

Subject Code: 35 (NS)

MATHEMATICS

(English Version)

Instructions:

1. The question paper has five parts namely A, B, C, D, and E. Answer all the parts.
2. Use the Graph Sheet for the question on Linear Programming Problem on Part –E.

PART – A

Answer all the ten questions:

1) Define Binary Operation.

1×10=10

Sol.

A binary operation $*$ on a set A is a function $*$: $A \times A \rightarrow A$. We denote $*$ (a,b) By $a * b$.

2) Find the principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$.

Sol.

Simplify the expression,

$$\pi - \cos^{-1}\left(-\frac{1}{2}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$

3) Define a scalar matrix.

Sol.

Scalar matrix – Scalar matrix is a diagonal matrix in which all the diagonal elements are equal.

4) Find a value of X for which $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$

Sol

Simplify the expression,

$$\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$$

$$3 - x^2 = 3 - 8 \Rightarrow x^2 = 8$$

$$x = \pm 2\sqrt{2}$$

5) If $y = \sin(x^2 + 5)$ then find $\frac{dy}{dx}$.

Sol.

Simplify the expression,

$$\begin{aligned} \frac{dy}{dx} \sin(x^2 + 5) \\ = \cos(x^2 + 5)(2x) \end{aligned}$$

6) Find $\int (1-x)\sqrt{x}.dx$

Sol.

Simplify the expression,

$$\begin{aligned} \int (1-x)\sqrt{x}.dx &= \int (1-x)(x)^{1/2}.dx \\ &= \frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2} + c \end{aligned}$$

7) Find a value of x for which $x(\hat{i} + \hat{j} + \hat{k})$ is a unit vector.

Sol.

Simplify the expression,

$$\begin{aligned} x(\hat{i} + \hat{j} + \hat{k}) \\ \sqrt{x^2 + x^2 + x^2} = 1 \\ = \sqrt{3x^2} = 1 \\ x = \pm \frac{1}{\sqrt{3}} \end{aligned}$$

8) If a line has direction ratios 2, -1, -2 then determine its direction cosines.

Sol.

Simplify the expression,

$$\text{The direction cosines are } \frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}.$$

9) Define objective function in Linear Programming Problem.

Sol.

A linear function $Z = ax + by$, where a, b are constants which can be optimized (maximized) in an LPP, is called objective function.

10) If $P(E) = 0.6$, $P(F) = 0.3$ and $P(E \cap F) = 0.2$ then find $P(F/E)$.

Sol.

Simplify the expression,

$$P(F/E) = \frac{P(E \cap F)}{P(E)} = \frac{0.2}{0.6}$$

$$P(F/E) = 0.34$$

PART-B

Answer any ten questions:

(10x2=20)

11) Show that the function $f : N \rightarrow N$ given by $f(x) = 2x$ one-one not onto.

Sol.

Simplify the expression,

$$\text{Let } x_1, x_2 \in N_r$$

$$f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2$$

$$x_1 = x_2 = 2x_1 = 2x_2$$

$$x_1 = x_2 \quad \therefore f \text{ is one-one}$$

Let $y \in N$ and $f(x) = y$ Then

$$f(x) = y \Rightarrow 2x = y \Rightarrow x = \frac{y}{2} \notin N \text{ (Domain) since } f \text{ is not onto.}$$

12) Prove that $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, $x \in [-1, 1]$.

Sol.

Prove the expression,

$$\text{Let } \sin^{-1} x = y$$

Then $x = \sin y$

$$\cos\left(\frac{\pi}{2} - y\right)$$

$$\cos^{-1} x = \frac{\pi}{2} - y;$$

$$\sin^{-1} x + \cos^{-1} = \frac{\pi}{2}$$

$$\cos^{-1} x + y = \frac{\pi}{2}$$

13. Write $\cos^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right), x > 1$ in the simplest form.

Sol.

