Walrasian Tatonnement Stability near Autarchy without Differentiability and Interiority*

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Abstract

We prove that, when the initial endowment is close to a Pareto optimal allocation, there is a locally tatonnement stable equilibrium, provided that the utility functions are strictly increasing and strictly concave, and every good is indispensable to some consumer.

1 Introduction

Since the emergence of general equilibrium models, the question - in an economy, if the market price changes according to the rule: the price of a commodity goes up whenever its demand is larger than its supply, and goes down whenever its demand is smaller than its supply, whether the market price will converge to an equilibrium price - has attracted much attention. Among various formalizations of this question, we study a price adjustment rule formalized by the standard Walrasian tatonnement process, in which the rate of change for any commodity's price is equal to the excess demand of the commodity. We say an equilibrium is locally tatonnement stable if, starting from

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any price close to the equilibrium, the price path generated by tatonnement process converges to the equilibrium price.

Tatonnement stability is regarded as a very good property for the equilibrium state, as it not only implies the equilibrium is determinate, but also provides a way to compute the equilibrium. However, it is well known that the equilibrium of an economy is not locally tatonnement stable in general: first, Scarf [1960] specifies an economy with three consumers and three commodities, in which the equilibrium is not locally tatonnement stable. Second, Dierker [1972] showed that, if an economy has multiple equilibria, at least one equilibrium is not locally tatonnement stable. In addition, following Sonnenschein [1972] and Mantel [1974], Debreu [1974] showed that the excess demand function in an economy can be highly arbitrary. As local tatonnement stability of an equilibrium implies that the vector field generated by the excess demand function must point to the equilibrium locally, in order to obtain a positive result on the tatonnement stability, we must impose additional assumptions on the model.

To my knowledge, there are two main categories of positive results on the tatonnement stability of equilibrium in economies. The first category of results studies economies with a unique equilibrium. The second category of results studies economies near autarchy, i.e., the initial endowment is close to an equilibrium (Pareto optimal) allocation.

In the first category, proposed by Arrow et al. [1959], the gross substitutability is one of the most renown sufficient conditions to guarantee the tatonnement stability. This condition assumes that if the price of one commodity increases while all other prices remain the same, there will be an increase in demand for every commodity whose price has not changed. In other words, gross substitutability imposes constraints on the Jacobian matrix of the demand. On the other hand, Bewley [1980a,b] showed that the tatonnement stability holds if all utility functions are quasi-linear. That is, there is a special commodity, money, such that all utility functions are linear with respect to this commodity.

In the second category, the assumption that there exists a special commodity has also been explored. For instance, Jofré et al. [2017] assumed that there is a special commodity, money, that is indispensable to all agents. That is, every utility function has an infinite growth rate whenever the consumption of a special commodity, money, is zero.

With this assumption, they proved the equilibrium is locally tatonnement stable in an economy near autarchy. Actually, the intuition that the equilibrium is locally tatonnement stable in an economy near autarchy has been established in Sattinger [1975]. Nevertheless, Sattinger's argument relies on the technique of Slutsky decomposition, thus assuming that demand functions are differentiable and the equilibrium allocation is in the interior of the consumption set. Consequently, this argument cannot be generalized to study an economy with infinitely many commodities.[1] While not studying the infinite dimensional economy directly, Jofré et al. [2017] introduced a method to study the stability of an economy which does not rely on the assumptions on differentiability and interiority. Their result suggests that the tatonnement process can fix small perturbations at an equilibrium and bring the economy back to the equilibrium. For more details about the literature, we refer to Negishi [1962], McKenzie [2005] and Bryant [2010].

The main purpose of this paper is to provide a read of Jofré et al. [2017]. In particular, we generalize their result by dropping the assumption that there exists a special commodity money that is indispensable to every agent. Consequently, our result implies the result in Jofré et al. [2017], thus implying that in Sattinger [1975].

In the proof, we show that the price path generated by the tatonnement process converges to an equilibrium price in three steps. First, we introduce a sequence of perturbed economies such that commodity 1 is indispensable to all agents in every perturbed economy, and the size of perturbations converges to zero. In each such perturbed economy, we apply the result in Jofré et al. [2017], and prove that the price path converges to an equilibrium price. Then, we prove the equilibria in perturbed economies converge to an equilibrium in unperturbed economy. Furthermore, we prove that the price paths in perturbed economies converge to the price path in unperturbed economy by applying the Gronwall's inequality in the differential equation literature.

Lastly, we mention that the tatonnement process is not a trading mechanism as observed in Arrow [1959]. We follow Scarf's idea by treating the process as a fictional computational mechanism. Our discussion is thus purely on the level of mathematical

[1] In an economy with infinitely many commodities, demand functions are in general not differentiable (Araujo [1988]) and the Inada condition is usually incompatible with the existence of equilibrium (Araujo [1985]).

model. Regarding to the connection between tatonnement process and the real world problem, we refer to the experimental works in Hirota et al. [2005], Crockett et al. [2011].

The paper is organized as follows. In section 2, we describe the model and state the main theorem. In section 3, we prove the main theorem. In section 4, we conclude.

2 Model

We study a pure exchange economy $\mathcal{E} = (I, (u_i, \omega_i)_{i\in I})$. Here, I is a finite set of agents and there are L commodities, $u_i : \mathbb{R}^L_+ \to \mathbb{R}$ is agent i's utility function and $\omega_i \in \mathbb{R}^L_+ - \{0\}$ is agent i's initial endowment. We will use ω to denote the tuple $(\omega_i)_{i\in I} \in \mathbb{R}^{L|I|}_+$.

