Small Income Effects in Economies with a Large Number of Commodities and Patient Consumers*

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Abstract

We prove that, at any equilibrium of an economy with additively separable utility functions, when the number of commodities is sufficiently large and all agents are sufficiently patient, all entries in the income derivative of the demand are arbitrarily small. By dropping the assumption that prices are uniformly bounded from below by a positive number, we extend the intuition in Vives (1987) on small income effects from partial equilibrium models to general equilibrium models. In addition, we propose a definition of sufficiently patient for non-separable utility functions.

1 Introduction

The smallness of income effects are usually assumed in partial equilibrium analysis to isolate the analysis of one commodity from the rest of commodities. To justify this assumption, Marshall [1920] suggested that, when there is a large set of commodities, each commodity represents a small part of the total expenditure, and the income effects are small. Marshall's observation was first formalized in Vives [1987]. In his paper, Vives showed that, in a partial equilibrium framework in which prices are given, when the

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number of commodities goes to infinity, income effects go to zero. Later, Hayashi [2013] extended Marshall's observation to a general equilibrium framework with a continuum set of commodities. In both works, the crucial assumption is that prices are uniformly bounded from below by a positive number. However, this assumption is inconsistent with many natural models. For example, in a dynamic general equilibrium model with a countable horizon, equilibrium prices are summable. Therefore, the equilibrium prices cannot be bounded from below by a positive number.

In this paper, we resolve this inconsistency issue, thus extending Marshall's observation to a wide class of general equilibrium models. In particular, we work on the finite truncations of a dynamic general equilibrium model with a countable horizon. In this infinite dimensional economy, there are countably many commodities. Every agent has an additively separable utility function in the form $\sum_{t=0}^{\infty} \beta^t u_i(x_t)$. The discount factor $\beta \in (0, 1)$ represents agents' degree of patience. We assume that all agents have the same degree of patience. In any truncation of this infinite dimensional economy, agents' utility functions are in the form $\sum_{t=0}^{T} \beta^t u_i(x_t)$ for some natural number T. When the number T is large, there is a large set of commodities in the truncated model.

Our main result suggests that, at any equilibrium of an truncated model defined as above, the income effects are small whenever the number of commodities is large and all agents are patient enough. By the term "income effects are small", we mean all entries in the income derivative of the demand function is uniformly bounded by a small number. The major difference between our work and other works is that we do not assume prices are bounded away from zero as the number of commodities becomes very large.

An economic intuition behind our main result is that additively separable utility functions studied in this paper behave like quasi-linear utility functions when agents are sufficiently patient. The relationship between separable utility functions and quasi-linear utility functions has been made precise in Bewley [1977] and Weretka [2018] in different settings. Since the income effects are zero when all agents have quasi-linear utility functions, it is reasonable to expect that the income effects are small whenever utility functions are close to quasi-linear utility functions. Our proof validates this argument and also provides an explicit bound for the income effects.

Lastly, we propose a definition of the degree of patience for non-separable utility functions. The definition serves as a first step to extend our result on small income effects to non-separable utility functions. As Brown and Lewis [1981] have observed, Mackey topology plays a key role in the analysis of patience. In particular, they showed an agent is impatient (myopic) if and only if his utility function is Mackey continuous. That is, the Mackey continuity offers a qualitative criterion on whether agents are patient or not. In our setting, in order to define the term "patient enough", we need to obtain a more refined classification of impatient agents. That is, for any Mackey continuous function, we need to find a discount factor β corresponding to it.

In this paper, we followed a hint in Lewis [1977] and defined a continuum class of topologies $\{\tau_s : s \in (0,1)\}$. Each topology τ_s is weaker than the Mackey topology and corresponds to a degree of patience s. A non-separable utility function is defined to be β-patient if it is τ_s-continuous for all $s > \beta$ but is not τ_s-continuous for all $s < \beta$. We justified our definition by proving, with reasonable assumptions on the per-period utility functions, all additively separable utility functions with a discount factor β are β-patient.

The paper is organized as follows. In Section 2, we state the model and the main theorem. In Section 3, we prove the main theorem. In Section 4, we illustrate the smallness of income effects by an example. In Section 5, we propose a definition of agents' degree of patience when utility functions are non-separable. Lastly, in Section 6, we summarize the results.

2 Model

In this paper, we study the finite truncations of an economy $\mathcal{E} = (I, X, (U_i, \omega_i)_{i\in N})$ with countably many commodities. In the following, we will first introduce the economy $\mathcal{E}$ with countably many commodities and then introduce its finite truncations.

In economy $\mathcal{E}$, the set of agents is denoted by a finite set I. The consumption set is denoted by a set $X = \ell^\infty_+(\mathbb{R})$. Consequently, a consumption plan $x \in X$ is represented by a vector with countably many entries $(x_0, x_1, ..., x_t, ...)$, where $x_t \geq 0$ for all natural numbers $t \in \mathbb{N}$. In this paper, we interpret a vector $x \in X$ as a dynamic consumption plan in a model with countably many periods and one consumption good in each period. Therefore, x_t specifies the consumption of the unique consumption good at date t. Moreover, the consumption at date 0 will be taken as a numeraire.