Simplify the expression,

Put $x = \sec \theta$

$$\cot^{-1}\left[\frac{1}{\sqrt{x^2-1}}\right] = \cot^{-1}\left(\frac{1}{\sqrt{\sec^2 \theta - 1}}\right)$$

$$\cot^{-1} \frac{1}{\sqrt{\tan^2 \theta}}$$

$$= \cot^{-1}\left(\frac{1}{\tan \theta}\right) = \cot^{-1}(\cot \theta) = \theta = \sec^{-1} x.$$

14. Find the area of the triangle with vertices (2,7), (1,1) and (10,8) using determinant method.

Sol.

Simplify the expression,

Area of triangle =

$$\frac{1}{2} \begin{vmatrix} 2 & 7 & 1 \\ 1 & 1 & 1 \\ 10 & 8 & 1 \end{vmatrix}$$

$$\begin{aligned}
 &= \frac{1}{2} [2(1-8) - 7(1-10) + (8-10)] \\
 &= \frac{1}{2} [-14 + 63 - 2] = \frac{47}{2} \text{ sq.units}
 \end{aligned}$$

15. Find $\frac{dy}{dx}$, if $y = (\log x)^{\cos x}$

Sol.

Simplify the expression,

$$y = (\log x)^{\cos x}$$

$$\log y = \cos x \cdot \log(\log x)$$

$$= \frac{1}{2} \frac{dy}{dx} = \cos x \frac{d}{dx} (\log(\log x)) \frac{d}{dx} (\cos x)$$

$$= \frac{\cos x}{x \log x} - \log(\log x) \sin x$$

$$\frac{dy}{dx} = y \left[\frac{\cos x}{x \log x} - \log(\log x) \sin x \right]; \frac{dy}{dx} (\log x)^{\cos x} \left[\frac{\cos x}{x \log x} - \log(\log x) \sin x \right]$$

16. if $ax^2 + by^2 = \cos y$

Sol.

Simplify the expression,

$$ax^2 + by^2 = \cos y$$

Differentiating w.r. to x

$$a + 2by \cdot \frac{dy}{dx} = -(\sin y) \frac{dy}{dx}$$

$$= (2by + \sin y) \frac{dy}{dx} = -a; \frac{dy}{dx} = \frac{-a}{2by + \sin y}$$

17. Find the approximate change in the value V of a cube of side x meters caused by increasing side by 2%.

Sol.

Simplify the expression,

$$V = x^3$$

$$\delta V = 3x^2 \times 0.02x = 0.06x^3 m^3$$

18. Find $\int \frac{1}{\cos^2 x (1 - \tan x)^2} dx$

Sol.

Simplify the expression,

$$I = \int \frac{\sec^2 x}{(1 - \tan^2 x)} dx$$

$$1 - \tan x = t \Rightarrow \sec^2 dx = -dt$$

Put $I = \int \frac{1}{t^2} (-dt) = \frac{1}{t} = \frac{1}{1 - \tan x} + c$

19. Find $\int \sin 2x \cos 3x dx$.

Sol.

Simplify the expression,

$$\int \sin 2x \cos 3x dx.$$

$$= \frac{1}{2} \int (\sin 5x - \sin x) dx = \frac{1}{2} \left[\cos x - \frac{\cos 5x}{5} \right] + c$$

20. Find the order and degree (if defined) of the differential equation

$$\left(\frac{d^2y}{dx^2}\right)^2 + \cos\left(\frac{dy}{dx}\right) = 0$$

Sol.

$$\left(\frac{d^2y}{dx^2}\right)^2 + \cos\left(\frac{dy}{dx}\right) = 0$$

Order =2, Degree is not defined.

21. If $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 8$ and $|\vec{a}| = 8|\vec{b}|$

Sol.

Simplify the expression,

$$(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 8 \Rightarrow |\vec{a}|^2 - |\vec{b}|^2 = 8$$

$$|\vec{b}|^2 = \frac{8}{63} \Rightarrow \sqrt{\frac{8}{63}} = \frac{2\sqrt{2}}{\sqrt{63}}$$

22. Find the projection of the vector.

Sol.

