RELATIONS AND FUNCTIONS

- A relation R from a set A to a set B is a subset of the cartesian product $A \times B$ obtained by describing a relationship between the first element x and the second element y of the ordered pairs in $A \times B$.
- **Function :** A function f from a set A to a set B is a specific type of relation for which every element x of set A has one and only one image y in set B. We write $f: A \to B$, where $f(x) = y$.
- A function $f: X \to Y$ is one-one (or injective) if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \ \forall \ x_1, x_2 \in X$.
- A function $f: X \to Y$ is onto (or surjective) if given any $y \in Y$, $\exists \ x \in X$ such that $f(x) = y$.
- **Many-One Function :** A function $f: A \to B$ is called many- one, if two or more different elements of A have the same f- image in B.
- **Into function :** A function $f: A \to B$ is into if there exist at least one element in B which is not the f - image of any element in A.
- **Many One -Onto function :** A function $f: A \to R$ is said to be many one- onto if f is onto but not one-one.
- **Many One -Into function :** A function is said to be many one-into if it is neither one-one nor onto.
- A function $f: X \to Y$ is invertible if and only if f is one-one and onto.

TRIGONOMETRIC FUNCTIONS AND EQUATIONS

- **General Solution of the equation $\sin\theta = 0$:** when $\sin\theta = 0$ $\theta = n\pi : n \in I$ i.e. $n = 0, \pm 1, \pm 2$..........

  General solution of the equation $\cos\theta = 0$: when $\cos\theta = 0$ $\theta = (2n+1)\pi/2, n \in I$. i.e. $n = 0, \pm 1, +2$.......

  General solution of the equation $\tan\theta = 0$: General solution of $\tan\theta = 0$ is $\theta = n\pi; n \in I$
- **General solution of the equation**
  - **(a)** $\sin\theta = \sin\alpha$ **:** $\theta = n\pi + (-1)^n\alpha \ ; n \in I$
  - **(b)** $\sin\theta = k$, **where** $-1 \le k \le 1$. $\theta = n\pi + (-1)^n\alpha$, where $n \in I$ and $\alpha = \sin^{-1}k$
  - **(c)** $\cos\theta = \cos\alpha$ **:** $\theta = 2n\pi \pm \alpha, n \in I$
  - **(d)** $\cos\theta = k$, **where** $-1 \le k \le 1$. $\theta = 2n\pi \pm \alpha$, where $n \in I$ and $\alpha = \cos^{-1}k$
  - **(e)** $\tan\theta = \tan\alpha$ **:** $\theta = n\pi + \alpha \ ; n \in I$
  - **(f)** $\tan\theta = k$, $\theta = n\pi + \alpha$, where $n \in I$ and $\alpha = \tan^{-1}k$
  - **(g)** $\sin^2\theta = \sin^2\alpha$ **:** $\theta = n\pi \pm \alpha; \ n \in I$
  - **(h)** $\cos^2\theta = \cos^2\alpha$ **:** $\theta = n\pi \pm \alpha \ ; n \in I$
  - **(i)** $\tan^2\theta = \tan^2\alpha$ **:** $\theta = n\pi \pm \alpha \ ; n \in I$

- $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots\ldots$ to n terms

$$= \frac{\sin\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right]\left[\sin\left(\frac{n\beta}{2}\right)\right]}{\sin(\beta/2)} \ ; \ \beta \ne 2n\pi$$

- $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots\ldots$ to n terms

$$= \frac{\cos\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right]\left[\sin\left(\frac{n\beta}{2}\right)\right]}{\sin\left(\frac{\beta}{2}\right)} \ ; \beta \ne 2n\pi$$

- $\tan\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{b+c}\right)\cot\left(\frac{A}{2}\right)$
- $\sin\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{bc}}$
- $\tan\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$
- $R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C}$
- $R = \frac{abc}{4\Delta}$
- $r = 4R\sin\left(\frac{A}{2}\right).\sin\left(\frac{B}{2}\right).\sin\left(\frac{C}{2}\right)$
- $a = c\cos B + b\cos C$
- Maximum value of $a\sin\theta + b\cos\theta = \sqrt{a^2+b^2}$ and minimum value of $a\sin\theta + b\cos\theta = -\sqrt{a^2+b^2}$

INVERSE TRIGONOMETRIC FUNCTIONS

- **Properties of inverse trigonometric function**

• $\tan^{-1}x + \tan^{-1}y$

$$= \begin{cases} \tan^{-1}\left(\frac{x+y}{1-xy}\right), & \text{if } xy < 1 \\ \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right), & \text{if } x > 0, y > 0 \text{ and } xy > 1 \\ -\pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right), & \text{if } x < 0, y < 0 \text{ and } xy > 1 \end{cases}$$

• $\tan^{-1}x - \tan^{-1}y$

$$= \begin{cases} \tan^{-1}\left(\frac{x-y}{1+xy}\right), & \text{if } xy > -1 \\ \pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right), & \text{if } x > 0, y < 0 \text{ and } xy < -1 \\ -\pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right), & \text{if } x < 0, y > 0 \text{ and } xy < -1 \end{cases}$$