A vector $x = (x_i)_{i\in I}$ is a feasible allocation if $x_i \in \mathbb{R}^L_+$ for all $i \in I$ and $\sum_{i\in I}(x_i - \omega_i) = 0$. Here, the i-th entry, x_i, is agent i's consumption, and, $x_{i\ell}$, is the amount of commodity ℓ consumed by agent i. A price vector p is a vector in $\mathbb{R}^L_+$. We normalize the price vector such that $\|p\|_2 = 1$.[2]

A pair of price and feasible allocation $(\bar{p}, (\bar{x}_i)_{i\in I})$ is an equilibrium of the economy $\mathcal{E}$ if for all agent $i \in I$, we have

$$\bar{x}_i \in \text{argmax}\{u_i(x) : \bar{p} \cdot x \leq \bar{p} \cdot \omega_i\}$$

Given an equilibrium, we say $\bar{p}$ is an equilibrium price and $\bar{x} = (\bar{x}_i)_{i\in I}$ is an equilibrium allocation.

Next, we define the tatonnement process and tatonnement stability. Given a price p, agent i's demand map is defined by

$$\xi_i(p) = \text{argmax}\{u_i(x) : p \cdot x \leq p \cdot \omega_i\}$$

agent i's excess demand map is defined by

$$Z_i(p) = \xi_i(p) - \omega_i$$

and the aggregate excess demand map is defined by

$$Z(p) = \sum_{i\in I} Z_i(p)$$

[2]For a vector $x = (x_1, ..., x_L) \in \mathbb{R}^L$, $\|x\|_2 = \sqrt{x_1^2 + ... + x_L^2}$.

As we will assume that utility functions are strictly concave, all demand maps defined above are vector-valued functions. In addition, an equilibrium price $\bar{p}$ satisfies $Z(\bar{p}) = 0$ due to feasibility condition, and the Walras' law, $p \cdot Z(p) = 0$, holds for all prices p if all utility function are strictly increasing.

The tatonnement process is defined by the following ordinary differential equation:

$$\nabla p(t) = Z(p(t)) \tag{1}$$

with an initial condition $p(0) \in \mathbb{R}^n_+$. This formal expression implies that for any $1 \leq \ell \leq L$, $\frac{d}{dt} p_\ell(t) = Z_\ell(p(t))$.

Following the definition in Jofré et al. [2017], an equilibrium $(\bar{p}, \bar{x} = (\bar{x}_i)_{i \in I})$ is defined to be *locally tatonnement stable* if there is a neighborhood N_p of $\bar{p}$ in $\mathbb{R}^L_+$ and a neighborhood N_x of $\bar{x}$ in $\mathbb{R}^{L|I|}_+$ such that, whenever the initial price $p(0)$ is in N_p and the initial endowment ω is in N_x, the solution $p(t)$ of the tatonnement process converges to $\bar{p}$ as $t \to +\infty$.

By the first welfare theorem, any equilibrium is a Pareto optimal allocation. Therefore, an equilibrium is tatonnement stable if the tatonnement process can fix small perturbations by leading the economy back to its equilibrium.

Lastly, we line out our assumptions.

(A1) for all $i \in I$, u_i is twice continuously differentiable.

(A2) for all $i \in I$, $\nabla u_i(x) >> 0$ for all $x \in \mathbb{R}^L_+$.

(A3) for all $i \in I$, $\nabla^2 u_i(x)$ is negative definite for all $x \in \mathbb{R}^L_+$.

(A4) $\sum_{i \in I} \omega_i >> 0$

(A5) every commodity is indispensable[3] to some consumer

Assumption (A1) is the regularity assumption on utility functions. Assumptions (A2) and (A3) imply that all utility functions are strictly increasing and strictly concave. Assumption (A4) implies the endowments are not trivial. Assumption (A5) ensures the equilibrium price will be strictly positive. This assumption helps to clarify the proofs, and plays no essential role to the theorem.

Now, we are ready to state our main theorems.

[3] A commodity ℓ is indispensable to agent i if, for any $x \in \mathbb{R}^L_+$, the ℓ-th entry of the vector $\nabla u_i(x)$ is $+\infty$ whenever $x_\ell = 0$.

Theorem 1. *For an economy $\mathcal{E}$ satisfying Assumptions (A1) - (A5), every equilibrium $(\bar{p}, (\bar{x}_i)_{i\in I})$ is locally tatonnement stable.*[4]

3 Proof

In this section, we introduce the idea of the proof. Our proof depends on the result in Jofré et al. [2017]. In addition to the assumptions (A1) - (A5) we require, they needs one more assumption:

(A6) There is one commodity indispensable to all consumers

This special commodity is referred to as money, which exists in earlier works on this problem such as Bewley [1980a,b]. Assumption (A6) ensures that, in equilibrium, there is one commodity that every consumer consumes a positive amount. The statement of their theorem is as follows.

Theorem (Jofre, Rockafellar and Wets, 2017)**.** *For an economy $\mathcal{E}$ satisfying assumptions (A1) - (A6), every equilibrium is locally tatonnement stable. Moreover, the convergence speed is bounded by the inequality*

$$\|p(t) - \bar{p}\|_2 \leq e^{-\rho t} \|p(0) - \bar{p}\|_2$$

for some $\rho > 0$ and the demand functions are locally Lipchitz continuous.

Proof. See Jofré et al. [2017] for the proof of tatonnement stability and Dontchev and Rockafellar [2012] for the proof of Lipchitz continuity. ■

In this paper, we drop assumption (A6). To do so, we perturb the economy $\mathcal{E}$. In more detail, we perturb all utility functions such that some commodity, indexed by commodity 1, is indispensable to all agents in every perturbed economy. Moreover, we control the size of the perturbations such that they are small in an appropriate sense.