Simplify the expression,

$$\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}, \vec{b} = \hat{i} + 2\hat{j} + \hat{k}$$

$$\begin{aligned} \text{Projection of } \vec{a} \text{ on } \vec{b} \text{ is } \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} &= \frac{(2\hat{i} + 3\hat{j} + \hat{k}) \cdot (\hat{i} + 2\hat{j} + \hat{k})}{\sqrt{1^2 + 2^2 + 1^2}} = \frac{2 + 6 + 2}{\sqrt{6}} \\ &= \frac{10}{\sqrt{6}} = \frac{10\sqrt{6}}{6} = \frac{5\sqrt{6}}{3} \end{aligned}$$

23. Find the distance of the point (3,-2,1) from the plane $2x - y + 2z + 3 = 0$

Sol.

Simplify the expression,

The distance of point (3,-2,1) from the plane $2x - y + 2z + 3 = 0$

$$\begin{aligned} p &= \left| \frac{ax_1 + by_1 + cz_1}{\sqrt{a^2 + b^2 + c^2}} \right| \\ &= \left| \frac{2 \cdot 3 - (-2) + 2(1) + 3}{\sqrt{4 + 1 + 4}} \right| \\ &= \frac{13}{3} \end{aligned}$$

24. Probability of solving specific problem independently by A and B are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. If both try to solve the problem independently. Find the probability that the problem is solved.

Sol.

Let E: A solves the problem

F: B Solves the problem

E and F are independent events, then

$$P(E \text{ or } F) = P(E \cup F) = P(E) + P(F) - P(E \cap F) = \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$$

PART –C

Answer any ten questions:

(10x3=30)

25. Check whether the relation R in IR of real numbers defined by $R = \{(a, b) : a \leq b^3\}$ is reflexive or transitive.

Sol.

$$R = \{(a, b) : a \leq b^3\}$$

R is not reflexive, for $\left(\frac{1}{2}, \frac{1}{2}\right) \notin R \left(\because \frac{1}{2} \neq \left(\frac{1}{2}\right)^3 \right)$

R is not symmetric, for $(a, b) \in R$, but $(a, c) \notin R$

$$(a, b) = (1, 2) \in R \left(\because 1 \leq 2^3 \right)$$

$$\text{But } (b, a) = (2, 1) \notin R \left(\because 1 \not\leq 2^3 \right)$$

R is not transitive, for $(a, b) \in R$ and $(b, c) \in R$, but $(a, c) \notin R$

$$(a, b) = (9, 3) \in R \left(\because 9 \leq 3^3 \right)$$

$$(b, c) = (3, 2) \in R \left(\because 3 \leq 2^3 \right)$$

$$\text{But } (a, c) = (9, 2) \notin R \left(\because 9 \not\leq 2^3 \right)$$

26. Prove that $\cos^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) = \cos^{-1}\left(\frac{33}{65}\right)$

Sol.

Prove the expression,

We know that,

$$\tan^{-1}(x + y) = \left(\frac{x + y}{1 - xy} \right)$$

$$\begin{aligned}
 &= \tan^{-1} \left(\frac{3/4 + 5/12}{1 - 3/4 \times 5/12} \right) \\
 &= \tan^{-1} \left(\frac{56}{33} \right) = \cos^{-1} \left(\frac{33}{65} \right)
 \end{aligned}$$

27 By using elementary transformations, find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

Sol.

Simplify the matrix expression,

$$A = IA$$

$$\begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} A \quad \therefore R_2 \rightarrow R_2 - 2R_1$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix} A \quad R_2 \rightarrow \frac{1}{5} R_2$$

$$\therefore A^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix}$$

28. If $x = a(\theta + \sin \theta)$, $y = a(1 - \cos \theta)$ then Show that $\frac{dy}{d\theta} = \tan \left(\frac{\theta}{2} \right)$.

Sol.

$$X = a(\theta + \sin \theta) \Rightarrow \frac{dy}{d\theta} a(1 + \cos \theta) = 2a \cos^2 \theta / 2$$

$$Y = a(1 - \cos \theta) \Rightarrow \frac{dy}{d\theta} = a(\sin \theta) = 2a \sin \theta / 2 \cos \theta / 2$$

$$\therefore \frac{dy}{d\theta} / \frac{dx}{d\theta} = \frac{2a \sin \theta / 2 \cos \theta / 2}{2 \cos^2 \theta / 2} = \tan \theta / 2$$

29. Verify Rolle's for the function $f(x) = x^2 + 2x - 8$, $X \in [-4, 2]$.

Sol.