In this section, agents will have additively separable utility functions. That is, for any $i \in I$, agent i's utility function $U_i : X \to \mathbb{R}$ is defined by

$$U_i(x) = \sum_{t=0}^{\infty} \beta^t u_i(x_t)$$

where $\beta \in (0,1)$ is agent i's *degree of patience* and $u_i : \mathbb{R}_+ \to \mathbb{R}$ is agent i's *per-period utility function*. In addition, agent i's initial endowment is denoted by a vector $(\omega_{i0}, \omega_i) \in X$, where $\omega_{i0} \in \mathbb{R}_+$ and $\omega_i = (\omega_{i1}, \omega_{i2}, ...) \in \ell^\infty_+(\mathbb{R})$.

We impose the following assumptions on economy $\mathcal{E}$. There are five assumptions on the per-period utility functions and two assumptions on the initial endowments.

(A1) for all $i \in I$, u_i is twice continuously differentiable on $\mathbb{R}_+$

(A2) for all $i \in I$, $u_i(0) = 0$

(A3) for all $i \in I$, $u_i' > 0$ on $\mathbb{R}_+$

(A4) for all $i \in I$, $u_i'' < 0$ on $\mathbb{R}_+$

(A5) for all $i \in I$, $u_i'(0) = +\infty$

(A6) There is a $d > 0$ such that $\sum_{i=1}^N \omega_{it} \in [1/d, d]$ for all $t \in \mathbb{N}$

(A7) There is an integer $s \geq 1$ such that $\omega_{it} + ... + \omega_{i(t+s)} \geq 1$ for all $i \in I$, $t \in \mathbb{N}$

Among assumptions on the utility functions, Assumption (A1) is a regularity condition on the utility functions. Assumption (A2) ensures that the utility functions are well-defined at the origin. Assumption (A3) and Assumption (A4) imply that the utility functions are strictly increasing and strictly concave. Assumption (A5) implies all optimal consumption are strictly positive. A per-period utility function satisfying all assumptions is given by $u(x) = x^{1/2}$.

Among assumptions on the initial endowments, Assumption (A6) ensures the market supply of every commodity is uniformly bounded from above and below by positive numbers. Assumption (A7) ensures that all agents have a nontrivial amount of endowments in the distant future. An initial endowment satisfying all assumptions is given by $\omega_i = (1, 1, ...)$ for all $i \in I$.

As noticed in Bewley [1972], and later explored by Balasko [1997] and Shannon [1999], an economy with an ℓ^∞ consumption set can usually be studied as a limit of its finite truncations. Therefore, in this paper, we study the finite truncations of economy $\mathcal{E}$.

Formally, for any natural number $T \geq 1$, we define a *T-truncated economy* $\mathcal{E}^T = (I, X^T, (U_i^T, (\omega_{i0}, \omega_i^T))_{i\in I})$ with discount factor β. In economy $\mathcal{E}^T$, the consumption set is $X^T = \mathbb{R}_+^{1+T}$. For any $i \in I$, agent i's utility function $U_i^T : \mathbb{R}_+^{1+T} \to \mathbb{R}$ is defined by

$$U_i^T(x) = U_i(x_0, ..., x_T, 0, 0, ...) = \sum_{i=0}^{T} \beta^t u_i(x_t)$$

for any $x = (x_0, ..., x_T) \in X^T$. Agent i's initial endowment is given by a vector $(\omega_{i0}, \omega_i^T) = (\omega_{i0}, \omega_{i1}, ..., \omega_{iT}) \in \mathbb{R}_+^{1+T}$, where $\omega_i^T = (\omega_{i1}, ..., \omega_{iT}) \in \mathbb{R}_+^T$.

We use commodity 0 as a numeraire. That is, the price of commodity 0 is assumed to be 1. Consequently, a *price* p is defined to be a vector in $\mathbb{R}_{++}^T$. Under a price p, the *budget set* of an agent i is given by

$$B_i(p) = \Big\{(x_0, x) \in \mathbb{R}_+^{1+T} : (1, p) \cdot (x_0, x) \leq (1, p) \cdot (\omega_{i0}, \omega_i)\Big\}$$

In addition, a *consumption plan* is denoted by a vector $(x_{i0}, x_i) \in \mathbb{R}_+^{1+T}$. A tuple of consumption plans $(x_{i0}, x_i)_{i\in I}$ is an *allocation* if

$$\sum_{i\in I}(\bar{x}_{i0}, \bar{x}_i) = \sum_{i\in I}(\omega_{i0}, \omega_i)$$

A pair of price and allocation $(\bar{p}, (\bar{x}_{i0}, \bar{x}_i)_{i\in I})$ is a *competitive equilibrium* if, for all $i \in I$,

$$U_i(\bar{x}_{i0}, \bar{x}_i) \geq U_i(x_{i0}, x_i), \forall (x_{i0}, x_i) \in B_i(\bar{p})$$

That is, at a competitive equilibrium, all agents consume their optimal consumption in their budget sets.