In the following, we call economy $\mathcal{E}$ as the original economy and define the n-th perturbed economy, or the n-th economy, by a tuple $\mathcal{E}^n = (I, (u_i^n, \omega_i)_{i\in I})$, where

$$u_i^n(x) = u_i(x) + \frac{1}{n^2}\eta(nx_1)$$

[4] We reemphasize that local tatonnement stability in this paper, different from some work in the literature, requires the initial endowment is close to the equilibrium allocation.

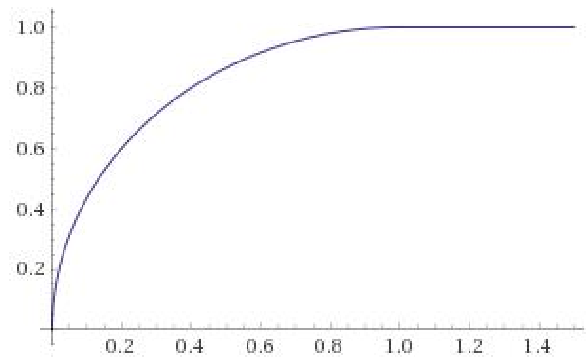

Figure 1: A Sketch of the function η

Here, $\eta \in \mathcal{C}^2(\mathbb{R}_+)$ is a concave and increasing real-valued function on $\mathbb{R}_+$ such that $\eta(y) = \sqrt{2y - y^2}$ in a neighborhood of 0 and $\eta'(y) = 0$ for all $y \geq 2$.[5] We note the n-th perturbation, $\frac{1}{n^2}\eta(nx)$, satisfies the following three properties:

- has an infinite growth rate at $x = 0$
- is bounded from above by $1/n^2$
- has an zero growth rate when $x \geq 2/n$

In particular, we set the coefficient to be $1/n^2$ rather than $1/n$ to ensure the demand of good 1 is about $1/n$ in perturbed economies when the demand of this good is zero in the original economy. We state and prove it formally in Lemma 2.

It is clear that, if the original economy $\mathcal{E}$ satisfies Assumptions (A1) - (A5), any n-th perturbed economy $\mathcal{E}^n$ satisfies Assumptions (A1) - (A6). In an perturbed economy $\mathcal{E}^n$, we denote its equilibrium by $(\bar{p}^n, (\bar{x}_i^n)_{i\in I})$, and the aggregate excess demand function by Z^n. Consequently, the tatonnement process in $\mathcal{E}^n$ is defined by

$$\nabla p^n(t) = Z^n(p^n(t)), p^n(0) = p(0) \tag{2}$$

To prove our main theorem, we prove three properties (P1) to (P3) as follows:

[5]To see such function exists, we can define a piecewise function that equals $\sqrt{2y - y^2}$ for $y \in [0, 1]$ and equals 1 for $y \in [1, +\infty)$. However, Such function is in $\mathcal{C}^1$ but not in $\mathcal{C}^2$ as the second derivative does not exist at the connection point. To deal with this problem, we can convolute it by a compactly supported bump function centered at the connecting point. The new function after appropriate scaling would be a smooth function satisfying all required properties.

(P1) There is a neighborhood of the equilibrium $(\bar{p}, (\bar{x}_i)_{i\in I})$ such that, if $(p(0), \omega)$ is in this neighborhood, $p^n(t) \to \bar{p}^n$ as $t \to +\infty$ for all $n \in \mathbb{N}$.

(P2) A subsequence of $(\bar{p}^n, \bar{x}^n)$ converges to $(\bar{p}, \bar{x})$.

(P3) The function $p^n(t)$ converges uniformly to $p(t)$ as $n \to +\infty$. [6]

(P1) implies that, given the initial price is close enough to the equilibrium, the tatonnement process in every perturbed economy converges. (P2) implies that perturbed economies and the original economy have equilibrium prices close to each other. Lastly, (P3) implies the price paths of perturbed economies and the original economy are close to each other. With the aid of these three properties, we are able to assert $p(t)$ converges to $\bar{p}$ by using the facts that $p(t)$ is close to $p^n(t)$, $p^n(t)$ is close to $\bar{p}^n$ and $\bar{p}^n$ is close to $\bar{p}$ for large t. Formally,

Lemma 1. *Given the original economy $\mathcal{E}$ and perturbed economies $\mathcal{E}^n$, if price paths $p(t)$ and $p^n(t)$ satisfy properties (P1) - (P3), then we have $p(t) \to \bar{p}$ as $t \to +\infty$.*

Proof. Firstly, by selecting subsequence of $\mathcal{E}^n$, we can replace (P2) by $(\bar{p}^n, \bar{x}^n)$ converges to $(\bar{p}, \bar{x})$ as $n \to +\infty$. Then, we note, for any $n \in \mathbb{N}$,

$$\|p(t) - \bar{p}\|_2 \le \|p(t) - p^n(t)\|_2 + \|p^n(t) - \bar{p}^n\|_2 + \|\bar{p}^n - \bar{p}\|_2$$

Thus, for any $\varepsilon > 0$, by (P3), there is an $N_1 \in \mathbb{N}$ such that for any $n \ge N_1$, we have $\|p(t) - p^n(t)\|_2 < \varepsilon/3$ for all $t \ge 0$. By (P2), there exist an $N_2 \in \mathbb{N}$ such that for all $n \ge N_2$, $\|\bar{p}^n - \bar{p}\|_2 < \varepsilon/3$. Now, we take n to be any number greater than $\max(N_1, N_2)$. By (P1), we can find a $T \ge 0$ such that for all $t \ge T$, $\|p^n(t) - \bar{p}^n\|_2 \le \varepsilon/3$. Therefore, for any $t \ge T$, we have $\|p(t) - \bar{p}\|_2 \le \varepsilon$. ■

Hence, to prove the main theorem, we verify properties (P1) to (P3) in the following subsections.