$$f(x) = x^2 + 2x - 8 \text{ is continuous in } [-4, 2]$$

$$f'(x) = 2x + 2 \Rightarrow f(x) \text{ is differential in } [-4, 2]$$

$$f(a) = f(-4) = 0; f(b) = f(2) = 0$$

Conditions of Rolle's Theorem are satisfied.

$$\begin{aligned} \text{Now } f'(c) &= 0 \Rightarrow 2c + 2 \\ c &= -1 \in (-4, 2) \end{aligned}$$

30. Find the intervals in which the function f given by $f(x) = 2x^3 - 3x^2 - 36x + 7$ is strictly increasing.

Sol.

$$f(x) = 2x^3 - 3x^2 - 36x + 7$$

$$f'(x) = 6x^2 - 6x - 36 = 6(x^2 - x - 6)$$

$$= 6(x+2)(x-3) = 6(x-(-2))(x-3) \Rightarrow x = -2, 3$$

$$f(x) \text{ is strictly increasing } \Rightarrow f'(x) > 0$$

$$X \in (-\infty, -2) \cup (3, +\infty)$$

31. Find $\int x \cdot \log x dx$.

Sol.

Simplify the expression,

$$I = \int (\log_e x) x dx = (\log_e x) \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} dx = (\log x) \frac{x^2}{2} - \frac{x^2}{4} + c$$

$$I = \int (\log_e x) d\left(\frac{1}{2} X^2\right) = (\log_e X) \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x} dx$$

$$= (\log_e X) \frac{x^2}{2} - \frac{x^2}{4} + c$$

32. Evaluate: $\int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x} dx.$

Sol.

Simplify the expression,

$$\begin{aligned} I &= \int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x} dx. = \int_0^{\pi/2} \frac{-d(\cos x)}{1 + \cos^2 x} = - \int_0^{\pi/2} \frac{d(\cos x)}{1 + \cos^2 x} \\ &= \left[-\tan^{-1}(\cos x) \right]_0^{\pi/2} = -\tan^{-1}\left(\cos \frac{\pi}{2}\right) + \tan^{-1}(\cos(0)) = 0 + \frac{\pi}{4} = \frac{\pi}{4} \end{aligned}$$

33. Find the area of the region bounded by the curve $y^2 = 4x$ and the line $X=3$.

Sol.

Figure

Required area,

$$\begin{aligned} A &= 2 \int_2^3 y dx = 2 \int_0^3 2x^{1/2} dx = 4 \times \frac{2}{3} (X^{3/2})_2^3 \\ &= \frac{8}{3} (3^{3/2}) = 8\sqrt{3} \text{sq.units} \end{aligned}$$

34. Form the differential equation of the family of curves $y = ae^{3x} + be^{-2x}$ by eliminating arbitrary constants a and b.

Sol.

$$y = ae^{3x} + be^{-2x} \quad \text{---(1)}$$

Multiplying both sides by e^{2x}

$$ye^{2x} = ae^{5x} + b$$

Diff. w.r.t x we get,

$$ye^{2x}(2) + e^{2x}(y') = ae^{5x}(5)$$

Multiplying both sides by e^{-5x}

$$2ye^{2x} \cdot e^{-5x} + y'e^{2x}e^{-5x} = 5ae^{5x}e^{-5x}$$

$$2ye^{-3x} + y'e^{-3x} = 5a$$

$$2ye^{-3x}(-3) + 2e^{-3x}(y') + y'(e^{-3x})(y'') = 0$$

$$e^{-3x}(-6y + 2y' - 3y' + y'') = 0$$

$$\therefore y'' - y' - 6y = 0$$

35. Find a unit vector perpendicular to each of the vector $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ where

$$\vec{a} = \hat{i} + \hat{j} + \hat{k}, \vec{b} = \hat{i} + 2\hat{j} + 3\hat{k},$$

Sol

$$\vec{a} + \vec{b} = (2, 3, 4); \vec{a} - \vec{b} = (0, -1, -2)$$

Required unit vectors perpendicular to both $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ are

$$\hat{n} = \frac{(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})}{\left| (\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) \right|}$$

$$(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 0 & -1 & -2 \end{vmatrix} = -2\hat{i} + 4\hat{j} - 2\hat{k}$$

$$\therefore \left| (\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) \right| = \sqrt{(-2)^2 + 4^2 + (-2)^2} = \sqrt{24} = 2\sqrt{6}$$

36 Show that the four points with position vectors;

$4\hat{i} + 8\hat{j} + 12\hat{k}, 2\hat{i} + 4\hat{j} + 6\hat{k}, 3\hat{i} + 5\hat{j} + 4\hat{k}$ and $5\hat{i} + 8\hat{j} + 5\hat{k}$ are coplanar.