Furthermore, given a price $p \in \mathbb{R}_{++}^T$ and an income level $W \in \mathbb{R}_+$, agent i's optimal consumption is defined by a function[1] $f_i : \mathbb{R}_{++}^T \times \mathbb{R}_+ \to \mathbb{R}_+^{1+T}$, where

$$f_i(p, W) = \underset{x\in\mathbb{R}_+^T : x_0 + p\cdot x \leq W}{\operatorname{argmax}} \sum_{t=0}^{T} \beta^t u_i(x_t)$$

Coordinatewisely, $f_i(p, W) = (f_{i0}(p, W), f_{i1}(p, W), ..., f_{iT}(p, W))$, where $f_{it}(p, W) \in \mathbb{R}_+$ for all $t \in \{0, 1, ..., T\}$. When the income level is given by the value of initial endowment,

[1] By Assumption A4, f_i is a single-valued.

$W = \omega_0 + p \cdot \omega$, the function $\xi_i(p) = f_i(p, \omega_{i0} + p \cdot \omega_i)$, as a function in p, is agent i's demand function in the T-truncated economy $\mathcal{E}^T$.

Next, we state the main theorem: the income effects are sufficiently small whenever the number of commodities is sufficiently large and all agents are sufficiently patient.

Theorem 1 (Main Theorem)**.** *For any economy $\mathcal{E}$ satisfying assumptions (A1) - (A7), there are thresholds $T^* \in \mathbb{N}$ and $\beta^* \in (0,1)$ such that, for any $T > T^*$ and $\beta \in (\beta^*, 1)$, at any equilibrium $(\bar{p}, (\bar{x}_{i0}, \bar{x}_i)_{i \in I})$ of the T-truncated economy $\mathcal{E}^T$ with discount factor β, we have*

$$\left| \frac{d}{dW} f_{it}(\bar{p}, \omega_{i0} + \bar{p} \cdot \omega_i) \right| \leq C \frac{1}{\sum_{t=0}^{T} \beta^t}, \forall t \in \{1, 2, ..., T\}$$

where C does not depend on either β or T.

Here, $\frac{d}{dW} f_{it}(\bar{p}, \omega_{i0} + \bar{p} \cdot \omega_i)$ is the income effect of commodity t at the equilibrium. An immediate corollary of the main theorem is that income effects are arbitrarily small whenever the number of commodities is sufficiently large and all agents are sufficiently patient.

Corollary 1. *For any economy $\mathcal{E}$ satisfying assumptions (A1) - (A7) and any $\varepsilon > 0$, there are thresholds $T^{**} \in \mathbb{N}$ and $\beta^{**} \in (0,1)$ such that, for any $T > T^{**}$ and $\beta \in (\beta^{**}, 1)$, at any equilibrium $(\bar{p}, (\bar{x}_{i0}, \bar{x}_i)_{i \in I})$ of the T-truncated economy $\mathcal{E}^T$ with discount factor β, we have*

$$\left\| \frac{d}{dW} f_{it}(\bar{p}, \omega_{i0} + \bar{p} \cdot \omega_i) \right\|_\infty \leq \varepsilon, \forall t \in \{1, 2, ..., T\}$$

Proof. Take $\beta^{**} = \max(\beta^*, 1 - \frac{\varepsilon}{2C})$ and $T^{**} \geq T^*$ such that $\sum_{t=0}^{T^{**}} (\beta^{**})^t \geq \frac{1}{2(1-\beta^{**})}$. ■

2.1 Discussions

The only restrictive assumption in our model is that all discount factors are the same. This assumption is crucial in the proof of Lemma 2, which states agents' consumption are bounded away from zero. When agents have different discount factors, it is easy to prove that the least patient agent's consumption in the distant future will go to zero.

Suggested by the above observation, when agents have different discount factors, in order to prove income effects are small, we need to strengthen our assumptions. One way to do so is to replace the Inada condition, Assumption (A5), by the uniform concavity assumption introduced in Shannon [1999]. The uniform concavity assumption assumes that $|u_i''(0)/u_i'(0)| < 0$ for all agent $i \in I$. However, this assumption is incompatible with the Inada condition[2]. Thus, at the equilibrium, the consumption of some commodity might be zero. In such cases, in order to define the income effects as the income derivative of demand function, we may need to study directional derivatives. Without defining income effects by directional derivatives, we can prove the same result for all interior equilibria when agents have different discount factors and Assumption (A5) is replaced by the uniform concavity assumption.