3.1 Verification of (P1)

Property (P1) follows essentially from Jofré et al. [2017].

[6] Actually, as Lemma 3 suggested, it is sufficient to prove the pointwise convergence of $p^n(t)$ to $p(t)$. Here we prove a stronger version for potential future use.

Proposition 1. *For a sequence of perturbed economies $\mathcal{E}^n$ with equilibria $(\bar{p}^n, (\bar{x}_i^n))_{i\in I}$ such that $\bar{p}^n \to \bar{p}$ for some $\bar{p} \in \mathbb{R}_+^L$, there are constants $\delta_x > 0$, $\delta_p > 0$ and $\rho > 0$, independent of n, such that whenever $\|\omega - \bar{x}^n\|_2 < \delta_x$*[7] *and $\|p(0) - \bar{p}^n\|_2 < \delta_p$, we have*

$$\|p^n(t) - \bar{p}^n\|_2 \leq e^{-\rho t} \|p(0) - \bar{p}^n\|_2$$

To prove this proposition, we start by estimate the demand of commodity 1 in a perturbed economy. There are two observations. Firstly, when the demand of commodity 1 is positive in the original economy, the demand of commodity 1 is the same in any perturbed economy if the perturbation is small. Secondly, when the demand of commodity 1 is zero in the original economy, the demand of commodity 1 is approximately $1/n$ in the n-th perturbed economy. These two observations is summarized in Lemma 2.

Lemma 2. *There is a neighborhood N_p of $\bar{p}$ and a constant $c > 0$ such that for all $p \in N_p$, all $i \in I$,*

$$\lim_{n\to+\infty} n\xi_{i1}^n(p) \geq c$$

To prove this lemma, we extend our perturbations to a continuum setting. Formally, for every $i \in I$, we define a function $U_i : \mathbb{R}_+^n \times \mathbb{R} \to \mathbb{R}$ by

$$U_i(x, \varepsilon) = u_i(x) + \varepsilon^2 \eta \left(\frac{x_1}{\varepsilon}\right)$$

Here, we oddly extend function η to the whole real line by setting $\eta(-x) = -\eta(x)$ for any $x > 0$ to allow the case that $\varepsilon < 0$. As $|\eta| \leq 1$, when $\varepsilon = 0$, $\varepsilon^2\eta(x_1/\varepsilon)$ is set to be zero.

Moreover, we note $U^i(\cdot, 0) = u_i(\cdot)$ and $U^i(\cdot, \frac{1}{n}) = u_i^n(\cdot)$ for all $n \in \mathbb{N}$. The corresponding aggregate excess demand map is thus defined by $Z(p, \varepsilon)$ for price vectors $p \in \mathbb{R}_{++}^L$ and $\varepsilon > 0$. Formally,

$$Z(p, \varepsilon) = \sum_{i\in I} (\xi_i(p, \varepsilon) - \omega_i)$$

where $\xi_i(p, \varepsilon) = \operatorname{argmax}\{U^i(x, \varepsilon) : p \cdot x \leq p \cdot \omega_i\}$.

Proof. By given by Lemma 4.2 in Gould et al. [2016], we have

$$\nabla_\varepsilon \xi_i(p, \varepsilon) = \left(H^{-1}p(p^T H^{-1} p)^{-1} p^T H^{-1} - H^{-1}\right) U_{\varepsilon x}^i(\varepsilon, \xi_i(p, \varepsilon))$$

[7] It means $\sum_{i\in I} \|\omega_i - \bar{x}_i^n\|_2 < \delta_x$

where

$$H = U^i_{xx}(\varepsilon, \xi_i(p,\varepsilon)) = \left[\nabla^2 u_i(x) + \eta''\left(\frac{x_1}{\varepsilon}\right) e_{11}\right]_{x=\xi_i(p,\varepsilon)}$$

$$U^i_{\varepsilon x}(\varepsilon, \xi_i(p,\varepsilon)) = \left[\eta'\left(\frac{x_1}{\varepsilon}\right) - \left(\frac{x_1}{\varepsilon}\right)\eta''\left(\frac{x_1}{\varepsilon}\right)\right]_{x=\xi_i(p,\varepsilon)} e_1$$

and e_1 is the unit vector having 1 in the first coordinate, and e_{11} is the matrix having 1 at entry (1,1) and 0 elsewhere.

Below, we will prove two claims. Firstly, when $\xi_{i1}(p,0) > 0$, we have $\xi_i(p,1/n) = \xi_i(p,0)$ for all large enough n. Therefore, $n\xi_{i1}(p,1/n) \to +\infty$. Secondly, when $\xi_{i1}(p,0) = 0$, we prove the fraction $\frac{\xi_{i1}(p,\varepsilon)}{\varepsilon}$ converges to some positive number $\varepsilon \to 0$. Moreover, the limit is bounded away from zero around the equilibrium price $\bar{p}$. Therefore, $n\xi_{i1}(p,1/n) \geq c$ if $\xi_i(p,0) = 0$ and p is in a neighborhood of $\bar{p}$.

