\hat{\gamma}^* \in \hat{\Gamma}$ solves the maximization problem

$$\sup_{\hat{\gamma}\in\hat{\Gamma}} \int_{I^{N!}} \frac{S}{N} d\hat{\gamma}$$

By the construction in Lemma 25, we have a maximizer $\hat{\gamma}^*_{sym} \in \hat{\Gamma}_{sym}$. Then we repeat the construction in Equation 15, and obtain a $\tau^* = (\tau^*_{N'}, ..., \tau^*_N) \in \mathrm{T}$ such that

$$\sum_{n=N'}^{N} \int_{\mathcal{G}_n} s_n d\tau^*_n = \int_{I^{N!}} \frac{S}{N} d\hat{\gamma}^*_{sym} = \sup_{\hat{\gamma}\in\hat{\Gamma}_{sym}} \int_{I^{N!}} \frac{S}{N} d\hat{\gamma} = \sup_{\tau\in\mathrm{T}} \sum_{n=N'}^{N} \int_{\mathcal{G}_n} s_n d\tau_n$$

i.e. τ^* solves the maximization problem

$$\sup_{\tau\in\mathrm{T}} \sum_{n=N'}^{N} \int_{\mathcal{G}_n} s_n d\tau_n$$

B.11 Proof of Lemma 8

In this subsection, we prove Lemma 8. Given any maximizer τ^* and minimizer u^* in the duality equation Equation 3, by $u^* \in \mathcal{U}$, we have the no-blocking condition:

$$\sum_{i \in G} u^*(i) \geq s(G), \forall G \in \mathcal{G}$$

For every permitted group size n, we define γ_n^* by τ_n^* by Equation 9. By Lemma 3, we have

$$\sum_{n=N'}^{N} \int_{\mathcal{G}_n} \left(u^*(i_1) + \ldots + u^*(i_n)\right) d\tau_n^* = \sum_{n=N'}^{N} \int_{I^n} \frac{u^*(i_1) + \ldots + u^*(i_n)}{n} d\gamma_n^*$$

Moreover, by Lemma 4, $\gamma^* \in \Gamma$. Therefore, we have

$$\sum_{n=N'}^{N} \int_{I^n} \frac{u^*(i_1) + \ldots + u^*(i_n)}{n} d\gamma_n^* = \int_I u^* d\mu$$

Consequently,

$$\int_I u^* d\mu = \sum_{n=N'}^{N} \int_{\mathcal{G}_n} \left(u^*(i_1) + \ldots + u^*(i_n)\right) d\tau_n^* \geq \sum_{n=N'}^{N} \int_{\mathcal{G}_n} s_n d\tau_n^*$$

By the duality relation, Equation 3,

$$\int_I u^* d\mu = \sum_{n=N'}^{N} \int_{\mathcal{G}_n} s_n d\tau_n^*$$

Therefore, $\sum_{i \in G} u^*(i) \leq s(G)$ for τ^*-almost all $G \in \operatorname{supp}(\tau^*)$. By the continuity of u and s, we have the feasibility condition

$$\sum_{i \in G} u^*(i) \leq s(G), \forall G \in \operatorname{supp}(\tau^*)$$

That is, τ^* is a stable assignment with imputation given by u^*.

Conversely, given a stable assignment τ^* with an imputation u^*, we show τ^* solves the maximization problem and u^* solves the minimization problem. Firstly, since $u^* \in \mathcal{U}$ is an imputation such that there is no blocking coalitions, for any assignment $\tau \in \mathrm{T}$, we have

$$\sum_{n=N'}^{N} \int_{\mathcal{G}_n} s_n d\tau_n \leq \int_I u^* d\mu$$

However, since τ^* is stable, by feasibility condition,

$$\sum_{n=N'}^{N} \int_{\mathcal{G}_n} s_n d\tau_n^* \geq \int_I u^* d\mu$$

Therefore, τ^* solves the maximization problem in Equation 3. Secondly, by the no-blocking condition of stable assignments, for any imputation $u \in \mathcal{U}$, we have

$$\sum_{n=N'}^{N} \int_{\mathcal{G}_n} s_n d\tau_n^* \leq \int_I u d\mu$$

Again by feasibility condition, we have

$$\sum_{n=N'}^{N} \int_{\mathcal{G}_n} s_n d\tau_n^* \geq \int_I u^* d\mu$$

Therefore, u^* solves the minimization problem in Equation 3.

C Omitted Proofs in Section 3

In this section, we prove Theorem 3 and Theorem 4. We use the same three-step procedure introduced Gretsky et al. [1992] to show the existence of a stable assignment. Firstly, we study the optimization problems and show the optimizers exist. Secondly, we prove the duality property to link the assignment and the imputation. Lastly, we prove the equivalence between the solutions of optimization problems and the set of stable assignments.

Lemma 28. *$\Gamma_{\mathcal{G}}$ is a compact set in $\mathcal{M}_+(\mathcal{G})$.*

Proof. Since I is a Polish space, $\mathcal{M}_+(I)$, the set of nonnegative Borel measures on I, with weak topology on it is a Polish space by Theorem 8.9.4 in Bogachev [2007]. By the same argument, $\mathcal{M}_+(\mathcal{M}_+(I))$ is a Polish space.

On the other hand, since I is a Polish space and μ is a probability measure, by Ulam's tightness theorem, $\{\mu\} \subset \mathcal{M}(I)$ is tight. That is, for any $\delta > 0$, there is a compact set $K_\delta \subset I$ such that $\mu(I - K_\delta) < \delta$.

Next, we show $\mathcal{G}$, as a subset of $\mathcal{M}_+(I)$, is compact. Firstly, $\mathcal{G}$ is uniformly tight. Since for any $\delta > 0$, there is a compact set $K_\delta \subset I$ such that $|\nu|(I - K_\delta) \leq \mu(I - K_\delta) < \delta$, for all $\nu \in \mathcal{G}$. Moreover, $\mathcal{G}$ is uniformly bounded in variation norm as $\|\nu\| \leq \varepsilon$ for all $\nu \in \mathcal{G}$. By Prohorov's theorem for Borel measures (Theorem 8.6.2 in Bogachev [2007]), $\mathcal{G}$ has compact closure. It is routine to check $\mathcal{G}$ is closed. Therefore, $\mathcal{G}$ is compact.

Furthermore, we show $\Gamma_{\mathcal{G}}$ is compact in $\mathcal{M}_+(\mathcal{M}_+(I))$. Firstly, $\Gamma_{\mathcal{G}}$ is uniformly tight since $\gamma(\mathcal{M}_+(I) - \mathcal{G}) = 0$ for all $\gamma \in \Gamma_{\mathcal{G}}$. Moreover, the total variation norm of γ is uniformly bounded as for all $\gamma \in \Gamma_{\mathcal{G}}$

$$\gamma(\mathcal{M}_+(I)) = \gamma(\mathcal{G}) \leq \frac{1}{\varepsilon'} \int_{\mathcal{G}} \|\nu\| \, d\gamma = \gamma(\mathcal{G}) \leq \frac{1}{\varepsilon'} \int_{\mathcal{G}} \nu(I) d\gamma = \frac{\mu(I)}{\varepsilon'} = \frac{\|\mu\|}{\varepsilon'}$$

Again by Prohorov's theorem for Borel measures, $\mathcal{G}$ has compact closure in $\mathcal{M}_+(\mathcal{M}_+(I))$. Moreover, since the map $\nu \to \nu(A)$ is continuous and bounded on $\mathcal{M}_+(I)$ for any Borel set $A \subset I$, for any convergent sequence γ_n in $\Gamma_{\mathcal{G}}$ and any Borel set $A \subset I$,

$$\mu(A) = \lim_{n\to\infty} \int_{\mathcal{G}} \nu(A) d\gamma_n = \int_{\mathcal{G}} \nu d\gamma$$