• $\sin^{-1}x+\sin^{-1}y$

$$=\begin{cases}\sin^{-1}\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\}, & \text{if } -1\le x,\ y\le 1 \text{ and } x^2+y^2\le 1 \\ & \text{or if } xy<0 \text{ and } x^2+y^2>1 \\ \pi-\sin^{-1}\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\}, & \text{if } 0<x,\ y\le 1 \\ & \text{and } x^2+y^2>1 \\ -\pi-\sin^{-1}\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\}, & \text{if } -1\le x,\ y<0 \text{ and } x^2+y^2>1\end{cases}$$

• $\cos^{-1}x+\cos^{-1}y$

$$=\begin{cases}\cos^{-1}\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\}, & \text{if } -1\le x,y\le 1 \text{ and } x+y\ge 0 \\ 2\pi-\cos^{-1}\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\}, & \text{if } -1\le x,y\le 1 \text{ and } x+y\le 0\end{cases}$$

$$2\sin^{-1}x=\begin{cases}\sin^{-1}(2x\sqrt{1-x^2}), & \text{if } -\frac{1}{\sqrt2}\le x\le\frac{1}{\sqrt2} \\ \pi-\sin^{-1}(2x\sqrt{1-x^2}), & \text{if } \frac{1}{\sqrt2}\le x\le 1 \\ -\pi-\sin^{-1}(2x\sqrt{1-x^2}), & \text{if } -1\le x\le-\frac{1}{\sqrt2}\end{cases}$$

$$2\tan^{-1}x=\begin{cases}\tan^{-1}\left(\frac{2x}{1-x^2}\right), & \text{if } -1<x<1 \\ \pi+\tan^{-1}\left(\frac{2x}{1-x^2}\right), & \text{if } x>1 \\ -\pi+\tan^{-1}\left(\frac{2x}{1-x^2}\right), & \text{if } x<-1\end{cases}$$

QUADRATIC EQUATIONS AND INEQUALITIES

Roots of a Quadratic Equation : The roots of the quadratic equation are given by

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Nature of roots : In Quadratic equation $ax^2+bx+c=0$. The term b^2-4ac is called discriminant of the equation. It is denoted by Δ or D.

(A) Suppose $a, b, c \in R$ and $a\neq 0$

(i) If $D>0 \Rightarrow$ Roots are Real and unequal

(ii) If $D=0 \Rightarrow$ Roots are Real and equal and each equal to $-b/2a$

(iii) If $D<0 \Rightarrow$ Roots are imaginary and unequal or complex conjugate.

(B) Suppose $a, b, c \in Q$ and $a\neq 0$

(i) If $D>0$ and D is perfect square $\Rightarrow$ Roots are unequal and Rational

(ii) If $D>0$ and D is not perfect square $\Rightarrow$ Roots are irrational and unequal.

Condition for Common Root(s)

Let $ax^2+bx+c=0$ and $dx^2+ex+f=0$ have a common root α (say).

Condition for both the roots to be common is $\frac{a}{d}=\frac{b}{e}=\frac{c}{f}$

If $p+iq$ (p and q being real) is a root of the quadratic equation, where $i=\sqrt{-1}$, then $p-iq$ is also a root of the quadratic equation.

Every equation of n^{th} degree ($n\ge 1$) has exactly n roots and if the equation has more than n roots, it is an identity.

COMPLEX NUMBERS

Exponential Form: If $z=x+iy$ is a complex number then its exponential form is $z=re^{i\theta}$ where r is modulus and θ is amplitude of complex number.

(i) $|z_1|+|z_2|\ge|z_1+z_2|$; here equality holds when $\arg(z_1/z_2)=0$ i.e. z_1 and z_2 are parallel.

(ii) $||z_1|-|z_2||\le|z_1-z_2|$; here equality holds when $\arg(z_1/z_2)=0$ i.e. z_1 and z_2 are parallel.

(iii) $|z_1+z_2|^2+|z_1-z_2|^2=2(|z_1|^2+|z_2|^2)$

- $\arg(z_1z_2)=\theta_1+\theta_2=\arg(z_1)+\arg(z_2)$
- $\arg\left(\frac{z_1}{z_2}\right)=\theta_1-\theta_2=\arg(z_1)-\arg(z_2)$
- For any integer k, $i^{4k}=1, i^{4k+1}=i, i^{4k+2}=-1, i^{4k+3}=-i$
- $|z-z_1|+|z-z_2|=\lambda$, represents an ellipse if $|z_1-z_2|<\lambda$, having the points z_1 and z_2 as its foci. And if $|z_1-z_2|=\lambda$, then z lies on a line segment connecting z_1 and z_2.
- **Properties of Cube Roots of Unity**

(i) $1+\omega+\omega^2=0$ (ii) $\omega^3=1$

(iii) $1+\omega^n+\omega^{2n}=3$ (if n is multiple of 3)

(iv) $1+\omega^n+\omega^{2n}=0$ (if n is not a multiple of 3).