Lastly, we pointed out that, when prices are bounded from below by a positive number, Vives [1987] proved that the income effects are bounded by some constant times $1/T$, where T is the number of commodities. Consequently, Vives showed that the ℓ^2-norm of the income derivative of the demand goes to zero as the number of commodities T goes to infinity. By Theorem 1, we could have the same result if the threshold on discount factors β^* is taken as a function of the number of commodities T. However, as far as I can understand, ℓ^2-norm is not extremely helpful in applications. Instead, if the ℓ^1-norm of the income derivative of the demand goes to zero, we can immediately establish many nice properties of the equilibrium, such as the downward sloping demand, local determinacy, local stability and so on. Unfortunately, our current assumptions on the model are insufficient to establish the ℓ^1 convergence. A counter example is given in Section 4.

3 Proof

In this section, we prove the main theorem. We start by proving two lemmas. Firstly, we prove that the equilibrium price decays at the same rate as β^t.

[2]The uniform concavity assumption implies that $u''(0)/u'(0) \neq 0$. We prove by contradiction by assuming $u''(0)/u'(0) = 0$. Therefore, $u'(x) \to +\infty$ as $x \to 0$. Define $f(x) = \log(u'(x))$ for $x > 0$. Therefore, $f(x) \to +\infty$ as $x \to 0$. Moreover, $f'(x) = \frac{u''(x)}{u'(x)} \to 0$ as $x \to 0$. However, it is not possible: take a small enough $c > 0$ such that, at all $0 < x < c$, we have $f'(x)$ is very small. Consequently, $|f(x) - f(c)|$ on $(0, c)$ cannot be very large. In particular, $f(x)$ cannot goes to $+\infty$ as $x \to 0$. Contradiction. Actually, by the same argument, we can show $u''(0)/u'(0) = -\infty$ whenever $u'(0) = +\infty$. That is, the uniform concavity assumption is incompatible with the Inada condition.

Lemma 1. *For economy $\mathcal{E}$ satisfying assumptions Assumption (A3), Assumption (A4) and Assumption (A6), there is a number $B > 0$, such that for any $T \in \mathbb{N}$, any equilibrium price $\bar{p}$ of the T-truncated economy $\mathcal{E}^T$, we have*

$$\beta^t/B \leq \bar{p}_t \leq B\beta^t, \forall t \in \{1, 2, ..., T\}$$

Proof. Firstly, we prove the upper bound of the equilibrium price. In this part, we fix a commodity $t \in \{1, 2, ..., T\}$. By the market clear condition, we have $\sum_{i \in I} \bar{x}_{it} = \sum_{i \in I} \omega_{it}$. Therefore, for any $k \in I$, we have $\bar{x}_{kt} \leq \sum_{i \in I} \omega_{it}$. By Assumption (A6), we have $\bar{x}_{kt} \leq d$. On the other hand, there is an agent $j \in I$ consuming a more than average amount of commodity t in equilibrium. That is $\bar{x}_{jt} \geq \frac{1}{|I|} \sum_{i \in I} \omega_{it}$. By Assumption (A6), we have $\bar{x}_{jt} \geq \frac{1}{|I|d}$. For agent $j \in I$ defined above, $(\bar{x}_{jt})_{t \in \{0,1,...,T\}}$ solves his utility maximization problem at the equilibrium price $\bar{p}$:

$$\max_{(x_{jt})_{t=0,1,...,T} \in B_j(\bar{p})} \sum_{t=0}^{T} \beta^t u_j(x_{jt})$$

For some $\lambda > 0$, we have

$$\lambda \bar{p}_t = \beta^t u_j'(\bar{x}_{jt})$$

And by Assumption (A3), we have

$$\lambda \geq u_j'(\bar{x}_{j0}) > 0$$

Thus,

$$\bar{p}_t = \frac{\beta^t u_j'(\bar{x}_{jt})}{\lambda} \leq \frac{\beta^t u_j'(\bar{x}_{jt})}{u_j'(\bar{x}_{j0})} \leq \beta^t \frac{u_j'(\frac{1}{|I|d})}{u_j'(d)}$$

Secondly, we prove the lower bound of the equilibrium price. We define agent j to be an agent consuming a more than average amount of commodity 0 in equilibrium. That is, $\bar{x}_{j0} \geq \frac{1}{|I|} \sum_{i \in I} \omega_{i0} \geq \frac{1}{|I|d}$. Similarly, as $(\bar{x}_{jt})_{t \in \{0,1,...,T\}}$ solves agent j's utility maximization problem at equilibrium price $\bar{p}$, we have

$$\lambda = u_j'(\bar{x}_{j0}) > 0$$

$$\lambda \bar{p}_t \geq \beta^t u_j'(\bar{x}_{jt}), \forall t \in \{1, 2, ..., T\}$$

for some positive λ. Thus,

$$\bar{p}_t = \frac{\beta^t u_j'(\bar{x}_{jt})}{\lambda} \geq \frac{\beta^t u_j'(\bar{x}_{jt})}{u_j'(\bar{x}_{j0})} \geq \beta^t \frac{u_j'(d)}{u_j'(\frac{1}{|I|d})}, \forall t \in \{1, 2, ..., T\}$$