Therefore, $\Gamma_{\mathcal{G}}$ is closed. Hence, $\Gamma_{\mathcal{G}}$ is compact. ∎

Lemma 29. *For the cooperative game $((I, \mu), s, \varepsilon', \varepsilon)$ satisfying Assumption (B1) and Assumption (B2), there is an assignment $\gamma \in \Gamma_{\mathcal{G}}$ solving the welfare maximization problem*

$$\sup_{\gamma \in \Gamma_{\mathcal{G}}} \int_{\mathcal{G}} s d\gamma$$

Proof. Firstly, we prove the map $\gamma \to \int_{\mathcal{G}} s d\gamma$ is upper-semi continuous on $\mathcal{M}_+(\mathcal{M}_+(I))$ if s is non-positive. Since s is a upper semi-continuous function and has a uniform bound, we take a decreasing sequence of s_l converging to s pointwisely, where s_l is continuous. Let $\gamma_n \to \gamma$ in $\mathcal{M}_+(\mathcal{G})$, by the monotone convergence theorem and the fact s_l is decreasing, we have

$$\int_{\mathcal{G}} s d\gamma = \lim_{l\to\infty} \int_{\mathcal{G}} s_l d\gamma = \lim_{l\to\infty} \lim_{n\to\infty} \int_{\mathcal{G}} s_l d\gamma_n \geq \limsup_{n\to\infty} \int_{\mathcal{G}} s d\gamma_n$$

Moreover, define a function $h : \mathcal{G} \to \mathbb{R}$ by $h(\nu) = \int_I a d\nu$. By Assumption (B2), $s - h$ is non-positive in $\mathcal{G}$. Therefore, taking a sequence of $\gamma_n \in \Gamma_{\mathcal{G}}$ approaching the maximum, by compactness of the choice set $\Gamma_{\mathcal{G}}$, a subsequence of it converges to some $\gamma \in \Gamma_{\mathcal{G}}$. By selecting the subsequence, we suppose $\gamma_n \to \gamma$ in $\mathcal{M}_+(\mathcal{G})$. Therefore,

$$\int_{\mathcal{G}} s d\gamma - \int_{\mathcal{G}} h d\gamma \geq \limsup_{n\to\infty} \left(\int_{\mathcal{G}} s d\gamma_n - \int_{\mathcal{G}} h d\gamma_n \right)$$

However, we note h is continuous and bounded on $\mathcal{G}$ since $h(\nu) = \int_I a d\nu \leq \|a\|_\infty \|\nu\|_0$ and $h(\nu) \leq \|a\|_\infty \varepsilon$. Therefore, by the definition of weak convergence on in $\mathcal{M}_+(\mathcal{G})$, we have $\lim_{n\to\infty} \int_{\mathcal{G}} h d\gamma_n = \int_{\mathcal{G}} h d\gamma$. Therefore,

$$\int_{\mathcal{G}} s d\gamma \geq \limsup_{n\to\infty} \int_{\mathcal{G}} s d\gamma_n$$

i.e. γ is a maximizer. ∎

Lemma 30. *For the game $((I, \mu), s, \varepsilon', \varepsilon)$ satisfying Assumption (B1) and Assumption (B2), there is an imputation $u \in \mathcal{U}$ solving the payoff minimization problem*

$$\inf_{u \in \mathcal{U}} \int_I u d\mu$$

Proof. By By Assumption (B2), there is a function $a \in L^1(I, \mu)$ such that $\int_I a d\nu \geq s(\nu)$ for all $\nu \in \mathcal{G}$. Therefore, $a \in \mathcal{U}$. Note that for any $u \in \mathcal{U}$, $\min(u, a) \in \mathcal{U}$. Therefore, we define a subset $\mathcal{U}_a \subset \mathcal{U}$ where

$$\mathcal{U}_a = \{u \in \mathcal{U} : 0 \leq u \leq \|a\|_\infty\}$$

Clearly, $\inf_{u \in \mathcal{U}} \int_I u d\mu = \inf_{u \in \mathcal{U}_a} \int_I u d\mu$ as all $u \in \mathcal{U}$ is pointwisely non-negative.

Next, we define a sequence of spaces $(\mathcal{U}_a(k))_{k \in \mathbb{N}}$ in $L^1(\mu)$ where

$$\mathcal{U}_a(k) = \left\{u \in \mathcal{U}_a : \int_I u d\mu \leq \inf_{u \in \mathcal{U}} \int_I u d\mu + 1/k\right\}$$

We note for any $k \in \mathbb{N}$, $\mathcal{U}_a(k)$ is non-empty. Moreover, it is uniformly integrable: for any $c > 0$, we take $\delta = c/\|a\|_\infty > 0$. Then, for any Borel measurable $E \subset I$ with $\mu(E) < \delta$, $\int_E u d\mu \leq c$. By Dunford-Pettis theorem (Theorem 3 in Diestel and Uhl Jr [1978]), the weak closure of $\mathcal{U}_a(k)$ is weakly compact in $L^1(\mu)$. But it is easy to see $\mathcal{U}_a(k)$ is closed in weak topology. Therefore, $\mathcal{U}_a(k)$ is non-empty, closed and compact in $L^1(\mu)$ endowed with weak topology. Note the sequence $\mathcal{U}_a(k)$ is decreasing, by Cantor's intersection theorem,

$$\bigcap_{k \in \mathbb{N}} \mathcal{U}_a(k) \neq \emptyset$$

The element $u \in \bigcap_{k \in \mathbb{N}} \mathcal{U}_a(k)$ has the property that $\int_I u d\mu \leq \inf_I u d\mu$. ■

Next, we prove the Theorem 4. The proof is based on the proof of Theorem 11.8.2 in Dudley [2002].

Proof. For any $u \in L^1(I, \mu)$, we define $F_u : \mathcal{G} \to \mathbb{R}$ by

$$F_u(\nu) = \int_I u d\nu$$

Since I is a metric space and μ is a finite Borel measure, $C_b(I)$ is dense in $L^1(I, \mu)$. Therefore, for any $\nu_1, \nu_2 \in \mathcal{G}$ and $g \in C_b(I)$, we have

$$|F_u(\nu_1) - F_u(\nu_2)| \leq 2\|u - g\|_1 + \int_I g d(\nu_1 - \nu_2) \leq 2\|u - g\|_1 + \|g\|_\infty \|\nu_1 - \nu_2\|_0$$

Therefore, F_u is weak continuous in $\mathcal{G}$. i.e. $F_u \in C(\mathcal{G})$.

We define L to be the space containing all such function F_u. That is,

$$L = \{F_u \in C(\mathcal{G}) : u \in L^1(I, \mu)\}$$

Since, for any $c \in \mathbb{R}$, $u_1, u_2 \in L^1(I, \mu)$, we have $F_{cu_1 + u_2} = cF_{u_1} + F_{u_2}$, L is a linear subspace of $C(\mathcal{G})$.

Next, define $H \subset C(\mathcal{G})$ by

$$H = \{F \in C(\mathcal{G}) : F(\nu) \geq s(\nu), \forall \nu \in \mathcal{G}\}$$

It is easy to see H is convex. Moreover, H is nonempty since, by Assumption (B2), $F_a \in H$. Consequently, we also have $F_a + 1 \in int(H)$.