PERMUTATIONS AND COMBINATIONS

The number of permutations of n different things, taken r at a time, where repetition is allowed, is n^r.

Selection of Objects with Repetition : The total number of selections of r things from n different things when each thing may be repeated any number of times is ${}^{n+r-1}C_r$

Selection from distinct objects :
The number of ways (or combinations) of n different things selecting at least one of them is ${}^nC_1+{}^nC_2+{}^nC_3+.....+{}^nC_n$ $=2^n-1$. This can also be stated as the total number of combination of n different things.

Selection from identical objects :
The number of ways to select some or all out of $(p+q+r)$ things where p are alike of first kind, q are alike of second kind and r are alike of third kind is $(p+1)(q+1)(r+1)-1$

Selection when both identical and distinct objects are present:
If out of $(p+q+r+t)$ things, p are alike one kind, q are alike of second kind, r are alike of third kind and t are different, then the total number of combinations is $(p+1)(q+1)(r+1)2^t-1$

Circular permutations:

(a) Arrangements round a circular table :
The number of circular permutations of n different things taken all at a time is $\frac{{}^nP_n}{n}=(n-1)!$, if clockwise and anticlockwise orders are taken as different.

(b) Arrangements of beads or flowers (all different) around a circular necklace or garland:
The number of circular permutations of 'n' different things taken all at a time is $\frac{1}{2}(n-1)!$, if clockwise and anticlockwise orders are taken to be some.

Sum of numbers :

(a) For given n different digits $a_1, a_2, a_3 ......a_n$ the sum of the digits in the unit place of all numbers formed (if numbers are not repeated) is $(a_1+a_2+a_3+.....+a_n)(n-1)!$

(b) Sum of the total numbers which can be formed with given n different digits $a_1, a_2, ...a_n$ is
$(a_1+a_2+a_3+.........+a_n)(n-1)!.(111 ....n \text{ times})$

BINOMIAL THEOREM

Greatest binomial coefficients : In a binomial expansion binomial coefficients of the middle terms are called as greatest binomial coefficients.

(a) If n is even : When $r=\frac{n}{2}$ i.e. ${}^nC_{n/2}$ takes maximum value.

(b) If n is odd : $r=\frac{n-1}{2}$ or $\frac{n+1}{2}$

i.e. ${}^nC_{\frac{n-1}{2}} = {}^nC_{\frac{n+1}{2}}$ and take maximum value.

Important Expansions :
If $|x|<1$ and $n \in Q$ but $n \notin N$, then

(a) $(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2 + ...+\frac{n(n-1).....(n-r+1)}{r!}x^r + ...$

(b) $(1-x)^n = 1-nx+\frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + ...+\frac{n(n-1).....(n-r+1)}{r!}(-x)^r + ...$

SEQUENCE AND SERIES

Properties related to A.P. :

(i) Common difference of AP is given by $d = S_2 - 2S_1$ where S_2 is sum of first two terms and S_1 is sum of first term.

(ii) If for an AP sum of p terms is q, sum of q terms is p, then sum of (p + q) term is (p + q).

(iii) In an A.P. the sum of terms equidistant from the beginning and end is constant and equal to sum of first and last terms.

(iv) If terms $a_1, a_2, ..., a_n, a_{n+1}, ..., a_{2n+1}$ are in A.P., then sum of these terms will be equal to $(2n + 1)a_{n+1}$.

(v) If for an A.P. sum of p terms is equal to sum of q terms then sum of (p + q) terms is zero

(vi) Sum of n AM's inserted between a and b is equal to n times the single AM between a and b i.e. $\sum_{r=1}^{n} A_r = nA$

where $A=\frac{a+b}{2}$

The geometric mean (G.M.) of any two positive numbers a and b is given by $\sqrt{ab}$ i.e., the sequence a, G, b is G.P.

n GM's between two given numbers: If in between two numbers 'a' and 'b', we have to insert n GM $G_1, G_2, ... G_n$ then $a_1, G_1, G_2, ...G_n$, b will be in G.P.
The series consist of (n + 2) terms and the last term is b and first term is a.

$$\Rightarrow ar^{n+2-1} = b \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$G_1 = ar, G_2 = ar^2 ....G_n = ar^n$ or $G_n = b/r$

Use of inequalities in progression :

(a) Arithmetic Mean $\geq$ Geometric Mean

(b) Geometric Mean $\geq$ Harmonic Mean :

$A \geq G \geq H$

STRAIGHT LINES

An acute angle (say θ) between lines L_1 and L_2 with slopes m_1 and m_2 is given by

$$\tan\theta = \left|\frac{m_2-m_1}{1+m_1m_2}\right|, \quad 1+m_1m_2 \neq 0$$

Three points A, B and C are collinear, if and only if slope of AB = slope of BC.

The equation of the line having normal distance from origin is p and angle between normal and the positive x-axis is ω, is given by $x\cos\omega + y\sin\omega = p$.