To prove the first claim, we have a pair $(\lambda, x = \xi_i(p))$ which solves the following system for some $\lambda \geq 0$,

$$\begin{cases} \lambda p - \nabla u_i(x) \geq 0 \\ x \cdot (\lambda p - \nabla u_i(x)) = 0 \end{cases} \tag{3}$$

and $(\lambda^n, x^n = \xi_i^n(p))$ solves the following system for some $\lambda^n \geq 0$,

$$\begin{cases} \lambda^n p - \nabla u_i^n(x^n) \geq 0 \\ x^n \cdot (\lambda^n p - \nabla u_i^n(x^n)) = 0 \end{cases}$$

By definition of $u_i^n(x)$, it is equivalent to

$$\begin{cases} \lambda^n p - \nabla u_i(x^n) - \frac{1}{n}\eta'(nx_1^n) \geq 0 \\ x^n \cdot (\lambda^n p - \nabla u_i^n(x^n) - \frac{1}{n}\eta'(nx_1^n)) = 0 \end{cases} \tag{4}$$

We note the solution (λ, x) of the system (3) satisfies $x_1 > 2/n$ for all large n. Thus, $\eta'(nx_1) = 0$ for all large n. Therefore, $(\lambda^n = \lambda, x^n = x)$ is a solution of system (4) for all large n. In other words, $\xi_i(p) = \xi_i^n(p)$ for all large n.

To prove the second claim, we first show $\lim_{\varepsilon\to 0} \frac{\xi_{i1}(p,\varepsilon)}{\varepsilon} \geq c$ for some small number c determined later. By the formula in the beginning of the proof, such limit exists (but might be infinity). Moreover, such limit must be nonnegative, as the demand is always non-negative. We prove by contradiction. Suppose

$$c \geq \lim_{\varepsilon\to 0} \frac{\xi_{i1}(p,\varepsilon) - \xi_{i1}(p,0)}{\varepsilon} = \lim_{\varepsilon\to 0} \frac{\xi_{i1}(p,\varepsilon)}{\varepsilon}$$

for some some small c. Since $\xi_{i1}(p,\varepsilon)$ is a maximizer solving the constraint maximization problem, for some $\lambda(\varepsilon) \geq 0$, we have

$$\lambda(\varepsilon)p_1 = [\nabla u(\xi_i(p,\varepsilon))]_1 + \varepsilon\eta'(\xi_{i1}(p,\varepsilon)/\varepsilon)$$

By the strict concavity assumption (A2) and implicit function theorem, $\lambda(\varepsilon)$ is continuous in ε. Take ε goes to zero. The left hand side goes to $\lambda(0)p_1$, the first term on the right goes to $[\nabla u(\xi_{i1}(p,0))]_1 \in [0, \lambda(0)p_1]$ by the optimality condition, but the second term on the right goes to a very large number since

$$\varepsilon\eta'\left(\frac{x_1}{\varepsilon}\right) \simeq \varepsilon\left(\frac{x_1}{\varepsilon}\right)^{-1/2} = \left(\frac{\varepsilon}{x_1}\right)^{1/2} \geq (1/c)^{1/2}$$

as $\eta'(x) \simeq x^{1/2}$ when x is small. Take $p = \bar{p}$. We note the marginal utility of money $\lambda(0)$ is a positive number and $\lambda(0)$ is continuous in p. By

$$(1/c)^{1/2} \lesssim \lambda_0\bar{p} + [\nabla u(\xi_i(p,\varepsilon))]_1 \leq 2\lambda_0\bar{p}_1$$

As c goes to zero, the left hand side goes to infinity while the right hand side is a constant. Contradiction. Consequently, there is a neighborhood of $\bar{p}$ and a constant $c > 0$ such that $\lim_{\varepsilon\to 0} \frac{\xi_{i1}(p,\varepsilon)}{\varepsilon} \geq c$ for all p in this neighborhood.

Secondly, we show $\lim_{\varepsilon\to 0} \frac{\xi_{i1}(p,\varepsilon)}{\varepsilon} \neq +\infty$. We prove by contradiction. Suppose the limit is positive infinity, then

$$\lim_{\varepsilon\to 0} U_{\varepsilon x}(\varepsilon, \xi_{i1}(p,\varepsilon)) = \lim_{\varepsilon\to 0}\left[\eta'\left(\frac{x_1}{\varepsilon}\right) - \left(\frac{x_1}{\varepsilon}\right)\eta''\left(\frac{x_1}{\varepsilon}\right)\right]_{x=\xi_i(p,\varepsilon)} e_1 = \vec{0}$$

That is, $\lim_{\varepsilon\to 0} \nabla_\varepsilon \xi_i(p,\varepsilon) = \vec{0}$ by the formula at the beginning of the proof. Contradiction.

Hence, we proved that there is a neighborhood of $\bar{p}$ and a constant $c > 0$ such that $\lim_{\varepsilon\to 0} \frac{\xi_{i1}(p,\varepsilon)}{\varepsilon} \in [c, +\infty)$ for all p in this neighborhood. ■

Proof of Proposition 1. By Jofré et al. [2017], for each purturbed economy $\mathcal{E}^n$, the size of the neighborhood and the convergence speed ρ only depend on the largest and the smallest eigenvalue of the Hessian of utility functions at equilibrium. i.e. the eigenvalues of the matrices $\nabla^2 u_i^n(\bar{x}_i^n)$, which is $\nabla^2 u_i(\bar{x}_i^n) + \eta''(n\bar{x}_{i1}^n)e_{11}$. As implied by Lemma 2, $n\bar{x}_i^n$ does not go to zero as n goes to infinity.[8] Therefore, the eigenvalues of

[8] Assume the contrary, we have $n\bar{x}_{i1}^n(\bar{p}^n) < \varepsilon$ for all large n. On the other hand, take $p = \bar{p}^n$ in Lemma 2, $n\bar{x}_{i1}^n(p) \to c > 0$. Contradiction.

the matrices $\nabla^2 u_i^n(\bar{x}_i^n)$ are bounded from above and below . Hence, we know the sizes of the neighborhoods of $(\bar{p}^n, \bar{x}^n)$ and the convergence speed ρ^n can be bounded by some finite numbers $\delta_x > 0$, $\delta_p > 0$ and $\rho > 0$. ■

3.2 Verification of (P2)

In this section, we verify property (P2):