Now, we define a linear form r on $L \subset C(\mathcal{G})$ by

$$r(F_u) = \int_I u d\mu$$

It is easy to check r is a linear map. By Assumption (B2), $L \cap H \neq \emptyset$ as $F_a \in L \cap H$. Moreover, r is bounded from below in $L \cap H$, since, for any $\gamma \in \Gamma_{\mathcal{G}}$,

$$r(F_u) = \int_I u d\mu = \int_{\mathcal{G}} \int_I u d\nu d\gamma \geq \int_{\mathcal{G}} s(\nu) d\gamma \geq \left(\inf_{\nu \in \mathcal{G}} s(\nu) \right) \gamma(\mathcal{G})$$

By By Assumption (B1), s is upper-semi continuous on a compact set $\mathcal{G}$, Therefore, $\inf_{\nu \in \mathcal{G}} s(\nu) > -\infty$. In addition, we have $1 = \mu(I) = \int_{\mathcal{G}} \nu(I) d\gamma \leq \varepsilon \gamma(\mathcal{G})$. Therefore, r is bounded from below by $(\inf_{\nu \in \mathcal{G}} s(\nu)) / \varepsilon$. By Hahn-Banach theorem (Theorem 6.2.11 in Dudley [2002]), r can be extended to a linear functional $\tilde{r}$ on $C(\mathcal{G})$ such that,

$$\inf_{F \in H} \tilde{r}(F) = \inf_{F \in L \cap H} r(F)$$

We claim $\tilde{r}$ is a bounded positive functional on $C(\mathcal{G})$. To see the r is positive for any $F \geq 0$ in $C(\mathcal{G})$ and real number $c \geq 0$, we have $\tilde{s} + cF + 1 \in H$, where $\tilde{s}$ is a continuous approximation of s. Note we have

$$\inf_{F \in H} \tilde{r}(F) = \inf_{F \in H \cap L} r(F) \geq \frac{\inf_{\nu \in \mathcal{G}} s(\nu)}{\varepsilon}$$

Therefore, by taking c large enough, we get $\tilde{r}(F) \geq 0$. To see $\tilde{r}$ is bounded, we note for any $F \in C(\mathcal{G})$, $|\tilde{r}(F)| \leq |\tilde{r}(1)| \, \|F\|_\infty$.

Hence, by Riesz representation theorem (Theorem 7.4.1 in Dudley [2002]), there exists a positive Borel measure ρ on $\mathcal{G}$ such that

$$\tilde{r}(F) = \int_{\mathcal{G}} F d\rho$$

for any $F \in C(\mathcal{G})$. Furthermore, we show $\rho \in \Gamma_{\mathcal{G}}$. Note $\tilde{r} = r$ on L, for any Borel measurable set $A \subset I$, we take $u = \mathbb{1}_A$, $F_u(\nu) = \nu(A)$ and we have

$$\int_{\mathcal{G}} \nu(A) d\rho = \tilde{r}(F_u) = r(F_u) = \int_I u d\mu$$

Therefore,

$$\inf_{u \in \mathcal{U}} \int_I u d\mu = \inf_{F \in H \cap L} r(F) = \inf_{F \in H} \tilde{r}(F) = \inf_{F \in H} \int_{\mathcal{G}} F d\rho = \int_{\mathcal{G}} s d\rho \leq \sup_{\gamma \in \Gamma_{\mathcal{G}}} \int_{\mathcal{G}} s d\gamma$$

Conversely, for any $u \in L^1(I, \mu)$, we approximate u by step functions such that $\int_I u d\nu \geq s(\nu)$ for all $\nu \in \mathcal{G}$. Therefore, for any indicate function $u = 1_A$, we have

$$\int_I u d\mu = \int_{\mathcal{G}} \int_I u d\nu d\gamma \geq \int_{\mathcal{G}} s d\gamma$$

Consequently,

$$\sup_{\gamma\in\Gamma_{\mathcal{G}}}\int_{\mathcal{G}} v d\gamma \leq \inf_{u\in\mathcal{U}}\int_{I} u d\mu$$

■

Lastly, to prove the existence of a stable assignment, we prove establish the equivalence relation between duality theorem Theorem 4.

Lemma 31. *Any maximizer $\gamma \in \Gamma_{\mathcal{G}}$ in the duality equation is a stable assignment with the imputation given by the minimizer $u \in \mathcal{U}$.*

Conversely, any stable assignment γ solves the maximization problem and the imputation u associated with the assignment solves the minimization problem.

Proof. Firstly, we suppose $\gamma \in \Gamma_{\mathcal{G}}$ and $u \in \mathcal{U}$ are the solutions of the maximization and minimization problem respectively. By definition, $\int_I u d\nu \geq s(\nu)$ for all groups $\nu \in \mathcal{G}$. In contrast, by Riesz representation theorem,

$$\int_I u d\mu = \int_{\mathcal{G}} \left(\int_I u d\nu\right) d\gamma = \int_{\mathcal{G}} s d\gamma$$

Therefore, $\int_I u d\nu \leq s(\nu)$ for $\gamma-$almost all $\nu \in \mathcal{G}$.

Conversely, we take a stable assignment $\gamma \in \Gamma_{\mathcal{G}}$ with its imputation $u \in \mathcal{U}$. By definition, $\int_I u d\nu \geq s(\nu)$ for all $\nu \in \mathcal{G}$, and $\int_I u d\nu \leq v(\nu)$, for γ-almost all $\nu \in \mathcal{G}$. Therefore, by Riesz representation theorem,

$$\int_{\mathcal{G}} s d\gamma = \int_{\mathcal{G}}\int_I u d\nu d\gamma = \int_I u d\mu$$

For any $y \in \mathcal{U}$, we have that $\int_I y d\nu \geq s(\nu)$ for all $\nu \in \mathcal{G}$. As a result,

$$\int_I u d\mu = \int_{\mathcal{G}} s d\gamma \leq \int_{\mathcal{G}}\int_I y d\nu d\gamma = \int_I y d\mu$$

That is, u solves the minimization problem. For any $\tau \in \Gamma_{\mathcal{G}}$,

$$\int_{\mathcal{G}} s d\tau \leq \int_{\mathcal{G}}\int_I u d\nu d\tau = \int_I u d\mu = \int_{\mathcal{G}} s d\gamma$$

That is, γ solves the maximization problem. ■

D Omitted Proofs in Section 4

D.1 Proof of Lemma 9

In this subsection, we prove Lemma 9. We fixed the group size n in the proof.

To start with, we give a few definitions. Firstly, the produce space $X = [0, NM\vec{e}]^I$ consists of all tuples $(x_i)_{i\in I}$ such that $x_i \in \mathbb{R}^L_+$ and $x_i \leq NM\vec{e}$. By Assumption (C3) and the market clear condition, no agent can consume more than $NM\vec{e}$. Moreover, since $[0, NM\vec{e}] = \{x \in \mathbb{R}^L_+ : x \leq NM\vec{e}\}$ is compact, by Tychonoff's theorem (Theorem 2.61 in Aliprantis and Border [2007]), X is compact in the product topology.

Moreover, we define a function $f : \mathcal{G}_n \times X \to \mathbb{R}$ by

$$f(G, x) = \sum_{i \in G} u_i(x_i)$$

We claim f is continuous: taking any sequence (G^k, x^k) converging to (G, x), we have $G^k = [i_1^k, ..., i_n^k] \to [i_1, ..., i_n] = G$ and $x_i^k \to x_i$. By Lemma 24, there is a permutation $\sigma_k \in S_n$ such that $i^k_{\sigma_k(j)} \to i_j$ for all $1 \leq j \leq n$. Moreover, by Assumption (C2) and Assumption (C4), all utility functions u_i are continuous and $u_i(x)$ is continuous in I for any fixed x. Therefore,

$$\lim_{k\to+\infty} f(G^k, x^k) = \lim_{k\to+\infty} \sum_{j=1}^n u_{i_j^k}(x^k_{i_j^k}) = \lim_{k\to+\infty} \sum_{j=1}^n u_{i^k_{\sigma_k(j)}}(x^k_{i^k_{\sigma_k(j)}}) = \sum_{j=1}^n u_{i_j}(x^k_{i_j}) = f(G, x)$$

In addition, we define a correspondence $\varphi : \mathcal{G}_n \rightrightarrows X$ by

$$\varphi(G) = \left\{ x \in X : \sum_{i \in G} x_i = \sum_{i \in G} \omega_i^x \right\} \tag{25}$$

We claim φ is a continuous correspondence with nonempty compact values. For every fixed $G \in \mathcal{G}_n$, $\varphi(G) \neq \emptyset$ since $\omega_i^x \in X$. To see $\varphi(G) \subset X$ is compact, as X is compact, it is sufficient to prove $\varphi(G)$ is closed: taking any $x^k \to x$ in X such that $\sum_{i\in G} x_i^k = \sum_{i \in G} \omega_i^x$ for every k, we have $\sum_{i \in G} x_i = \sum_{i\in G} \omega_i^x$.