Sol.

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (2, 4, 6) - (4, 8, 12) = (-2, -4, -6)$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (3, 5, 4) - (4, 8, 12) = (-1, -3, -8)$$

$$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = (5, 8, 5) - (4, 8, 12) = (1, 0, -7)$$

$$\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD}) = [\overrightarrow{AB}, \overrightarrow{AC} \times \overrightarrow{AD}] = \begin{vmatrix} -2 & -4 & -6 \\ -1 & -3 & -8 \\ 1 & 0 & -7 \end{vmatrix}$$

$$= (-2)(21) + 4(15) - 6(0 + 3) = 0$$

$$= -42 + 60 - 18 = 0 \quad \text{The points are coplanar}$$

37. Find the vector equation of the plane passing through the points R(2,5,-3), S(-2,-3,5) and T(5,3,-3).

Sol.

Required equation of the plane is

$$[\vec{r} - \vec{a}, \vec{b} - \vec{a}, \vec{c} - \vec{a}] = 0$$

$$(\vec{r} - \vec{a}) \cdot (\vec{b} - \vec{a}) \times (\vec{c} - \vec{a}) = 0$$

$$(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a}) = \begin{vmatrix} i & j & k \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix} = 16i + 24j + 32k = (16, 24, 32)$$

$$(\vec{r} - (2, 5, -3)) \cdot (16, 24, 32) = 0$$

$$\vec{r} \cdot (16i + 24j + 32k) = 56$$

$$\vec{r} \cdot (2i + 3j + 4k) = 7$$

38. An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accident is 0.01, 0.03 and 0.15 respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver?

Sol.

Let A be the event that one of the insured persons meets with an accident.

$$\text{Total} = 2000 + 4000 + 6000 = 12000$$

Let E_1, E_2, E_3 be the events that the insured scooter, car and truck drivers meet with an accident respectively.

$$P(E_1) = \frac{2000}{12000} = \frac{1}{6}; P(E_2) = \frac{4000}{12000} = \frac{1}{3}; P(E_3) = \frac{6000}{12000} = \frac{1}{2}$$

$$P(E_1 / A) = \frac{P(E_1)P(A|E_1)}{P(E_1)P(A|E_1) + P(E_2)P(A|E_2) + P(E_3)P(A|E_3)}$$

$$\begin{aligned} \text{Now, } &= \frac{\frac{1}{6} \times 0.01}{\frac{1}{6} \times 0.01 + \frac{1}{3} \times 0.03 + \frac{1}{2} \times 0.15} \\ &= \frac{0.01}{0.52} = \frac{1}{52} \end{aligned}$$

PART- D

Answer any six questions:

(6 x 5=30)

39. Let $f; N \rightarrow Y$ be a function defined as $f(x) = 4x + 3$, where $y = \{y \in N : y = 4x + 3 \text{ for some } x \in N\}$. Show that f is invertible. Find the inverse of f .

Sol.

Let X_1, X_2, X_1 Now, $f(X_1) = f(X_2)$

$$\Rightarrow 4X_1 + 3 = 4X_2 + 3 \Rightarrow 4X_1 = 4X_2 \Rightarrow X_1 = X_2 \text{ } f \text{ is one-one}$$

$$y \in N \text{ and } f(x) = y$$

$$4x + 3 = y \Rightarrow 4x = y - 3$$

$$\text{Let } X = \frac{Y-3}{4} \in N(\text{domain}) \quad \therefore f \text{ is onto}$$

$$\text{Let } f^{-1}(y) = X \Rightarrow Y = f(X)$$

$$Y = 4X + 3 \Rightarrow X = \frac{Y-3}{4}; f^{-1}(y) = \frac{Y-3}{4}$$

40 If $A = \begin{vmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{vmatrix}$ then show that $A^3 - 23A - 40I$

Sol:

Prove the expression,

$$A = \begin{vmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{vmatrix}$$

$$A^2 = A \times A = \begin{vmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{vmatrix} \begin{vmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{vmatrix} = A = \begin{vmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{vmatrix}$$

$$A^3 = A^2 \times A = \begin{vmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{vmatrix} \begin{vmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{vmatrix} = \begin{vmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{vmatrix}$$

$$A^3 - 23A - 40I = \begin{vmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{vmatrix} - 23 \begin{vmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{vmatrix} - 40 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

41. Solve the following system of linear equation by matrix method.

$$3x - 2y + 3z = 8$$

$$2x + y - z = 1$$

$$4x - 3y + 2z = 4$$

Sol:

Let A

$$\begin{vmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{vmatrix}, X = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \text{ and } B = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

$$|A| = 3(2-3) + 2(4+4) + 3(-6-4) = -17$$

$$\text{Co-factor matrix of } A = \begin{vmatrix} -1 & -8 & 10 \\ -5 & -6 & 1 \\ -1 & 9 & 7 \end{vmatrix} \quad \therefore \text{Adj}(A) = \begin{vmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{vmatrix}$$

$$X = A^{-1}B, \text{ where, } A^{-1} = \frac{\text{Adj}(A)}{|A|}$$

$$A^{-1} = \frac{-1}{17} = \begin{vmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{vmatrix} \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix} = \frac{-1}{17} \begin{vmatrix} -8 & (-5) & (-4) \\ -64 & (-6) & 36 \\ -80 & 1 & 28 \end{vmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow X = 1, Y = 2, Z = 3$$

42. If $y = \sin^{-1} x$, show that $(1-x^2) \frac{d^2 y}{dx^2} - x \left(\frac{dy}{dx} \right) = 0$.

Sol.

$$y = \sin^{-1} x \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \sqrt{1-x^2} \frac{dy}{dx} = 1 \Rightarrow \sqrt{1-x^2} \frac{d^2 y}{dx^2} + \frac{dy}{dx} \cdot \frac{1(-2x)}{2\sqrt{1-x^2}} = 0 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} = 0$$

43. The length x of a rectangle is decreasing at the rate of 3 cm/min and the width y is increasing at the rate of 2 cm/min. when $X=10$ cm and $y=6$ cm, find the rates of change of

i) The perimeter and

ii) The area of the rectangle.

Sol.

$$\frac{dy}{dx} = -3 \text{ cm / min} \quad \text{and} \quad \frac{dy}{dx} = 2 \text{ cm / min}$$

(i) The perimeter $P=2(x+y)$

$$\begin{aligned} \frac{dp}{dt} &= 2 \left(\frac{dx}{dt} + \frac{dy}{dt} \right) \\ &= 2(-3 + 2) = -2 \text{ cm / min} \end{aligned}$$

(ii) The area A of the rectangle $A=x.y$

$$\frac{dA}{dt} = x \frac{dy}{dx} + y \frac{dx}{dy} = 10(2) + (6)(-3) = 2 \text{ cm}^2 / \text{min}$$

44. Find the integral of $\frac{1}{x^2 - a^2}$ With respect to x and hence evaluate $\int \frac{1}{x^2 - 16} . dx$

Sol.

$$\begin{aligned} \frac{1}{x^2 - a^2} &= \frac{1}{(x+a)(x-a)} \\ &= \frac{1}{2a} \left[\frac{(x+a) - (x-a)}{(x+a)(x-a)} \right] = \frac{1}{2a} \left[\frac{1}{(x-a)} - \frac{1}{(x+a)} \right] \\ &= \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \left[\int \frac{dx}{x^2 - 4} - \int \frac{dx}{x-a} \right] = \frac{1}{2a} \left[\log|x-a| - \log|x+a| \right] + c = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c \\ &= \int \frac{dx}{x^2 - 16} = \int \frac{dx}{x^2 - 4^2} = \frac{1}{8} \log_e \left| \frac{x-4}{x+4} \right| + c \end{aligned}$$

45. Using the method of integration, find the smaller area enclosed by the circle $x^2 + y^2 = 4$ and the line $x + y = 2$.