Taking $B = \max\left\{\frac{u_i'(d)}{u_i'(\frac{1}{|I|d})}, \frac{u_i'(\frac{1}{|I|d})}{u_i'(d)} : i \in I\right\}$, we have $\beta^t/B \leq \bar{p}_t \leq B\beta^t$ for all $t \in \{0, 1, ..., T\}$. ■

Secondly, we prove that all equilibrium consumption are uniformly bounded from below by a positive number when the number of commodities is sufficiently large and agents are sufficiently patient.

Lemma 2. *For an economy $\mathcal{E}$ satisfying assumptions (A1) - (A7), there are thresholds $x^* > 0$, $\beta^* \in (0, 1)$, $T^* \in \mathbb{N}$ such that, for any $T \geq T^*$ and $\beta \in (\beta^*, 1)$, we have*

$$\bar{x}_{it} \geq x^*, \forall i \in I, 0 \leq t \leq T$$

where $(\bar{p}, (\bar{x}_{i0}, \bar{x}_i)_{i\in I})$ is a competitive equilibrium of the T-truncated economy $\mathcal{E}^T$ with discount factor β.

Proof. Since there are only finitely many agents, we fix an agent $i \in I$ and find the thresholds x^*, T^*, β^* for agent i. In order to make these thresholds work for all agents, we only need to repeat the following procedure $|I|$ times.

For a fixed agent $i \in I$, we will show that agent i can not consume too little at date 0, as otherwise, by Lemma 1, agent i will consume very little in all dates. However, it contradicts the fact that he has a nontrivial amount of endowment in the long run.

Formally, the first order condition of agent i's utility maximization problem gives

$$\lambda_i \bar{p}_t = \beta^t u_i'(\bar{x}_{it}), \forall t \in \{0, 1, ..., T\}$$

for some $\lambda_i > 0$. For some small $c > 0$ determined later, if $\bar{x}_{i0} \leq c$, by taking $t = 0$ in the first order condition, we have $\lambda_i = u_i'(\bar{x}_{i0}) \geq u_i'(c)$. Therefore, by Lemma 1,

$$u_i'(\bar{x}_{it}) = \frac{\bar{p}_t}{\beta^t}\lambda_i \geq u_i'(c)/B$$

As utility functions are assumed to be strictly concave in Assumption (A4), u_i' is a strictly decreasing function. In particular, it is invertible. Therefore

$$\bar{x}_{it} \leq (u_i')^{-1}(u_i'(c)/B), \forall t \in \{0, 1, 2, ..., T\} \tag{1}$$

It is easy to see that $(u_i')^{-1}(u_i'(c)/B) \to 0$ as $c \to 0$. On the other hand, as utility functions are assumed to be strictly increasing in Assumption (A3), the budget constraint is binding at equilibrium. That is,

$$\sum_{t=0}^{T} \bar{p}_t \bar{x}_{it} = \sum_{t=0}^{T} \bar{p}_t \omega_{it}$$

By Equation 1 and Lemma 1,

$$\sum_{t=0}^{T} \bar{p}_t \bar{x}_{it} \leq (u_i')^{-1}(u_i'(c)/B) \sum_{t=0}^{T} \bar{p}_t \leq B(u_i')^{-1}(u_i'(c)/B) \sum_{t=0}^{T} \beta^t \to \frac{B(u_i')^{-1}(u_i'(c)/B)}{1-\beta}$$

as $T \to \infty$. On the other hand, by Assumption (A7) and Lemma 1,

$$\sum_{t=0}^{T} \bar{p}_t \omega_{it} \geq \frac{1}{B} \sum_{t=1}^{[T/s]} \beta^{st} \to \frac{\beta^s}{B(1-\beta^s)}$$

as $T \to \infty$. Consequently, there is a $T^* \in \mathbb{N}$ such that for all $T \geq T^*$,

$$\frac{\beta^s}{2B(1-\beta^s)} \leq \sum_{t=0}^{T} \bar{p}_t \omega_{it} = \sum_{t=0}^{T} \bar{p}_t \bar{x}_{it} \leq \frac{2B(u_i')^{-1}(u_i'(c)/B)}{1-\beta}$$

We take $\beta^* = (1/2)^{1/s}$. For all $\beta \in (\beta^*, 1)$,

$$(u_i')^{-1}(u_i'(c)/B) \geq \frac{1}{8B^2(1+\beta+...\beta^{s-1})} \geq \frac{1}{8B^2 s}$$