Regarding the continuity of the correspondence, to see φ is upper hemi-continuous, it is sufficient to prove φ has a closed graph (by Theorem 17.11 in Aliprantis and Border [2007]). Take $G^k = [i_1^k, ..., i_n^k] \to [i_1, ..., i_n] = G$ and $x^k \to x$ in X such that $x^k \in \varphi(G^k)$, we have $x_i^k \to x_i$ for all $i \in I$. By Lemma 24, there are permutations $\sigma_k \in S_n$ such that $i^k_{\sigma_k(j)} \to i_j$ for all $1 \leq j \leq n$. Therefore, by Assumption (C3),

$$\sum_{i \in G} x_i = \sum_{j=1}^n x_{i_j} = \lim_{k\to\infty} \sum_{j=1}^n x_{\sigma_k(i_j^k)} = \lim_{k\to\infty} \sum_{j=1}^n \omega^x_{\sigma_k(i_j^k)} = \sum_{j=1}^n \omega^x_{i_j} = \sum_{i\in G} \omega_i^x$$

Lastly, to see φ is lower hemi-continuous, we take any $G^k \to G$ in $\mathcal{G}_n$ and any $x \in \varphi(G)$. If $G^k = [i_1^k, ..., i_n^k]$ and $G = [i_1, ..., i_n]$, then

1. by Lemma 24, there are permutations $\sigma_k \in S_n$ such that $i^k_{\sigma_k(j)} \to i_j$ for all $1 \leq j \leq n$.
2. $\sum_{j=1}^n x_{i_j} = \sum_{j=1}^n \omega^x_{i_j}$.

Now we construct $x^k \in \varphi(G^k)$ such that $x^k \to x$. To start with, we note $i^k_{\sigma_k(j_1)} = i^k_{\sigma_k(j_2)}$ only if $i_{j_1} = i_{j_2}$ for all large k: for any $\varepsilon > 0$, we take k large enough so that $|i^k_{\sigma_k(j)} - i_j| < \varepsilon$ for all $\varepsilon > 0$. If $i_{j_1} \neq i_{j_2}$, then we have $|i^k_{\sigma_k(j_1)} - i^k_{\sigma_k(j_2)}| \geq |i_{j_1} - i_{j_2}| - |i^k_{\sigma_k(j_1)} - i_{j_1}| - |i^k_{\sigma_k(j_2)} - i_{j_2}| \geq |i_{j_1} - i_{j_2}| - 2\varepsilon$. Since we wish to show the convergence, we take k large enough to have this property. Next, for every such large k and every $m \in I$, we define

$$x^k_m = \begin{cases} \dfrac{\sum_{j=1}^n \omega^x_{i^k_j}}{\sum_{j=1}^n \omega^x_{i_j}} x_{i_j}, & \text{if } m = i^k_{\sigma_k(j)} \text{ for some } j \\ x_m, & \text{otherwise} \end{cases}$$

The function $(x^k_m : m \in I)$ has three properties:

1. it is well defined since $i^k_{\sigma_k(j_1)} = i^k_{\sigma_k(j_2)}$ only if $i_{j_1} = i_{j_2}$.
2. $(x^k_m : m \in I) \in \varphi(G^k)$, since
$$\sum_{j=1}^n x^k_{i^k_j} = \sum_{j=1}^n x^k_{i^k_{\sigma_k(j)}} = \frac{\sum_{j=1}^n \omega^x_{i^k_j}}{\sum_{j=1}^n \omega^x_{i_j}} \sum_{j=1}^n x_{i_j} = \sum_{j=1}^n \omega^x_{i^k_j}$$
3. by Assumption (C3), $x^k_{i^k_{\sigma_k(j)}} \to x_{i_j}$ for all $1 \leq j \leq n$ as k goes to infinity.

Now, we prove $x^k_m \to x_m$ for all $m \in I$. Firstly, when $m \neq i_j$ for any $1 \leq j \leq n$, it is easy to see $m \neq i^k_{\sigma_k(j)}$ for any $1 \leq j \leq n$ for all large enough k. Therefore, $x^k_m = x_m$. Thus, $x^k_m \to x_m$. When $m = i_j$ for some $1 \leq j \leq n$, we consider a subsequence of $(x^k_m)_{k\in\mathbb{N}}$ such that $m = i^k_{\sigma_k(j_k)}$ for some permutations $\sigma_k \in S_n$ and $1 \leq j_k \leq n$. (If such sequence does not exists, the convergence is automatically satisfied.) But we note, as $i^k_{\sigma_k(j_1)} = i^k_{\sigma_k(j_2)}$ only if $i_{j_1} = i_{j_2}$, $j_k = j$ for all $k \in \mathbb{N}$. Therefore, by the third property of the function $(x^k_m : m \in I)$, $x^k_m = x^k_{i^k_{\sigma_k(j)}} \to x_{i_j}$.

In sum, the correspondence φ defined in Equation 25 is continuous. By Berge's maximization theorem (Theorem 17.31 in Aliprantis and Border [2007]) and Assumption (C3), $s(G) = \max_{x\in\varphi(G)} f(G, x) + \sum_{i\in G} \omega^m_i$ is continuous.

Lastly, taking $a_n(i) = u_i(NM\vec{e})$, by Assumption (C4), a_n is continuous and bounded. Therefore, since μ is a finite measure, $a_n \in L^1(I, \mu)$. By the monotonicity assumption Assumption (C1), $s(G) \leq \sum_{i\in G} a_n(i)$ for all groups $G \in \mathcal{G}_n$.

D.2 Proof of Theorem 5

In this subsection, we prove Theorem 5. By Lemma 9 and Theorem 1, there is an assignment τ and an imputation $(v_i)_{i\in I}$ such that

$$\sum_{i\in G} v_i = s(G), \forall G \in \mathrm{supp}(\tau)$$

$$\sum_{i\in G} v_i \geq s(G), \forall G \in \mathcal{G}$$

For any $G \in \text{supp}(\tau)$, as we have proved in Appendix D.1, there are $(x_i^G)_{i\in G}$ such that

$$x_i^G \geq 0, \sum_{i\in G} u_i(x_i^G) = s(G) - \sum_{i\in G} \omega_i^m$$

Moreover, we define $(m_i^G)_{i\in G}$ such that

$$m_i^G \in \mathbb{R}, m_i^G = v_i - u_i(x_i^G)$$

Therefore,

$$\sum_{i\in G} U_i(m_i^G, x_i^G) = s(G), \forall G \in \text{supp}(\tau)$$

$$\sum_{i\in G} U_i(m_i^G, x_i^G) \geq s(G), \forall G \in \mathcal{G}$$

i.e. $(\tau, ((m_i^G, x_i^G)_{i\in G})_{G\in\text{supp}(\tau)})$ is a stable state.