Co-ordinate of some particular points :
Let $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ are vertices of any triangle ABC, then

Incentre : Co-ordinates of incentre

$$\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$$

where a, b, c are the sides of triangle ABC

Area of a triangle : Let (x_1, y_1), (x_2, y_2) and (x_3, y_3) respectively be the coordinates of the vertices A, B, C of a triangle ABC. Then the area of triangle ABC, is

$$\frac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)]$$

Or

$$=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

CONIC SECTIONS

- **Condition of Tangency :** Circle $x^2 + y^2 = a^2$ will touch the line.

$$y = mx + c \text{ if } c = \pm a\sqrt{1+m^2}$$

- **Pair of Tangents :** From a given point $P(x_1, y_1)$ two tangents PQ and PR can be drawn to the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$. Their combined equation is $SS_1 = T^2$.
- **Condition of Orthogonality :** If the angle of intersection of the two circle is a right angle ($\theta = 90°$) then such circle are called Orthogonal circle and conditions for their orthogonality is $2g_1g_2 + 2f_1f_2 = c_1 + c_2$
- **Tangent to the parabola :**

  Condition of Tangency : If the line $y = mx + c$ touches a parabola $y^2 = 4ax$ then $c = a/m$
- **Tangent to the Ellipse:**

  Condition of tangency and point of contact :

  The condition for the line $y = mx + c$ to be a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is that $c^2 = a^2m^2 + b^2$ and the coordinates of the points of contact are $\left(\pm\frac{a^2m}{\sqrt{a^2m^2+b^2}}, \mp\frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$
- **Normal to the ellipse**

  (i) Point Form : The equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$

  (ii) Parametric Form : The equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $(a\cos\theta, b\sin\theta)$ is $ax\sec\theta - by\,\mathrm{cosec}\theta = a^2 - b^2$
- **Tangent to the hyperbola :**

  Condition for tangency and points of contact : The condition for the line $y = mx + c$ to be a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is that $c^2 = a^2m^2 - b^2$ and the coordinates of the points of contact are $\left(\pm\frac{a^2m}{\sqrt{a^2m^2-b^2}}, \pm\frac{b^2}{\sqrt{a^2m^2-b^2}}\right)$
- **Chord of contact :**

  The equation of chord of contact of tangent drawn from a point $P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $T = 0$

  where $T \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$
- **Equation of normal in different forms :**

  Point Form : The equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$

THREE DIMENSIONAL GEOMETRY

- **Slope Form :** The equation of normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in terms of slope 'm' is

$$y = mx \pm \frac{m(a^2+b^2)}{\sqrt{a^2-b^2m^2}}$$

- **Conditions of Parallelism and Perpendicularity of Two Lines:**

  Case-I : When dc's of two lines AB and CD, say ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 are known.

  $AB \,||\, CD \Leftrightarrow \ell_1 = \ell_2, m_1 = m_2, n_1 = n_2$

  $AB \perp CD \Leftrightarrow \ell_1\ell_2 + m_1m_2 + n_1n_2 = 0$

  Case-II : When dr's of two lines AB and CD, say a_1, b_1, c_1 and a_2, b_2, c_2 are known

  $AB \,||\, CD \Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

  $AB \perp CD \Leftrightarrow a_1a_2 + b_1b_2 + c_1c_2 = 0$
- If ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 are the direction cosines of two lines; and θ is the acute angle between the two lines; then $\cos\theta = |\ell_1\ell_2 + m_1m_2 + n_1n_2|$.
- Equation of a line through a point (x_1, y_1, z_1) and having direction cosines ℓ, m, n is $\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$
- Shortest distance between $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ is $\left|\frac{(\vec{b}_1\times\vec{b}_2).(\vec{a}_2-\vec{a}_1)}{|\vec{b}_1\times\vec{b}_2|}\right|$
- Let the two lines be

$$\frac{x-\alpha_1}{\ell_1} = \frac{y-\beta_1}{m_1} = \frac{z-\gamma_1}{n_1} \quad \text{..........(1)}$$

and

$$\frac{x-\alpha_2}{\ell_2} = \frac{y-\beta_2}{m_2} = \frac{z-\gamma_2}{n_2} \quad \text{..........(2)}$$

These lines will coplanar if

$$\begin{vmatrix} \alpha_2-\alpha_1 & \beta_2-\beta_1 & \gamma_2-\gamma_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} = 0$$

The plane containing the two lines is

$$\begin{vmatrix} x-\alpha_1 & y-\beta_1 & z-\gamma_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} = 0$$

- The equation of a plane through a point whose position vector is $\vec{a}$ and perpendicular to the vector $\vec{N}$ is $(\vec{r} - \vec{a}).\vec{N} = 0$

- Vector equation of a plane that passes through the intersection of planes $\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$ is $\vec{r}.(\vec{n}_1 + \lambda\vec{n}_2) = d_1 + \lambda d_2$, where λ is any nonzero constant.
- Two planes $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ are coplanar if $(\vec{a}_2 - \vec{a}_1) + (\vec{b}_1 \times \vec{b}_2) = 0$