Therefore, taking $y^* = (u_i')^{-1}(Bu_i'(\frac{1}{8B^2 s}))$, we have that any small number $c < y^*$ will violate the above inequality. i.e. $\bar{x}_{i0} \geq y^*$. Revisiting the first order condition and Lemma 1, we have

$$u_i'(\bar{x}_{it}) = \frac{p_t \lambda_i}{\beta^t} \leq Bu_i'(y^*)$$

Therefore, we take $x^* = (u_i')^{-1}(Bu_i'(y^*))$. In sum, for all $T \geq T^*$ and discount factor $\beta \in (\beta^*, 1)$, the competitive equilibrium $(\bar{p}, (\bar{x}_{i0}, \bar{x}_i)_{i\in I})$ of the T-truncated economy $\mathcal{E}^T$ satisfies

$$\bar{x}_{it} \geq x^*, \forall i \in I, 0 \leq t \leq T$$

In particular, x^* does not depend on either β or T. ■

Next, we prove the main theorem. Around an equilibrium $(\bar{p}, (\bar{x}_{i0}, \bar{x}_i)_{i\in I})$ of the T-truncated economy $\mathcal{E}^T$, by Assumption (A5) and the first order condition of agent i's consumption maximization problem, there is a positive λ_i such that $x_i = f_i(p, W)$ solves the system of equations

$$\lambda_i p_t = \beta^t u_i'(x_{it}), \forall t \in \{0, 1, 2, ..., T\}$$

Moreover, by Assumption (A3), the budget constraint is binding:

$$\sum_{t=0}^{T} p_t x_{it} = W$$

By Assumption (A1) and Debreu [1972], the income derivative of x_t and λ_i is continuous differentiable in a neighborhood of $(\bar{p}, \bar{W} = \omega_{i0} + \bar{p} \cdot \omega_i)$. Therefore, in this neighborhood, we have

$$\frac{d\lambda_i}{dW} p_t = \beta^t u_i''(x_{it}) \frac{dx_{it}}{dW}, \forall t \in \{0, 1, 2, ..., T\}$$
$$\sum_{t=0}^{T} p_t \frac{dx_{it}}{dW} = 1$$

Consequently,

$$\frac{dx_{it}}{dW} = \left(\sum_{t=0}^{T} (\beta^t u_i''(x_{it}))^{-1} p_t^2 \right)^{-1} (\beta^t u_i''(x_{it}))^{-1} p_t$$

Our goal is to give an upper bound on the fraction

$$\frac{(\beta^t u_i''(\bar{x}_{it}))^{-1} \bar{p}_t}{\sum_{t=0}^{T} (\beta^t u_i''(\bar{x}_{it}))^{-1} \bar{p}_t^2} \tag{2}$$

By Lemma 2 and the market clear condition, we have $x^* \leq \bar{x}_{it} \leq d$ for all $i \in I$ and $t \in \{0, 1, ..., T\}$. Therefore, as $u \in C^2(\mathbb{R}_+)$, there is a positive $M > 0$ such that

$$-M \leq u_i''(\bar{x}_{it}) \leq -1/M, \forall i \in I, t \in \{0, 1, ..., T\}$$

By Lemma 1, the numerator of the fraction in Equation 2 is bounded from above uniformly:

$$\left| \frac{\bar{p}_t}{\beta^t u_i''(\bar{x}_{it})} \right| \leq BM, \forall i \in I, t \in \{0, 1, ..., T\}$$

On the other hand, the denominator of the fraction in Equation 2 goes to infinity as $\beta \to 1$ and $T \to \infty$:

$$\left| \sum_{t=0}^{T} (\beta^t u_i''(\bar{x}_{it}))^{-1} \bar{p}_t^2 \right| \geq \frac{1}{B^2 M} \sum_{t=0}^{T} \beta^t \to +\infty$$

Both bounds above are independent of the index of commodity t. Therefore, the fraction in Equation 2 is arbitrarily small whenever T and β are sufficiently large.

4 Example

We illustrate that the income effects are small in a two-person economy with $2T$ commodities when T is large and all agents are patient. The utility functions are given by

$$U_1(x) = U_2(x) = \sum_{t=0}^{2T-1} \beta^t \sqrt{x_t}$$

where $x = (x_0, ..., x_{2T-1}) \in \mathbb{R}^{2T}_+$. The initial endowments are given by

$$\omega_1 = (2, 0, 2, 0, ..., 2, 0)$$

$$\omega_2 = (0, 2, 0, 2, ..., 0, 2)$$

It is straightforward to check that $(\bar{p}, \bar{x}_1, \bar{x}_2)$ is a competitive equilibrium, where $\bar{p} = (1, \beta, ..., \beta^{2T-1})$, $\bar{x}_1 = (\frac{2}{1+\beta}, ..., \frac{2}{1+\beta})$ and $\bar{x}_2 = (\frac{2\beta}{1+\beta}, ..., \frac{2\beta}{1+\beta})$.