Proposition 2. *If $(\bar{p}^n, (\bar{x}_i^n)_{i\in I})$ is a sequence of equilibria of perturbed economies $\mathcal{E}^n$, then there is an equilibrium $(\bar{p}, (\bar{x}_i)_{i\in I})$ of the original economy $\mathcal{E}$ such that a subsequence of $(\bar{p}^n, \bar{x}^n)$ converges to $(\bar{p}, \bar{x})$.*

Proof. By our definition, $\|\bar{p}^n\|_2 = 1$. Since the unit sphere is compact in R^L_+, there is a subsequence of $\bar{p}^n$ converging to some element $\bar{p} \in R^L_+$. Since $\|\bar{p}^n\|_2 - \|\bar{p} - \bar{p}^n\|_2 \leq \|\bar{p}\|_2 \leq \|\bar{p}^n\|_2 + \|\bar{p} - \bar{p}^n\|_2$, we have $\|\bar{p}\|_2 = 1$.

Similarly, for any agent $i \in I$, $\bar{x}_i^n$ is in the ball of centered at zero with radius $\left\|\sum_{i\in I} \omega_i\right\|_2 > 0$. Therefore, for each agent $i \in I$, there is a subsequence of $\bar{x}_i^n$ converging to some $\bar{x}_i$. By choosing the subsequence, we assume $(\bar{p}^n, (\bar{x}_i^n)_{i\in I})$ converges to $(\bar{p}, (\bar{x}_i)_{i\in I})$. It remains to show $(\bar{p}, (\bar{x}_i)_{i\in I})$ is an equilibrium of the economy $\mathcal{E}$.

Firstly, we show $(\bar{x}_i)_{i\in I}$ is a feasible allocation. To begin with, $\bar{x}_i = \lim_{n\to+\infty} \bar{x}_i^n$ and $\bar{x}_i^n \geq 0$ implies $\bar{x}_i \geq 0$. Moreover, $\sum_{i\in I}(\bar{x}_i - \omega_i) = \lim_{n\to\infty} \sum_{i\in I}(\bar{x}_i^n - \omega_i) = 0$. Therefore, $(\bar{x}_i)_{i=1}^I$ is a feasible allocation.

Secondly, we show $\bar{x}_i$ maximize agent i's payoff within his budget set $\{x \in \mathbb{R}^L_+ : p \cdot x \leq p \cdot \omega_i\}$. By definition, for all $i \in I$, we have

$$\bar{x}_i^n \in \operatorname{argmax} \{u_i^n(x) : \bar{p}^n \cdot x \leq \bar{p}^n \cdot \omega_i\}$$

and we wish to show,

$$\bar{x}_i \in \operatorname{argmax} \{u_i(x) : \bar{p} \cdot x \leq \bar{p} \cdot \omega_i\}$$

We prove by contradiction. Firstly,

$$\bar{p} \cdot (\bar{x}_i - \omega_i) = \lim_{n\to\infty} \bar{p}^n \cdot (\bar{x}_i^n - \omega_i) \leq 0$$

Thus, $\bar{x}_i$ is in agent i's budget set $\{x : p \cdot x \leq p \cdot \omega_i\}$ at price $p = \bar{p}$. By strict monotonicity assumption (A2), the inequality sign must be equality. Suppose there is a $y \in R^n_+$ such that $\bar{p} \cdot y = \bar{p} \cdot \omega_i$ and $u_i(y) > u_i(\bar{x}_i)$. We wish to find a y' close to y and in the budget set of agent i in some perturbed economy $\mathcal{E}^n$ such that $u_i^n(y') > u_i^n(\bar{x}_i^n)$. Firstly, we take n large enough such that $|u_i(\bar{x}_i^n) - u_i(\bar{x}_i)| < 1/4(u_i(y) - u_i(\bar{x}_i))$.

Now we scale y linearly to fit in the budget set of perturbed economy $\mathcal{E}^n$. By the convergence of $\bar{p}^n$, $\bar{p}^n \cdot y \to \bar{p} \cdot y = \bar{p} \cdot \omega_i$. So we can find n large enough such that the fraction $|\frac{\bar{p}^n \cdot y}{\bar{p} \cdot \omega_i} - 1|$ as small as we want. For each large n, we define $y^n \in R_+^L$ to be the unique vector colinear to y and satisfying the budget constraint $\bar{p}^n \cdot y^n = \bar{p}^n \cdot \omega_i$. We note y^n can be controlled to be arbitrarily close to y in this construction. Therefore, $|u_i(y^n) - u_i(y)| < 1/4(u_i(y) - u_i(\bar{x}_i))$ for large enough n.

Lastly, $|u_i^n(y^n) - u_i(y^n)| = \frac{1}{n^2}\eta(ny_1^n) \leq 1/n^2 < 1/4(u_i(y) - u_i(\bar{x}_i))$ for n large enough. Therefore, $u_i^n(y^n) - u_i^n(\bar{x}_i^n) > 1/4(u_i(y) - u_i(\bar{x}_i)) > 0$. Since, by construction, y^n is in the budget set of agent i in the perturbed economy $\mathcal{E}^n$, we have a contradiction. ■

3.3 Verification of (P3)

In this section, we verify property (P3):

Proposition 3. *For an economy $\mathcal{E}$ satisfying assumptions (A1)-(A5), if the price paths $p(t)$ and $p^n(t)$ are the solution of the differential equations in Equation (1) and Equation (2), we have $p^n(t)$ converges uniformly to $p(t)$ on $[0, \infty)$.*

To prove Proposition 3, we observe it is sufficient to show a convergence result that is slightly stronger than the pointwise convergence. The reason we can do so is that the sequences $p^n(t)$ decay uniformly fast according to Proposition 1.