D.3 Proof of Theorem 7

In this subsection, we prove Theorem 7. Firstly, at any competitive equilibrium represented by the triplet $(p, \tau, ((m_i^G, x_i^G)_{i\in G})_{G\in\text{supp}(\tau)})$, the same type of agent gets the same payoff due to the individual maximization. That is, for any $i \in I$, there is a function $v : I \to \mathbb{R}$ such that

$$U_i(m_i^G, x_i^G) = v(i), \forall G \in \text{supp}(\tau), i \in G$$

Now we prove the allocation satisfies no-blocking condition by contradiction. Suppose there is an n-person group $[i_1, ..., i_n] \in \mathcal{G}_n$ such that $\sum_{k=1}^n v(i_k) < s([i_1, ..., i_n])$. Then, every agent in the group $G = [i_1, ..., i_n]$ can be strictly better off by some reallocation. That is, there are consumption bundles $(m_{i_k}, x_{i_k}) \in \mathbb{R}\times\mathbb{R}_+^L$, for every $1 \leq k \leq n$, such that

$$x_{i_k} \geq 0, \forall 1 \leq k \leq n, \sum_{k=1}^n (m_{i_k}, x_{i_k}) = \sum_{k=1}^n (\omega_{i_k}^m, \omega_{i_k}^x)$$

and $U_{i_k}(m_{i_k}, x_{i_k}) > v(i_k)$. On the other hand, since p is the equilibrium price, no one in this group can afford the bundle (m_{i_k}, x_{i_k}), i.e. $m_{i_k} > \omega_{i_k}^m + p(\omega_{i_k}^x - x_{i_k}), \forall 1 \leq k \leq n$. Summing over k, we have $\sum_{k=1}^n p(\omega_{i_k}^x - x_{i_k}) < 0$. However, when $n = 2$, by the oddness of the price function p, $\sum_{k=1}^2 p(\omega_{i_k}^x - x_{i_k}) = 0$. Contradiction.

D.4 Proof of Lemma 11

In this subsection, we prove Lemma 11. For any $(\Delta m, \Delta x) \in B$, by the definition of p_B, we have $\Delta m \leq -p_B(\Delta x)$. Therefore, $B \subset \{(\Delta m, \Delta x) \in \mathbb{R}\times\mathbb{R}^L : \Delta m \leq -p_B(\Delta x)\}$. Conversely, for any $(\Delta m, \Delta x) \notin B$, by the definition of p_B, we have $\Delta m \geq -p_B(\Delta x)$. On the other hand, since B is a closed set, $(-p_B(\Delta x), \Delta x)$ is in B. Therefore, $\Delta m > -p_B(\Delta x)$. That is, $B \supset \{(\Delta m, \Delta x) \in \mathbb{R}\times\mathbb{R}^L : \Delta m \leq -p_B(\Delta x)\}$. In sum, we have $B = \{(\Delta m, \Delta x) \in \mathbb{R}\times\mathbb{R}^L : \Delta m \leq -p_B(\Delta x)\}$.

D.5 Proof of Lemma 12

In this subsection, we prove Lemma 12. Suppose the function p_B is non-decreasing, for any $\Delta x' \leq \Delta x$, $p_B(\Delta x') \leq p_B(\Delta x)$. Therefore, if $\Delta m' \leq \Delta m$, by $(\Delta m, \Delta x) \in B$, we have $\Delta m' \leq \Delta m \leq -p_B(\Delta x) \leq -p_B(\Delta x')$. That is, if $(\Delta m, \Delta x) \in B$, we have $(\Delta m', \Delta x') \in B$. Conversely, we suppose, for any point $(\Delta m, \Delta x) \in B$, $(\Delta m', \Delta x') \leq (\Delta m, \Delta x)$ implies $(\Delta m', \Delta x') \in B$. Then, for any $\Delta x' \leq \Delta x$, $(\Delta m, \Delta x') \in B$ if $(\Delta m, \Delta x) \in B$. Hence, $\sup_{(\Delta m, \Delta x') \in B} \Delta m \geq \sup_{(\Delta m, \Delta x) \in B} \Delta m$, which implies $p_B(\Delta x') \leq p_B(\Delta x)$.

D.6 Proof of Lemma 16

In this subsection, we prove Lemma 16. We prove by contradiction. We start by supposing that there is a pair $(\Delta m, \Delta x) \in \mathbb{R} \times \mathbb{R}^L$ such that $(\Delta m_{i_k}^{G_k}, \Delta x_{i_k}^{G_k}) \to (\Delta m, \Delta x)$ and $\Delta m > \Delta m_i^G, \Delta x = \Delta x_i^G$ for some $G \in \text{supp}(\tau)$, $i \in G$ and a sequence of pairs $(\Delta m_{i_k}^{G_k}, \Delta x_{i_k}^{G_k})$ in Δ. Thus, we have

$$-(\Delta m_{i_k}^{G_k}, \Delta x_{i_k}^{G_k}) + (\Delta m_i^G, \Delta x_i^G) \to -(\Delta m - \Delta m_i^G, 0)$$

That is, when k is large enough, all trading partners of a type i_k agent in group G_k and a type i agent in group G can form a blocking group, if they are willing to give up some small amount of consumption goods consumption in exchange for a large monetary boost.

To make this argument rigorous, we study the vector of the total net trade on consumption goods in the new group $-\Delta x_{i_k}^{G_k} + \Delta x_i^G$. When $-\Delta x_{i_k}^{G_k} + \Delta x_i^G \leq 0$, it is feasible for everyone to consume their consumption at the stable state with some additional monetary goods. Therefore, all trade partners of a type i_k agent in group G_k and a type i agent in group G form a blocking group. When $(-\Delta x_{i_k}^{G_k} + \Delta x_i^G)_l > 0$ for some commodity $l \in \{1, 2, ..., L\}$, there are some excess demand for good l in the new group. When k is very large, this excess demand is very small so that a subset of the group members are willing to give up a little bit in exchange of a nontrivial monetary boost. Formally, to be clear, we relabel the new group by $[i_1, .., i_n] \in \mathcal{G}_n$ for some group size n. Their net trades at the stable state are $(\Delta m_{i_j}, \Delta x_{i_j})_{j=1,...,n}$. For every l such that $(\sum_{j=1}^n \Delta x_{i_j})_l > 0$, we define, for each $j \in \{1, 2, ..., n\}$,

$$(\Delta y_{i_j})_l = \begin{cases} (\Delta x_{i_j})_l, & \text{if } (\Delta x_{i_j})_l \leq 0 \\ (\Delta x_{i_j})_l - \frac{(\Delta x_{i_j})_l}{\sum_{j:(\Delta x_{i_j})_l > 0} (\Delta x_{i_j})_l} \sum_{j=1}^n (x_{i_j})_l, & \text{if } (\Delta x_{i_j})_l > 0 \end{cases}$$

It is routine to check $y_{i_j} = \omega_{i_j}^x + \Delta y_{i_j} \geq 0$ and $\sum_{j=1}^n \Delta y_{i_j} = 0$. Moreover, for any $\varepsilon > 0$, we can take k to be large enough such that y_{i_j} is in an ε-neighborhood of x_{i_j}. Taking

ε small enough, by the uniform equicontinuity lemma, Lemma 15, we have

$$\frac{\Delta m - \Delta m_i{}^G}{n} + u_{i_j}(y_{i_j}) > u_{i_j}(x_{i_j}), \forall j = 1, 2, ..., n$$

Therefore, the trade partners of a type i_k agent in group G_k and a type i agent in group G form a blocking group, for large enough k.

D.7 Proof of Lemma 17

In this subsection, we prove Lemma 17. We verify the properties in order. (P1) follows from the definition.