Next, we compute the income derivative of demand at the equilibrium for both agents. By Equation 2, agent i's income effect of commodity t is given by

$$\frac{(\beta^t u_i''(\bar{x}_{it}))^{-1}\bar{p}_t}{\sum_{t=0}^{2T-1}(\beta^t u_i''(\bar{x}_{it}))^{-1}\bar{p}_t^2} = \frac{1}{\sum_{t=0}^{2T-1}\beta^t} \to 0$$

as $\beta \to 1$ and $T \to \infty$.

5 General notion of patience

In the previous sections, the term "patient enough" plays an important role in establishing the smallness of the income effects at the equilibrium. Actually, the term "patient enough" can be also found in other context, such as the folk's theorem in repeated games. When utility functions are additively separable, it is clear that the term "patient enough" means that the discount factor $\beta \to 1$. However, when utility functions are not separable, it is not clear how to measure the patience of an agent based on his utility function. In this section, we provide a method to measure to patience of agents with non-separable utility functions based on the Mackey topology.

Introduced by Bewley [1972] to economic models, the Mackey topology is known to be the right topology for economies with an infinite set of commodities.[3] In $\ell^\infty(\mathbb{R})$, the Mackey topology is defined as below[4].

Definition 1. *Given a set of convergent sequences $\Phi = \{(\phi_t)_{t\in\mathbb{N}} : \phi_t \in \mathbb{R}, \phi_t \to 0\}$, for each $\phi \in \Phi$, we define a semi-norm on ℓ^∞ by*

$$\|f\|_\phi = \sup_t |\phi_t f_t|$$

[3]See Araujo [1985] for more details.

[4]The definition we give is the strict topology. Buck et al. [1958] proved that the Mackey topology is the same as the strict topology in $\ell^\infty(\mathbb{R})$.

where $f \in \ell^\infty$. The Mackey topology τ_{MA} is generated by the set of semi-norms $\{\|\cdot\|_\phi : \phi \in \Phi\}$.

By this definition, $f^n \to f$ in $\ell^\infty(\mathbb{R})$ with respect to the Mackey topology if and only if $\left\|f^k - f\right\|_\phi \to 0$ as $k \to +\infty$ for all $\phi \in \Phi$. A real-valued function u on $\ell^\infty(\mathbb{R})$ is continuous with respect to the Mackey topology if and only if $u(x^k) \to u(x)$ whenever $x^k \to x$ in τ_{MA}.

The relation between the Mackey topology and agents' patience was discovered in Lewis [1977] and Brown and Lewis [1981]. In particular, they proved that utility functions are Mackey continuous if and only if agents are impatient. In other words, Mackey continuity offers a qualitative criterion to distinguish whether an agent is patient or not. However, in order to define the term "patient enough", we need a quantitative criterion. Actually, in his thesis, Lewis [1977] hinted that an element $\phi \in \Phi$, which is a convergent sequence, looks like the discount factors in utility functions. Utilizing his observation, we define a continuum class of topologies, which offers an accurate measure for the degree of patience.

For any $s \in (0,1)$, we define a set Φ_s consisting of all sequences decaying faster than s^t:

$$\Phi_s = \{\phi \in \Phi : |\phi_t| \leq Cs^t \text{ for some } C > 0 \text{ and all } t \in \mathbb{N}.\}$$

According to Lewis's intuition, this set corresponds to the set of impatient agents who have discount factors decaying faster than s^t.

Definition 2. *For any $s \in (0,1)$, the topology τ_s on $\ell^\infty(\mathbb{R})$ is generated by the set of semi-norms $\{\|\cdot\|_\phi : \phi \in \Phi_s\}$.*

By definition, $x^n \to x$ in $\ell^\infty(\mathbb{R})$ with respect to τ_s if and only if $\|x^n - x\|_\phi \to 0$ for all $\phi \in \Phi_s$. Moreover, a real-valued function u on $\ell^\infty(\mathbb{R})$ is continuous with respect to τ_s if and only if $u(x^k) \to u(x)$ whenever $x^k \to x$ in τ_s. Furthermore, τ_s is weaker than Mackey topology on $\ell^\infty(\mathbb{R})$.

Next, we prove an additively separable utility function $U(x) = \sum_{t=0}^\infty \beta^t u(x_t)$ is continuous with respect to some τ_s but not all τ_s.

Proposition 1. *For any increasing and concave function $u : \mathbb{R} \to \mathbb{R}$ such that $u(0) = 0$*

and any $\beta \in (0,1)$, a real-valued function on ℓ^∞ defined by

$$U(x) = \sum_{t=0}^{\infty} \beta^t u(x_t)$$

is τ_s-continuous for $s > \beta$, but is not τ_s-continuous for $s < \beta$.