DIFFERENTIAL CALCULUS

LIMIT

- **Existence of Limit :**

$\lim\limits_{x\to a} f(x)$ exists $\Rightarrow \lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x) = \ell$

Where ℓ is called the limit of the function

- (i) If $f(x) \le g(x)$ for every x in the deleted nbd of a, then $\lim\limits_{x\to a} f(x) \le \lim\limits_{x\to a} g(x)$

(ii) If $f(x) \le g(x) \le h(x)$ for every x in the deleted nbd of a and $\lim\limits_{x\to a} f(x) = \ell = \lim\limits_{x\to a} h(x)$ then $\lim\limits_{x\to a} g(x) = \ell$

(iii) $\lim\limits_{x\to a} fog(x) = f\left(\lim\limits_{x\to a} g(x)\right) = f(m)$ where $\lim\limits_{x\to a} g(x) = m$

(iv) If $\lim\limits_{x\to a} f(x) = +\infty$ or $-\infty$, then $\lim\limits_{x\to a} \dfrac{1}{f(x)} = 0$

CONTINUITY AND DIFFERENTIABILITY OF FUNCTIONS

- A function f(x) is said to be continuous at a point $x = a$ if

$\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^-} f(x) = f(a)$

- **Discontinuous Functions :**

(a) Removable Discontinuity:

A function f is said to have removable discontinuity at $x = a$ if $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x)$ but their common value is not equal to $f(a)$.

(b) Discontinuity of the first kind: A function f is said to have a discontinuity of the first kind at $x = a$ if $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ both exist but are not equal.

(c) Discontinuity of second kind: A function f is said to have a discontinuity of the second kind at x = a if neither $\lim\limits_{x\to a^-} f(x)$ nor $\lim\limits_{x\to a^+} f(x)$ exists.

Similarly, if $\lim\limits_{x\to a^+} f(x)$ does not exist, then f is said to have discontinuity of the second kind from the right at $x = a$.

- **For a function f :**

Differentiability $\Rightarrow$ Continuity;

Continuity $\not\Rightarrow$ derivability

Not derivibaility $\not\Rightarrow$ discontinuous ;

But discontinuity $\Rightarrow$ Non derivability

- **Differentiation of infinite series:**

(i) If $y = \sqrt{f(x) + \sqrt{f(x) + \sqrt{f(x) + \ldots\ldots\infty}}}$

$\Rightarrow \quad y = \sqrt{f(x) + y} \Rightarrow y^2 = f(x) + y$

$2y\dfrac{dy}{dx} = f'(x) + \dfrac{dy}{dx} \qquad \therefore \dfrac{dy}{dx} = \dfrac{f'(x)}{2y - 1}$

(ii) If $y = f(x)^{f(x)^{f(x)^{\ldots\infty}}}$ then $y = f(x)^y$.

$\therefore \log y = y \log[f(x)]$

$\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{y'.f'(x)}{f(x)} + \log f(x).\left(\dfrac{dy}{dx}\right)$

$\therefore \dfrac{dy}{dx} = \dfrac{y^2 f'(x)}{f(x)[1 - y\log f(x)]}$

(iii) If $y = f(x) + \cfrac{1}{f(x) + \cfrac{1}{f(x) + \cfrac{1}{f(x)}}}$ .... then

$\dfrac{dy}{dx} = \dfrac{y f'(x)}{2y - f(x)}$

DIFFERENTIATION AND APPLICATION

- **Interpretation of the Derivative :** If $y = f(x)$ then, $m = f'(a)$ is the slope of the tangent line to $y = f(x)$ at $x = a$
- **Increasing/Decreasing :**

(i) If $f'(x) > 0$ for all x in an interval I then f(x) is increasing on the interval I.

(ii) If $f'(x) < 0$ for all x in an interval I then f(x) is decreasing on the interval I.

(iii) If $f'(x) = 0$ for all x in an interval I then f(x) is constant on the interval I.

- **Test of Local Maxima and Minima –**

First Derivative Test – Let f be a differentiable function defined on an open interval I and $c \in I$ be any point. f has a local maxima or a local minima at $x = c$, $f'(c) = 0$.

Put $\dfrac{dy}{dx} = 0$ and solve this equation for x. Let $c_1, c_2 \ldots\ldots c_n$ be the roots of this.

If $\dfrac{dy}{dx}$ changes sign from +ve to –ve as x increases through c_1 then the function attains a local max at $x = c_1$

If $\dfrac{dy}{dx}$ changes its sign from –ve to +ve as x increases through c_1 then the function attains a local minimum at $x = c_1$

If $\dfrac{dy}{dx}$ does not changes sign as increases through c_1 then $x = c_1$ is neither a point of local max^m nor a point of local min^m. In this case x is a point of inflexion.

Rate of change of variable :

The value of $\frac{dy}{dx}$ at $x = x_0$ i.e. $\left(\frac{dy}{dx}\right)_{x=x_0}$ represents the rate of change of y with respect to x at $x = x_0$

If $x = \phi(t)$ and $y = \psi(t)$, then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, provided that $\frac{dx}{dt} \neq 0$

Thus, the rate of change of y with respect to x can be calculated by using the rate of change of y and that of x each with respect to t.