For (P2), by Lemma 12, it is sufficient to show if $(\Delta m, \Delta x) \in B_0$, then any $(\Delta m', \Delta x') \leq (\Delta m, \Delta x)$ is in B_0. Since $(\Delta m, \Delta x) \in B_0$, there are $(\Delta m^k, \Delta x^k) \to (\Delta m, \Delta x)$ and $(\Delta m^k, \Delta x^k) \in (-\mathcal{P}_{i_k}) \cap (\mathbb{R}_- \times \mathbb{R}^L)$ for some $i_k \in I$. By Assumption (C1), utility functions are weakly increasing. Therefore, $(\Delta m^k, \Delta x^k) + (\Delta m', \Delta x') - (\Delta m, \Delta x) \in (-\mathcal{P}_{i_k}) \cap (\mathbb{R}_- \times \mathbb{R}^L)$ as $(\Delta m^k, \Delta x^k) + (\Delta m', \Delta x') - (\Delta m, \Delta x) \leq (\Delta m^k, \Delta x^k)$. On the other hand,

$$(\Delta m^k, \Delta x^k) + (\Delta m', \Delta x') - (\Delta m, \Delta x) \to (\Delta m', \Delta x')$$

as $k \to \infty$. Therefore, $(\Delta m', \Delta x') \in B_0$.

We prove that B_0 satisfies (P3) by contradiction. If $int(\mathcal{P}_j) \cap B_0 \neq \emptyset$ for some $j \in I$, we have $int(\mathcal{P}_j) \bigcap \left[(-\mathcal{P}_i) \cap (\mathbb{R}_- \times \mathbb{R}^L)\right] \neq \emptyset$ for some $i \in I$. Therefore, we have $int(\mathcal{P}_j) \cap \mathcal{P}_i \neq \emptyset$, which contradicts Lemma 14.

(P4) follows from the definition that

$$B_0 = cl \left[\bigcup_{i \in I} \left((-\mathcal{P}_i) \cap (\mathbb{R}_- \times \mathbb{R}^L) \right) \right]$$

Lastly, for (P5), by definition of B_0, $-(\Delta m_i^G, \Delta x_i^G) \in B_0$ if $(\Delta m_i^G, \Delta x_i^G) \in \Delta$ and $\Delta m_i^G \geq 0$. It remains to show that $\Delta m_i^G = -p_{B_0}(\Delta x_i^G)$ for all $(\Delta m_i^G, \Delta x_i^G) \in \Delta$ such that $\Delta m_i^G \geq 0$. We prove by contradiction. If there is $(-\Delta m, -\Delta x_i^G) \in B_0$ for some $0 \leq \Delta m < \Delta m_i^G$, then, there is a sequence $(-\Delta m_k, -\Delta x_k) \to (-\Delta m, -\Delta x_i^G)$ and $(-\Delta m_k, -\Delta x_k) \in -\mathcal{P}_{i_k}$ for some $i_k \in I$. i.e. $(\Delta m_k, \Delta x_k) \to (\Delta m, \Delta x_i^G)$. Then, the trading partners of a type i agent in group G and some type i_k agent will form a blocking group, when k is large enough. See Lemma 16 for a proof.

D.8 Proof of Lemma 18

In this subsection, we prove Lemma 18. Firstly, to prove $B_- \in \mathcal{B}$, we check $B_- \subset \mathbb{R}_- \times \mathbb{R}^L$ and satisfies (P1) - (P5). By definition, $-\mathcal{P}_i \cap (\mathbb{R}_- \times \mathbb{R}^L) \subset B_- \subset \mathbb{R}_- \times \mathbb{R}^L$ for all $i \in I$, is closed set. Thus, B_- satisfies (P1) and (P4).

For (P2), to see p_{B_-} is non-decreasing, by Lemma 12, it is sufficient to check for any $(\Delta m', \Delta x') \leq (\Delta m, \Delta x)$ in $\mathbb{R} \times \mathbb{R}^L$, if $(\Delta m, \Delta x) \in B_-$, then $(\Delta m', \Delta x') \in B_-$. Given $(\Delta m, \Delta x) \in B_-$, by definition, there is a sequence $(\Delta m^k, \Delta x^k) \in B^k$, where $B^k \in \mathcal{B}$, such that $(\Delta m^k, \Delta x^k) \to (\Delta m, \Delta x)$. Since $B^k \in \mathcal{B}$ satisfies (P2), $(\Delta m', \Delta x') + (\Delta m^k, \Delta x^k) - (\Delta m, \Delta x) \in B^k$ as $(\Delta m', \Delta x') + (\Delta m^k, \Delta x^k) - (\Delta m, \Delta x) \leq (\Delta m^k, \Delta x^k)$. Moreover, $(\Delta m', \Delta x') + (\Delta m^k, \Delta x^k) - (\Delta m, \Delta x) \to (\Delta m', \Delta x')$ as $k \to +\infty$. Therefore, $(\Delta m', \Delta x') \in B_-$.

Furthermore, we verify B_- satisfies (P3) by contradiction. If $(\Delta m, \Delta x) \in B_- \cap int(\mathcal{P}_i)$ for some $i \in I$, there is a sequence $(\Delta m^k, \Delta x^k) \in B^k$ where $B^k \in \mathcal{B}$ such that $(\Delta m^k, \Delta x^k) \to (\Delta m, \Delta x)$. Since $(\Delta m, \Delta x) \in int(\mathcal{P}_i)$, for all large k, $(\Delta m^k, \Delta x^k) \in int(\mathcal{P}_i)$. Consequently, $(\Delta m^k, \Delta x^k) \in B^k \cap int(\mathcal{P}_i)$. That is, $B_k \in \mathcal{B}$ does not satisfy (P3). Contradiction.

Lastly, to verify (P5), we first observe that $p_{B_-} = \inf_{B \in \mathcal{B}} p_B$ on $\mathbb{R}^L$. By the definition of p_{B_-} in Equation 19, it is equivalent to prove

$$-\sup_{(\Delta m, \Delta x) \in B_-} \Delta m = \inf_{B \in \mathcal{B}} \left(-\sup_{(\Delta m, \Delta x) \in B} \Delta m \right)$$

Firstly, since $B \subset B_-$ for all $B \in \mathcal{B}$, we have $-\sup_{(\Delta m, \Delta x) \in B_-} \Delta m \leq -\sup_{(\Delta m, \Delta x) \in B} \Delta m$. Taking the infimum over all $B \in \mathcal{B}$, we have

$$-\sup_{(\Delta m, \Delta x) \in B_-} \Delta m \leq \inf_{B \in \mathcal{B}} \left(-\sup_{(\Delta m, \Delta x) \in B} \Delta m \right)$$

Conversely, since B_- is closed and in the lower half space, there is a pair $(\Delta m^*, \Delta x^*) \in B_-$ such that $-\Delta m^* = -\sup_{(\Delta m, \Delta x) \in B_-} \Delta m$. By definition of B_-, there is a sequence $(\Delta m^k, \Delta x^k) \to (\Delta m^*, \Delta x^*)$ where $(\Delta m^k, \Delta x^k) \in B^k$ for some $B^k \in \mathcal{B}$. Clearly, $-\sup_{(\Delta m, \Delta x) \in B^k} \Delta m \leq -\Delta m^k$. Therefore, the

$$\inf_{B \in \mathcal{B}} \left(-\sup_{(\Delta m, \Delta x) \in B} \Delta m \right) \leq \inf_{k \to +\infty} \left(-\sup_{(\Delta m, \Delta x) \in B^k} \Delta m \right) \leq \inf_{k \to +\infty} \left(-\Delta m^k \right) = -m^*$$

That is,

$$-\sup_{(\Delta m, \Delta x) \in B_-} \Delta m \geq \inf_{B \in \mathcal{B}} \left(-\sup_{(\Delta m, \Delta x) \in B} \Delta m \right)$$

Hence, we have proved $p_{B_-} = \inf_{B \in \mathcal{B}} p_B$ on $\mathbb{R}^L$. For any $(\Delta m_i^G, \Delta x_i^G) \in \Delta$ such that $\Delta m_i^G \geq 0$ and any $B \in \mathcal{B}$, we have $-(\Delta m_i^G, \Delta x_i^G) \in \partial B$. Therefore, $\Delta m_i^G = p_B(-\Delta x_i^G)$. On the other hand, which implies $p_{B_-}(-\Delta x_i^G) = \inf_{B \in \mathcal{B}} p_B(-\Delta x_i^G) = \Delta m_i^G$. Hence, $-(\Delta m_i^G, \Delta x_i^G) \in \partial B_-$.