Lemma 3. *If, for any $\varepsilon > 0$ and $T > 0$, there is an $N \in \mathbb{N}$ such that, $\|p^n(t) - p(t)\| \leq \varepsilon$ for any $0 \leq t \leq T$ and all $n \geq N$, then $p^n(t)$ converges uniformly to $p(t)$ on $[0, \infty)$.*

Proof. We fix $\varepsilon > 0$. By Proposition 1,

$$\|p^n(t) - \bar{p}^n\|_2 \leq e^{-\rho t} \|p(0) - \bar{p}^n\|_2$$

Moreover, the condition of the lemma implies pointwise convergence. Take n goes to infinity, we have

$$\|p(t) - \bar{p}\|_2 \leq e^{-\rho t} \|p(0) - \bar{p}\|_2$$

Note,

$$\|p^n(t) - p(t)\|_2 \leq \|p^n(t) - \bar{p}^n\|_2 + \|\bar{p}^n - \bar{p}\|_2 + \|p(t) - \bar{p}\|_2$$
$$\leq 2e^{-\rho t} \|p(0) - \bar{p}\|_2 + 2\|\bar{p}^n - \bar{p}\|_2$$

We take T and N_1 large enough such that the $\|p^n(t) - p(t)\|_2 \leq \varepsilon$ for all $t \geq T$ and $n \geq N_1$. On the other hand, by the condition, there is N_2 such that for all $t \leq T$ and $n \geq N_2$ such that $\|p^n(t) - p(t)\|_2 \leq \varepsilon$. Therefore, taking $N = \max(N_1, N_2)$, we have $\|p^n(t) - p(t)\|_2 \leq \varepsilon$ for all $t \geq 0$, $n \geq N$. ■

Now, we prove the condition of the lemma. i.e. for any $\varepsilon > 0$, $T \geq 0$, there is an $N \in \mathbb{N}$ such that $\|p^n(t) - p(t)\| \leq \varepsilon$ for all $t \geq 0$ and $n \geq N$.

The two ordinary differential equations we study are

$$\dot{p}(t) = Z(p(t)), p(0) \in \mathbb{R}^L_+, \|p(0)\|_2 = 1$$

$$\dot{p}^n(t) = Z^n(p^n(t)), p^n(0) = p(0)$$

We note we are trying to show the solutions of two ordinary differential equations are close to each other. Therefore, we use the Gronwall's inequality in Howard [1998] stated as below.

Theorem (the Gronwall inequality)**.** *Let X be a Banach space and $U \subset X$ be an open set in X. Let $f, g : [0, \infty) \times U \to X$ be continuous functions and let $y, z : [0, \infty) \times U \to X$ satisfy the initial value problems*

$$y'(t) = f(t, y(t)), y(0) = y_0$$

$$z'(t) = g(t, z(t)), z(0) = z_0$$

Also assume there is a constant $C \geq 0$ so that

$$\|g(t, x_2) - g(t, x_1)\| \leq C \|x_2 - x_1\|$$

and a continuous function $\phi : [0, \infty) \to [0, \infty)$ so that

$$\|f(t, y(t)) - g(t, y(t))\| \leq \varphi(t)$$

Then for any $t \geq 0$,

$$\|y(t) - z(t)\| \leq e^{Ct} \|y_0 - z_0\| + e^{Ct} \int_0^t e^{-Cs} \varphi(s) ds$$

Comparing our differential equations and the ones in Gronwall's inequality, we set $f = Z^n$ and $g = Z$. By Jofré et al. [2017], $g = Z$ is Lipchitz continuous.[9] Thus, we can apply Gronwall's inequality.

By Gronwall's inequality, for any $\varepsilon > 0$, to ensure $\|p(t) - p^n(t)\|_2 \leq \varepsilon$ for all $0 \leq t \leq T$, it is enough to show

$$\|Z^n(p(t)) - Z(p(t))\|_2 \leq \varepsilon$$

for any t in the compact set $[0, T]$.

It is sufficient to show the same relation for individial demand, as

$$\begin{aligned}\|Z^n(p(t)) - Z(p(t))\|_2 &\leq \sum_{i \in I} \|Z_i^n(p(t)) - Z_i(p(t))\|_2 \\ &= \sum_{i \in I} \|\xi_i^n(p(t)) - \xi_i(p(t))\|_2 \\ &\leq |I| \max_{i \in I} \|\xi_i^n(p(t)) - \xi_i(p(t))\|_2\end{aligned}$$

Hence, to prove the condition of Lemma 5 holds, it is sufficient to prove the following lemma:

Lemma 4. *For any $\varepsilon > 0, T > 0$, there is an $N \in \mathbb{N}$ such that for any $0 \leq t \leq T$, any $n \geq N$,*

$$\|\xi^n(p(t)) - \xi(p(t))\|_2 \leq \varepsilon$$

where ξ and ξ^n are some agent's demand in economies $\mathcal{E}$ and $\mathcal{E}^n$ respectively.

Proof. We first discretize the time interval $[0, T]$ into 2^K equal parts, where K is a large number determined later. We first show the statement is true on these discrete set of points in the time interval. That is, we define $p(k) = p(k2^{-K}T)$ for $k \in \{0, 1, ..., 2^K\}$. Then, we use the Lipchitz continuity to deal with the points in their small gaps.

Firstly, we wish to show for any $k \in \{0, 1, ..., 2^K\}$, there is an $N(k) \in \mathbb{N}$ such that for all $n \geq N(k)$,

$$\|\xi^n(p(k)) - \xi(p(k))\|_2 \leq \varepsilon/2$$

To prove the above inequality, we use Berge's maximization theorem. We define a function $f : \mathbb{R}_+^n \times \mathbb{R} \to \mathbb{R}$ by

$$f(x, \delta) = u(x) + \delta^2 \eta(x/\delta)$$

[9] The proof of Lipchitz continuity does not reply on assumption (A6).