Proof. This proof is developed based on a proof in Bewley [1972]. Firstly, we prove that U is τ_s-continuous for any $s > \beta$. We start by fixing $s \in (\beta, 1)$ and an sequence $x^k \to x$ in τ_s. By definition, we have

$$\sup_t |\phi_t(x_t^k - x_t)| \to 0$$

for all sequence ϕ decaying faster than s^{-t}. In addition, for any $x \in \ell_+^\infty(\mathbb{R})$, we have $\sum_{t=1}^{\infty} \beta^t u(x_t) < \infty$. Therefore, for any $\varepsilon > 0$, there is a positive $y > 0$ such that

$$\sum_{t:x_t<y} \beta^t u(x_t) < \varepsilon/2$$

By the triangle inequality,

$$\begin{aligned} |U(x) - U(x^k)| &= \left| \sum_{t=1}^{\infty} \beta^t (u(x_t) - u(x_t^k)) \right| \\ &\leq \sum_{t:x_t<y} \beta^t |u(x_t) - u(x_t^k)| + \sum_{t:x_t\geq y} \beta^t |u(x_t) - u(x_t^k)| \end{aligned}$$

Our goal is to show this sum is less than $\varepsilon > 0$. For the first term, by concavity, we have

$$\begin{aligned} \sum_{t:x_t<y} \beta^t |u(x_t) - u(x_t^k)| &\leq \sum_{t:x_t<y} \beta^t \frac{|u(x_t) - u(0)|}{|x_t|} |x_t - x_t^k| \\ &\leq \sum_{t:x_t<y} \beta^t |u(x_t)| \frac{|x_t - x_t^k|}{|x_t|} \\ &\leq \sum_{t:x_t<y} \beta^t |u(x_t)| \\ &< \varepsilon/2 \end{aligned}$$

For the second term, by concavity, when k is large enough,

$$|u(x_t) - u(x_t^k)| \leq C|x_t - x_t^k|$$

where $C = 2u'(y)$[5]. Moreover, since $s > \beta$, there is a number $\alpha \in (\beta, s)$ such that $(1, \alpha, \alpha^2, ...) \in \Phi_s$. Therefore, as $x^k \to x$ in τ_s,

$$\sup_t |\alpha^t (x_t^k - x_t)| \to 0$$

[5] When if u is not differentiable at y, we replace the derivative by the left derivative of u.

as $k \to +\infty$. Consequently,

$$\begin{aligned}
\sum_{t:x_t \geq y} \beta^t |u(x_t) - u(x_t^k)| &\leq C \sum_{t:x_t \geq y} \beta^t |x_t - x_t^k| \\
&= C \sum_{t:x_t \geq y} \frac{\beta^t}{\alpha^t} \alpha^t |x_t - x_t^k| \\
&\leq C \sum_{t:x_t \geq y} \frac{\beta^t}{\alpha^t} \sup_t \alpha^t |x_t - x_t^k| \\
&\to 0
\end{aligned}$$

That is, when k is large enough,

$$\sum_{t:x_t \geq y} \beta^t (u(x_t) - u(x_t^k)) \leq \varepsilon/2$$

In sum, we proved that U is τ_s-continuous for any $s > \beta$.

On the other hand, to prove U is not τ_s-continuous for any $s < \beta$, we take $x_t^k = \frac{1}{k\beta^t}$. It is easy to see $x^k \to x = 0$ in τ_s for all $s < \beta$. Moreover, by taking subsequences, all x_t^k are in the interval $[0,1]$. By the concavity of the function u in $[0,1]$, we have $|u(x_t^k) - u(x_t)| = |u(x_t^k) - u(0)| \geq D|x_t^k|$ where D is the left derivative of u at 1. Hence,

$$|U(x) - U(x^k)| = \sum_{t=0}^{\infty} \beta^t u(x_t^k) \geq D \sum_{t=0}^{\infty} \beta^t |x_t^k| = D \sum_{t=0}^{\infty} 1/k = +\infty$$

That is, U is not τ_s-continuous for any $s < \beta$. ■

Suggested by this proposition, we define the degree of patience for agents with non-separable utility functions as follows:

Definition 3. *A real-valued function U on $\ell_+^\infty(\mathbb{R})$ is β-patient if it is τ_s-continuous for any $s > \beta$ and not τ_s-continuous for any $s < \beta$.*

By Proposition 1, an additively separable utility function $U(x) = \sum_{t=0}^{\infty} \beta^t u(x_t)$ defined in Section 2 is $\beta-$patient.

6 Concluding remarks

In this paper, we proved that, at any equilibrium of a dynamic general equilibirum model, the income effects are arbitrarily small whenever the number of commodities is sufficiently large and all agents are sufficiently patient. By dropping the assumption that prices are bounded from below by a positive number, we extended Vives' intuition on small income effects from partial equilibrium models to general equilibrium models. In addition, we proposed a definition of agents' degree of patience when utility functions are non-separable.

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