D.9 Proof of Lemma 19

In this subsection, we prove Lemma 19. We prove by contradiction. Suppose there is a $\Delta x^* \in \mathbb{R}^L$ such that $p_{B_-}(\Delta x^*) \geq p_{B_-}(-\Delta x^*) > 0$, we will construct some other set such that B_- is the largest element in $\mathcal{B}$. For some small number $\varepsilon > 0$ determined later, we define

$$Z = \left\{(\Delta m, \Delta x) \in \mathbb{R} \times \mathbb{R}^L : (\Delta m, \Delta x) \leq (-p_{B_-}(-\Delta x^*) + \varepsilon, -\Delta x^*)\right\} \tag{26}$$

That is, Z contains all points smaller than or equal to $\left(-p_{B_-}(-\Delta x^*) + \varepsilon, -\Delta x^*\right) \notin B_-$. But $Z \subset \mathbb{R}_- \times \mathbb{R}^L$.

We will show $B_- \cup Z \in \mathcal{B}$, as it will yield a contradiction with that B_- is the maximal element in $\mathcal{B}$. That is, we verify $B_- \cup Z$ satisfies $(P1)$ to $(P5)$: for $(P1)$, $B_- \cup Z$ is closed as both B_- and Z are closed. For $(P2)$, we note both B_- and Z has the property that if a point is in the set, then any point less than this point is in the set. Therefore, by Lemma 6, $p_{B_- \cup Z}$ is non-decreasing.

To verify (P3), we only need to show, when ε small enough, $Z \cap int(\mathcal{P}_i) = \emptyset$ for all $i \in I$ since B_- has this property. To show, when ε small enough, $Z \cap int(\mathcal{P}_i) = \emptyset$ for all $i \in I$, it is sufficient to show $(-p_{B_-}(-\Delta x^*), -\Delta x^*) \notin cl(\cup_{i \in I} \mathcal{P}_i)$, which implies $Z \cap int(\mathcal{P}_i) = \emptyset$ for any $i \in I$ when $\varepsilon > 0$ is small enough. We prove by contradiction. Suppose $(-p_{B_-}(-\Delta x^*), -\Delta x^*) \in cl(\cup_{i \in I} \mathcal{P}_i)$, then there is a sequence $(\Delta m_k, \Delta x_k) \in \mathcal{P}_{i_k}$, for some $i_k \in I$, such that $(\Delta m_k, \Delta x_k) \to (-p_{B_-}(-\Delta x^*), -\Delta x^*)$. Therefore, $(-\Delta m_k, -\Delta x_k) \in -\mathcal{P}_{i_k}$ and $-\Delta m_k > 0$ for all large k. Thus, $(0, -\Delta x_k) \in (-\mathcal{P}_{i_k}) \cap \mathbb{R}_- \times \mathbb{R}^L$ for all large k. Consequently, taking $k \to \infty$, by the closedness of B_0, we have $(0, -\Delta x^*) \in B_0$. That is, $p_{B_0}(\Delta x^*) = 0$, which implies $p_{B_-}(\Delta x^*) = 0$. Contradiction with the assumption.

The fact that $B_- \cup Z$ satisfies (P4) follows from that, for all $i \in I$,

$$-\mathcal{P}_i \cap (\mathbb{R}_- \times \mathbb{R}^L) \subset B_- \subset B_- \cup Z$$

Lastly, when we verify (P3), we have shown $(-p_{B_-}(-\Delta x^*), -\Delta x^*) \notin cl(\cup_{i \in I} \mathcal{P}_i) \supset cl(\Delta)$. Therefore, $(-p_{B_-}(-\Delta x^*), -\Delta x^*) \notin cl(\Delta)$. Consequently, for small enough $\varepsilon > 0$, there is no net trade $(\Delta m_i^G, \Delta x_i^G) \in \Delta$ in the ε-neighborhood of the point $(-p_{B_-}(-\Delta x^*), -\Delta x^*)$. Therefore, all net trade vectors with monetary gains are on the boundary of $B_- \cup Z$. That is, $B_- \cup Z$ satisfies (P5).

D.10 Proof of Lemma 20

In this subsection, we prove Lemma 20. Firstly, we prove p_{B_-} is real valued by contradiction: suppose $p_{B_-}(\Delta x^*) = +\infty$ for some $\Delta x^* \in \mathbb{R}^L$. Define

$$Z = \left\{(\Delta m, \Delta x) \in \mathbb{R} \times \mathbb{R}^L : (\Delta m, \Delta x) \leq (y, -\Delta x^*)\right\}$$

where $y = -\sup_{i \in I} u_i(NM\vec{e})$. By Assumption (C4), $y \in \mathbb{R}$. By Assumption (C1) and Assumption (C3), $y < \inf_{(\Delta m, \Delta x) \in \cup_i \mathcal{P}_i} \Delta m$. Therefore, we have $B_- \subsetneqq B_- \cup Z \in \mathcal{B}$, which contradicts the definition of B_-.

Next, to prove the continuity of p_{B_-}, we verify, for all $\Delta x^* \in \mathbb{R}^L$,

$$\limsup_{\Delta x^k \to \Delta x^*} p_{B_-}(\Delta x^k) = \liminf_{\Delta x^k \to \Delta x^*} p_{B_-}(\Delta x^k) = p_{B_-}(\Delta x^*)$$

On the one hand, if $\liminf_{\Delta x^k \to \Delta x^*} p_{B_-}(\Delta x^k) < p_{B_-}(\Delta x^*)$, by taking a subsequence, we can assume $\lim_{k\to\infty} p_{B_-}(\Delta x^k) = q < p_{B_-}(\Delta x^*)$. Since B_- is closed, we have $(-p_{B_-}(\Delta x^k), \Delta x^k) \in B_-, \forall k \in \mathbb{N}$. Again by the closedness of B_-, $(-q, \Delta x^*) \in B_-$. But $q < p_{B_-}(\Delta x^*)$ contradicts the definition of p_{B_-}. Therefore, $\liminf_{\Delta x^k \to \Delta x} p_{B_-}(\Delta x^k) = p_{B_-}(\Delta x^*)$.

Conversely, if $\limsup_{\Delta x^k \to \Delta x} p_{B_-}(\Delta x^k) > p_{B_-}(\Delta x^*)$ for some $\Delta x^* \in \mathbb{R}^L$, we show B_- is not the largest element in $\mathcal{B}$. Firstly, as $p_{B_-} \geq 0$, by choosing the subsequence, we assume $\lim_{k\to+\infty} p_{B_-}(\Delta x^k) = q > p_{B_-}(\Delta x^*)$. We define

$$Z = \Big\{(\Delta m, \Delta x) \in \mathbb{R} \times \mathbb{R}^L : (\Delta m, \Delta x) \leq (-p_{B_-}(\Delta x^k) + 1/k, \Delta x^k)\Big\}$$