Length of Sub–tangent $= \left|y\frac{dx}{dy}\right|$; Sub–normal $= \left|y\frac{dy}{dx}\right|$;

Length of tangent $= \left|y\sqrt{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}}\right|$

Length of normal $= \left|y\left\{1+\left(\frac{dy}{dx}\right)^2\right\}\right|$

Equations of tangent and normal : The equation of the tangent at P (x_1, y_1) to the curve $y = f(x)$ is

$$y - y_1 = \left(\frac{dy}{dx}\right)_P (x - x_1)$$

The equation of the normal at P (x_1, y_1) to the curve $y = f(x)$ is

$$y - y_1 = -\frac{1}{\left(\frac{dy}{dx}\right)_P}(x - x_1)$$

INTEGRAL CALCULUS

Two standard forms of integral :

$$\int e^x [f(x) + f'(x)] \, dx = e^x f(x) + c$$

$$\Rightarrow \int e^x [f(x) + f'(x)] \, dx = \int e^x f(x)dx + \int e^x f'(x)dx$$

$$= e^x f(x) - \int e^x f'(x)\,dx + \int e^x f'(x)$$

(on integrating by parts) $= e^x f(x) + c$

Table shows the partial fractions corresponding to different type of rational functions :

S. No.	Form of rational function	Form of partial fraction
1.	$\frac{px+q}{(x-a)(x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-b)}$
2.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-b)}$
3.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{(x-a)} + \frac{Bx+C}{x^2+bx+c}$

Leibnitz rule : $\frac{d}{dx}\int_{f(x)}^{g(x)} F(t)\,dt = g'(x)F(g(x)) - f'(x)F(f(x))$

If a series can be put in the form

$\frac{1}{n}\sum_{r=0}^{r=n-1} f\left(\frac{r}{n}\right)$ or $\frac{1}{n}\sum_{r=1}^{r=n} f\left(\frac{r}{n}\right)$, then its limit as $n \to \infty$ is $\int_0^1 f(x)\,dx$

Area between curves :

$$y = f(x) \Rightarrow A = \int_a^b [\text{upper function}] - [\text{lower function}]\,dx$$

and $x = f(y) \Rightarrow A = \int_c^d [\text{right function}] - [\text{left function}]\,dy$

If the curves intersect then the area of each portion must be found individually.

Symmetrical area : If the curve is symmetrical about a coordinate axis (or a line or origin), then we find the area of one symmetrical portion and multiply it by the number of symmetrical portion to get the required area.

PROBABILITY

Probability of an event: For a finite sample space with equally likely outcomes Probability of an event is

$P(A) = \frac{n(A)}{n(S)}$, where n (A) = number of elements in the set A, n (S) = number of elements in the set S.

Theorem of total probability : Let $\{E_1, E_2, ..., E_n\}$ be a partition of a sample space and suppose that each of $E_1, E_2, ..., E_n$ has nonzero probability. Let A be any event associated with S, then $P(A) = P(E_1)\,P(A \mid E_1) + P(E_2)\,P(A \mid E_2) + ... + P(E_n)\,P(A \mid E_n)$

Bayes' theorem: If $E_1, E_2, ..., E_n$ are events which constitute a partition of sample space S, i.e. $E_1, E_2, ..., E_n$ are pairwise disjoint and $E_1 \cup E_2 \cup ... \cup E_n = S$ and A be any event with nonzero probability, then

$$P(E_i \mid A) = \frac{P(E_i)\,P(A \mid E_i)}{\sum_{j=1}^{n} P(E_j)\,P(A \mid E_j)}$$

Let X be a random variable whose possible values $x_1, x_2, x_3, ..., x_n$ occur with probabilities $p_1, p_2, p_3, ... p_n$ respectively.

The mean of X, denoted by μ, is the number $\sum_{i=1}^{n} x_i p_i$

The mean of a random variable X is also called the expectation of X, denoted by E (X).

- Trials of a random experiment are called Bernoulli trials, if they satisfy the following conditions :

  (a) There should be a finite number of trials. (b) The trials should be independent. (c) Each trial has exactly two outcomes : success or failure. (d) The probability of success remains the same in each trial.

  For Binomial distribution B (n, p),

  $P(X=x)={}^nC_x q^{n-x} p^x, x=0, 1,..., n \ (q=1-p)$

MATRICES

- **Properties of Transpose**

  (i) $(A^T)^T = A$

  (ii) $(A \pm B)^T = A^T \pm B^T$

  (iii) $(AB)^T = B^T A^T$ (iv) $(kA)^T = k(A)^T$

  (v) $I^T = I$ (vi) $tr(A) = tr(A)^T$

  (vii) $(A_1 A_2 A_3 .....A_{n-1} A_n)^T = A_n^T A_{n-1}^T .....A_3^T A_2^T A_1^T$

- **Symmetric Matrix :** A square matrix $A = [a_{ij}]$ is called symmetric matrix if

  $a_{ij} = a_{ji}$ for all i, j or $A^T = A$

- **Skew-Symmetric Matrix :** A square matrix $A = [a_{ij}]$ is called skew-symmetric matrix if

  $a_{ij} = -a_{ji}$ for all i, j or $A^T = -A$

  Also every square matrix A can be uniquely expressed as a sum of a symmetric and skew-symmetric matrix.