Since we wish to study the continuity of maximizer of f at $\delta = 0$, we extend the domain of η to be $\mathbb{R}$ by defining $\eta(-x) = -\eta(x)$ for $x \geq 0$.[10] Then, $\eta \in \mathcal{C}^2(\mathbb{R})$ is weakly increasing and concave. Consequently, $f : X \times \mathbb{R} \to \mathbb{R}$ is a continuous, strictly increasing and strictly concave function.

For each $\delta \in \mathbb{R}$, we define a choice set

$$C(\delta) = \{y \in \mathbb{R}^L_+ : p(k) \cdot y \leq p(k) \cdot \omega\}$$

Since the correspondence C from $\mathbb{R}$ to $\mathbb{R}^L_+$ is independent of δ, it is a continuous correspondence. Moreover, the strict concavity of f implies the maximizer is single-valued. Consequently, by Berge's maximization theorem,

$$C^*(\delta) = \text{argmax}\{f(x, \delta) : x \in C(\delta)\}$$

is a continuous function. By definition, $\xi^n(p(k)) = C^*(1/n)$ and $\xi(p(k)) = C^*(0)$, we have $\xi^n(p(k)) \to \xi(p(k))$ as $n \to \infty$. That is, there is an $N(k) \in \mathbb{N}$ such that for all $n \geq N(k)$,

$$\|\xi^n(p(k)) - \xi(p(k))\| \leq \varepsilon/2$$

Next, we take $N = \max_{0 \leq k \leq 2^K} N(k)$. In addition, for any $t \in [(k-1)2^{-K}T, k2^{-K}T]$, by the Lipchitz continuity of ξ and ξ^n, we have $\|\xi^n(p(t)) - \xi^n(p(k))\|_2 \leq \varepsilon/4$ and $\|\xi(p(t)) - \xi(p(k))\|_2 \leq \varepsilon/4$ for all large enough K. Hence, for any $\varepsilon > 0$, we can take K large enough and define $N = \max_{1 \leq k \leq 2^K} N(k)$ such that for any $n \geq N$, $\|\xi^n(p(t)) - \xi(p(t))\|_2 \leq \varepsilon$. ■

4 Concluding remarks

In this paper, we proved that, when the initial endowment is close to a Pareto optimal allocation, there is a locally tatonnement stable equilibrium, provided every utility function is strictly increasing and strictly concave, and every good is indispensable to some consumer. On the technical level, we use a perturbation argument to drop an additional assumption in Jofré et al. [2017], which assumes the existence of a commodity indispensable to all consumers.

[10] It is clear that $\lim_{\delta \to 0} \delta^2 \eta(x/\delta) = 0$ since $|\eta| \leq 1$.

References

A. Araujo. Lack of pareto optimal allocations in economies with infinitely many commodities: the need for impatience. *Econometrica: Journal of the Econometric Society*, pages 455–461, 1985.

A. Araujo. The non-existence of smooth demand in general banach spaces. *Journal of Mathematical Economics*, 17(4):309–319, 1988.

K. J. Arrow. Toward a theory of price adjustment. *The allocation of economic resources*, pages 41–51, 1959.

K. J. Arrow, H. D. Block, and L. Hurwicz. On the stability of the competitive equilibrium, ii. *Econometrica: Journal of the Econometric Society*, pages 82–109, 1959.

T. Bewley. The permanent income hypothesis and short-run price stability. *Journal of Economic Theory*, 23(3):323–333, 1980a.

T. F. Bewley. The permanent income hypothesis and long-run economic stability. *Journal of Economic Theory*, 22(3):377–394, 1980b.

W. D. Bryant. *General equilibrium: theory and evidence*. World Scientific, 2010.

S. Crockett, R. Oprea, and C. Plott. Extreme walrasian dynamics: The gale example in the lab. *American Economic Review*, 101(7):3196–3220, 2011.

G. Debreu. Excess demand functions. *Journal of mathematical economics*, 1(1):15–21, 1974.

E. Dierker. Two remarks on the number of equilibria of an economy. *Econometrica: Journal of the Econometric Society*, pages 951–953, 1972.

A. L. Dontchev and R. T. Rockafellar. Parametric stability of solutions in models of economic equilibrium. *Journal of Convex Analysis*, 19:975–993, 2012.

S. Gould, B. Fernando, A. Cherian, P. Anderson, R. S. Cruz, and E. Guo. On differentiating parameterized argmin and argmax problems with application to bi-level optimization. *arXiv preprint arXiv:1607.05447*, 2016.

M. Hirota, M. Hsu, C. R. Plott, and B. W. Rogers. Divergence, closed cycles and convergence in scarf environments: Experiments in the dynamics of general equilibrium systems. 2005.

R. Howard. The gronwall inequality. *lecture notes*, 1998.

A. Jofré, R. Rockafellar, and R. Wets. On the stability and evolution of economic equilibrium. 2017.

R. R. Mantel. On the characterization of aggregate excess demand. *Journal of economic theory*, 7(3):348–353, 1974.

L. W. McKenzie. Classical general equilibrium theory. *MIT Press Books*, 1, 2005.

T. Negishi. The stability of a competitive economy: a survey article. *Econometrica: Journal of the Econometric Society*, pages 635–669, 1962.

M. Sattinger. Local stability when initial holdings are near equilibrium holdings. *Journal of Economic Theory*, 11(2):161–167, 1975.

H. Scarf. Some examples of global instability of the competitive equilibrium. *International Economic Review*, 1(3):157–172, 1960.

H. Sonnenschein. Market excess demand functions. *Econometrica: Journal of the Econometric Society*, pages 549–563, 1972.