Clearly, $Z \not\subset B$. We will show for large enough k, $B_- \cup Z \in \mathcal{B}$. We verify $B_- \cap Z$ satisfies (P1) to (P5). (P1), (P2) and (P4) are direct consequences of the definition. To verify (P3), it is sufficient to show that $(-q, \Delta x^*) \notin cl(\cup_{i\in I} \mathcal{P}_i)$, as if it is true, there is a ε-neighborhood of $(-q, \Delta x^*)$ does not intersect any interior of translated better than set $\mathcal{P}_i$. We take a sequence $(\Delta m_k, \Delta x_k) \to (-q, \Delta x^*)$ and $(\Delta m_k, \Delta x_k) \in \mathcal{P}_{i_k}$ for some $i_k \in I$. Fixed k large enough, there are two cases: firstly, there is some $\Delta m^* \in \mathbb{R}$ such that $(\Delta m^*, \Delta x^*) \in \mathcal{P}_{i_k}$. We take Δm^* to be the smallest such choice. Since the closed set B_- satisfies (P3), $\Delta m^* \geq -p_{B_-}(\Delta x^*)$. That is, $\Delta m^* - \Delta m_k \geq q - p_{B_-}(\Delta x^*)$ but $\Delta x_k \to \Delta x^*$, contradicts the uniform equicontinuity lemma Lemma 15. Secondly, no there is some $\Delta m^* \in \mathbb{R}$ such that there is no $\Delta m^* \in \mathbb{R}$ such that $(\Delta m^*, \Delta x^*) \in \mathcal{P}_{i_k}$. Recall Equation 18, we note $-\omega^x_{i_k}$ must in a neighborhood of Δx_k. By the uniform equicontinuity lemma Lemma 15, $u(\omega^x_{i_k} - \Delta x_k) - u(0)$ is small. But $\inf_{(\Delta_m, -\omega^x_{i_k})} \Delta_m - v_{i_k} \geq 0$, contradict with the fact that $\inf_{(\Delta_m, \Delta x_k)} \Delta_m \leq \Delta m_k \to q < 0$. Contradiction. Therefore, $(-q, \Delta x^*) \notin cl(\cup_{i\in I} \mathcal{P}_i)$.

Lastly, to verify (P5), it is sufficient to prove $(q, -\Delta x^*)$ is not an accumulation point for the set $\{-(\Delta m_i^G, \Delta x_i^G) \in \Delta : \Delta m_i^G \geq 0\}$. Suppose it is true, then we know $(-q, \Delta x^*) \in cl(\Delta)$, contradicts Lemma 16.

D.11 Proof of Lemma 21

In this subsection, we prove Lemma 21. Since B_- satisfies (P3), $\Delta m_i^G \geq -p_{B_-}(\Delta x_i^G)$ for all $(\Delta m_i^G, \Delta x_i^G) \in \Delta$. Thus, it remains to show $\Delta m_i^G < -p_{B_-}(\Delta x_i^G)$ for all $(\Delta m_i^G, \Delta x_i^G) \in \Delta$. We prove by contradiction. Suppose $\Delta m_i^G > -p_{B_-}(\Delta x_i^G)$ for some $(\Delta m_i^G, \Delta x_i^G) \in \Delta$, we define

$$Z = \big\{(\Delta m, \Delta x) \in \mathbb{R} \times \mathbb{R}^L : (\Delta m, \Delta x) \leq (-p_{B_-}(\Delta x_i^G) + \varepsilon, \Delta x_i^G)\big\}$$

for some small $\varepsilon > 0$ determined later. It is clear $Z \not\subset B_-$. We will show $B_- \cup Z \in \mathcal{B}$ by verifying $(P1)$ to $(P5)$. Note it is easy to verify (P1), (P2) and (P4) by definition. To verify $B_- \cup Z$ satisfies (P3), it is sufficient to prove that $(-p_{B_-}(\Delta x_i^G), \Delta x_i^G) \notin cl(\cup_{i\in I}\mathcal{P}_i)$. We choose $\mathcal{P}_j$ arbitrarily close to $(-p_{B_-}(\Delta x_i^G), \Delta x_i^G)$, then, the trade partner of an type i agent in group G can improve their payoff by forming a coalition with an type j agent. To verify $B_- \cup Z$ satisfies (P5), it is sufficient to prove that $(-p_{B_-}(\Delta x_i^G), \Delta x_i^G) \notin cl(\Delta)$. The argument is similar.

D.12 Non-existence of a Linear Market Clearing Price (cont.)

In this subsection, we discuss the general case in Section 4.4.1.

In general, the market clearing price $p : \mathbb{R}^I \to \mathbb{R}$ is defined according to the number of non-negative entries in the net trade vector:

Case 1 when there are no less than $I-1$ non-negative entries, we define the function value to be the second smallest entry in the net-trade vector. (i.e., after reordering, when $\Delta x_1 \geq ... \geq \Delta x_I$, $p(\Delta x_1, ..., \Delta x_I) = \Delta x_{I-1}$)

Case 2 when there are no more than 1 non-negative entries, we define $p(\Delta x_1, ..., \Delta x_I)$ to be $-p(-\Delta x_1, ..., -\Delta x_I)$. (Note, $(-\Delta x_1, ..., -\Delta x_I)$ is necessarily in category 1, thus is well-defined.)

Case 3 for all other cases, set the function value to be zero.

Next, we check that such function is an odd, increasing and continuous function, and it is a market clearing price at any stable state.

First, the oddness is by definition.

Second, to see the function is increasing, we note any non-negative increment on the net trade bundle will not reduce the number of non-negative entries in the vector. For the net trade in Case 1, an non-negative increment will weakly increase the second smallest entry. For the net trade in Case 3, an increment will either maintain the function value to be zero, or make it positive. For the net trade in Case 2, we only need to check the increase in argument will not result in a smaller negative function value. This could be proved by noticing that $(\Delta x_1', ..., \Delta x_I') \geq (\Delta x_1, ..., \Delta x_I)$ in Case 2 implies $(-\Delta x_1', ..., -\Delta x_I') \leq (-\Delta x_1, ..., -\Delta x_I)$ in Case 1.

Third, to see the function is continuous, we first note that the function p is continuous in each of these three cases. (For instance, in the region of Case 1, the function p is continuous: define I functions $f_i(\Delta x_1, ..., \Delta x_I) = \Delta x_i$. Then, in this region, p is the maximum of I minimum of $I-1$ many functions in this set, thus is continuous.) Moreover, on the boundary of the Case 1 region, the second smallest entry must be zero: (Pick any point on the boundary. Case 3.1: if there is no zero entry in this net trade vector, k positive entries and $I - k$ negative entries, we must have $k \geq I - 1$, since there is a small neighborhood such that all points in the neighborhood has this property and some point in this neighborhood must be in the region specified by Case 1. However, it means the whole neighborhood is in the region of Case 1. Not boundary, contradiction. Case 3.2: If there is only 1 zero entry in this boundary point, k positive entries and $I - k - 1$ negative entries: if $k = I - 1$, there is a small neighborhood of this point in the region of Case 1. Not boundary, contradiction. If $k = I - 2$, the second smallest entry is zero. If $k < I - 2$, there is a neighborhood such that, in the neighborhood, all points have at least two negative entries. Contradiction.) That is, p is continuous on the boundary of the region specified by Case 1. By oddness, it is also continuous at the region specified by Case 3.

Lastly, by symmetry, we prove the individual optimality for a type 1 agent. It is sufficient to show that

4.1 he has no interest in trade $(\Delta x_1, ..., \Delta x_I)$, where all entries are positive

4.2 he has no interest in trade $(-\Delta x_1, ..., \Delta x_I)$, where $0 \leq \Delta x_1 \leq 1$ and all other entries are non-negative.

4.3 he is indifference in trading $(-\Delta x_1, ..., \Delta x_I)$, where $0 \leq \Delta x_1 \leq 1$ and there is an zero entry in the later entries.

as he does not have enough endowment to sell two goods. To show 4.1, by buying every commodities, the payoff is increased by the minimum of these entries, but he needs to pay the second minimum, which is no less than the first minimum. To show 4.2, by trading this net trade bundle, the payoff is increased by the minimum of the nonnegative entries, but he also needs to pay this amount. So no strict incentive to do so. To show 4.3, the cost of implementing this trade is always zero by definition and the payoff increment is also zero.