- **Differentiation of a matrix :** If $A = \begin{bmatrix} f(x) & g(x) \\ h(x) & \ell(x) \end{bmatrix}$ then

  $\frac{dA}{dx} = \begin{bmatrix} f'(x) & g'(x) \\ h'(x) & \ell'(x) \end{bmatrix}$ is a differentiation of Matrix A.

DETERMINANTS

- **Properties of adjoint matrix :** If A, B are square matrices of order n and I_n is corresponding unit matrix, then

  (i) $A(adj. A) = |A| I_n = (adj A) A$

  (ii) $|adj A| = |A|^{n-1}$ (Thus A (adj A) is always a scalar matrix)

  (iii) $adj(adj A) = |A|^{n-2} A$

  (iv) $|adj(adj A)| = |A|^{(n-1)^2}$

  (v) $adj(A^T) = (adj A)^T$

  (vi) $adj(AB) = (adj B)(adj A)$

  (vii) $adj(A^m) = (adj A)^m, m \in N$

  (viii) $adj(kA) = k^{n-1}(adj. A), k \in R$

  (ix) $adj(I_n) = I_n$

- **Properties of Inverse Matrix :** Let A and B are two invertible matrices of the same order, then

  (i) $(A^T)^{-1} = (A^{-1})^T$

  (ii) $(AB)^{-1} = B^{-1}A^{-1}$

  (iii) $(A^k)^{-1} = (A^{-1})^k, k \in N$

  (iv) $adj(A^{-1}) = (adj A)^{-1}$

  (v) $(A^{-1})^{-1} = A$

  (vi) $|A^{-1}| = \frac{1}{|A|} = |A|^{-1}$

  (vii) If $A = diag(a_1, a_2, ..., a_n)$, then $A^{-1} = diag(a_1^{-1}, a_2^{-1}, ..., a_n^{-1})$

  (viii) A is symmetric matrix $\Rightarrow$ A^{-1} is symmetric matrix.

- **Rank of a Matrix :** A number r is said to be the rank of a $m \times n$ matrix A if

  (a) Every square sub matrix of order (r + 1) or more is singular and (b) There exists at least one square submatrix of order r which is non-singular.

  Thus, the rank of matrix is the order of the highest order non-singular sub matrix.

- Using Crammer's rule of determinant we get

  $\frac{x}{\Delta_1} = \frac{y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{1}{\Delta}$ i. e. $x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}, z = \frac{\Delta_3}{\Delta}$

  Case-I : If $\Delta \neq 0$

  Then $x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}, z = \frac{\Delta_3}{\Delta}$

  $\therefore$ The system is consistent and has unique solutions.

  Case-II if $\Delta = 0$ and

  (i) If at least one of $\Delta_1, \Delta_2, \Delta_3$ is not zero then the system of equations a inconsistent i.e. has no solution.

  (ii) If $d_1 = d_2 = d_3 = 0$ or $\Delta_1, \Delta_2, \Delta_3$ are all zero then the system of equations has infinitely many solutions.

VECTOR ALGEBRA

- Given vectors $x_1\vec{a} + y_1\vec{b} + z_1\vec{c}$, $x_2\vec{a} + y_2\vec{b} + z_2\vec{c}$, $x_3\vec{a} + y_3\vec{b} + z_3\vec{c}$, where $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors, will be coplanar if and only if $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$

- **Scalar triple product :**

  (a) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ then

  $(\vec{a} \times \vec{b}).\vec{c} = [\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

  (b) [a b c] = volume of the parallelopiped whose coterminous edges are formed by $\vec{a}, \vec{b}, \vec{c}$

  (c) $\vec{a}, \vec{b}, \vec{c}$ are coplanar if and only if $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

  (d) Four points A, B, C, D with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively are coplanar if and only if $[\vec{AB}\ \ \vec{AC}\ \ \vec{AD}] = 0$ i.e. if and only if $[\vec{b}-\vec{a}\ \ \vec{c}-\vec{a}\ \ \vec{d}-\vec{a}] = 0$

  (e) Volume of a tetrahedron with three coterminous edges $\vec{a}, \vec{b}, \vec{c} = \frac{1}{6}\left|[\vec{a}\ \vec{b}\ \vec{c}]\right|$

  (f) Volume of prism on a triangular base with three coterminous edges $\vec{a}, \vec{b}, \vec{c} = \frac{1}{2}\left|[\vec{a}\ \vec{b}\ \vec{c}]\right|$

Lagrange's identity :

$$(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})=\begin{vmatrix}\vec{a}.\vec{c} & \vec{a}.\vec{d}\\ \vec{b}.\vec{c} & \vec{b}.\vec{d}\end{vmatrix}=(\vec{a}.\vec{c})(\vec{b}.\vec{d})-(\vec{a}.\vec{d})(\vec{b}.\vec{c})$$

Reciprocal system of vectors : If $\vec{a},\vec{b},\vec{c}$ be any three non coplanar vectors so that $[\vec{a}\,\vec{b}\,\vec{c}]\neq 0$ then the three vectors $\vec{a}'\vec{b}'\vec{c}'$ defined by the equations $\vec{a}'=\frac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]},\ \vec{b}'=\frac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]},\ \vec{c}'=\frac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$ are called the reciprocal system of vectors to the given vectors $\vec{a},\vec{b},\vec{c}$.

STATISTICS

Relation between A.M., G.M. and H.M.

A.M. $\geq$ G.M. $\geq$ H.M.

Equality sign holds only when all the observations in the series are same.

Relationship between mean, mode and median :

(i) In symmetrical distribution

Mean = Mode = Median

(ii) In skew (moderately symmetrical) distribution

Mode = 3 median – 2 mean

Mean deviation for ungrouped data

$$M.D.(\bar{x})=\frac{\sum|x_i-\bar{x}|}{n},\quad M.D.(M)=\frac{\sum|x_i-M|}{n}$$

Mean deviation for grouped data

$$M.D.(\bar{x})=\frac{\sum f_i|x_i-\bar{x}|}{N},\quad M.D.(M)=\frac{\sum f_i|x_i-M|}{N},$$

where $N=\sum f_i$

Variance and standard deviation for ungrouped data

$$\sigma^2=\frac{1}{n}\sum(x_i-\bar{x})^2,\quad \sigma=\sqrt{\frac{1}{n}\sum(x_i-\bar{x})^2}$$

Variance and standard deviation of a discrete frequency distribution

$$\sigma^2=\frac{1}{n}\sum f_i(x_i-\bar{x})^2,\quad \sigma=\sqrt{\frac{1}{N}\sum f_i(x_i-\bar{x})^2}$$

Variance and standard deviation of a continuous frequency distribution

$$\sigma^2=\frac{1}{n}\sum f_i(x_i-\bar{x})^2,\quad \sigma=\sqrt{\frac{1}{N}\sum f_i x_i^2-\left(\sum f_i x_i\right)^2}$$

Coefficient of variation (C.V.) $=\frac{\sigma}{\bar{x}}\times 100,\ \bar{x}\neq 0$

For series with equal means, the series with lesser standard deviation is more consistent or less scattered.

DIFFERENTIAL EQUATIONS

Methods of solving a first order first degree differential equation :

(a) Differential equation of the form $\frac{dy}{dx}=f(x)$

$$\frac{dy}{dx}=f(x)\Rightarrow dy=f(x)\,dx$$

Integrating both sides we obtain

$$\int dy=\int f(x)dx+c \text{ or } y=\int f(x)dx+c$$

(b) Differential equation of the form $\frac{dy}{dx}=f(x)\,g(y)$

$$\frac{dy}{dx}=f(x)\,g(y)\Rightarrow\int\frac{dy}{g(y)}=\int f(x)\,dx+c$$

(c) Differential equation of the form of $\frac{dy}{dx}=f(ax+by+c)$ **:**

To solve this type of differential equations, we put

$$ax+by+c=v \text{ and } \frac{dy}{dx}=\frac{1}{b}\left(\frac{dv}{dx}-a\right)$$

$$\therefore\ \frac{dv}{a+bf(v)}=dx$$

So solution is by integrating $\int\frac{dv}{a+bf(v)}=\int dx$

(d) Differential Equation of homogeneous type :

An equation in x and y is said to be homogeneous if it can be put in the form $\frac{dy}{dx}=\frac{f(x,y)}{g(x,y)}$ where $f(x,y)$ and $g(x,y)$ are both homogeneous functions of the same degree in x & y.

So to solve the homogeneous differential equation $\frac{dy}{dx}=\frac{f(x,y)}{g(x,y)}$, substitute $y=vx$ and so $\frac{dy}{dx}=v+x\frac{dV}{dx}$

Thus $v+x\frac{dv}{dx}=f(v)\Rightarrow\frac{dx}{x}=\frac{dv}{f(v)-v}$

Therefore solution is $\int\frac{dx}{x}=\int\frac{dv}{f(v)-v}+c$

Linear differential equations :

$$\frac{dy}{dx}+Py=Q \qquad ....(1)$$

Where P and Q are either constants or functions of x.

Multiplying both sides of (1) by $e^{\int P\,dx}$, we get

$$e^{\int P\,dx}\left(\frac{dy}{dx}+Py\right)=Q\,e^{\int P\,dx}$$

On integrating both sides with respect to x we get

$$y\,e^{\int P\,dx}=\int Q\,e^{\int P\,dx}+c$$

which is the required solution, where c is the constant and $e^{\int P\,dx}$ is called the integration factor.