Sol.

$$x^2 + y^2 = 4 \Rightarrow y\sqrt{4-x^2}$$

$$x + y = 2 \Rightarrow y = 2 - x$$

Required area =

$$\begin{aligned} & \int_0^2 \sqrt{4-x^2} - (2-x) dx \\ &= \left[\frac{x}{2} \sqrt{4-x^2} + \frac{4}{2} \sin^{-1} \frac{x}{2} - 2x + \frac{x^2}{2} \right]_0^2 \\ &= [0 + 2 \sin^{-1} 1 - 4 + 2] - [0] \\ &= 2 \left(\frac{\pi}{2} \right) - 2 = (\pi - 2) \text{ sq. units} \end{aligned}$$

46. Find the general solution of the differential equation

$$\frac{dy}{dx} + (\sec x)y = \tan x \quad \left(0 \leq x < \frac{\pi}{2} \right)$$

Sol.

$$\text{Given, } \frac{dy}{dx} + y \sec x = \tan x \Rightarrow P = \sec x, Q = \tan x$$

$$\text{I.F} = e^{\int p dx} = e^{\int \sec x dx} = e^{\log_e (\sec x + \tan x)} = \sec x + \tan x$$

The general solution is

$$y(I.F) = \int Q(I.F) dx + c$$

$$y(\sec x + \tan x) = \int \tan x (\sec x + \tan x) dx + c$$

$$= \int \tan x \sec x dx + \int \tan^2 x dx + c \Rightarrow \sec x + \int (\sec^2 x - 1) dx + c$$

$$= y(\sec x + \tan x) = \sec x + \tan x - x + c$$

47. Derive the equation of a line in space passing through a given point and parallel to a given vector in both vector and Cartesian form.

Sol.

Let 'l' be the line which passes through the point A and is parallel to given vector $\vec{b}$

Let $\vec{r}$ be the position vector of any arbitrary point P on the line 'l'

Then $\overrightarrow{AP}$ is parallel to $\vec{b}$

That $\overrightarrow{AP} = \lambda \vec{b}$ where λ is some real number

$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \lambda \vec{b}$$

$$\vec{r} = \vec{a} + \lambda \vec{b} \text{ is the vector form.}$$

Let $A(x_1, y_1, z_1)$ be the given point and (x, y, z) be the arbitrary point on the line.

Let a, b, c be the direction ratios of $\vec{b}$

$$\overrightarrow{OA} = \vec{a} = x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k}$$

$$\overrightarrow{OP} = \vec{r} = x \hat{i} + y \hat{j} + z \hat{k}$$

$$\text{and } \vec{b} = a \hat{i} + b \hat{j} + c \hat{k}$$

Substitute these values in vector equation

$$\text{We get } x \hat{i} + y \hat{j} + z \hat{k} = x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k} + \lambda (a \hat{i} + b \hat{j} + c \hat{k})$$

Equating the coefficients of $\hat{i}, \hat{j}, \hat{k}$

$$\text{We get } x - x_1 = \lambda a, \quad y - y_1 = \lambda b, \quad z - z_1 = \lambda c$$

By eliminating λ we get Cartesian form equation of the line.

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

48. Five cards are drawn successively with replacement from a well shuffled deck of 52 cards. What is the probability that

i) All the five cards are spades?

ii) Only three cards are spades?

iii) None is a spade?

Sol.

Let 'X' denote the number of spade cards among the 5 cards drawn.

In a well shuffled deck of 52 cards, there are 13 spade cards.

$P = p$ (success) = P (a spade card drawn)

$$\frac{13}{52} = \frac{1}{4} \text{ and } q = 1 - \frac{1}{4} = \frac{3}{4} \text{ and } n=5$$

$$P(X = r) = {}^nC_r P^r q^{n-r}, \text{ where } r=0, 1, 2, 3, 4, 5$$

$$P(X = r) = {}^5C_r \left(\frac{1}{4}\right)^r \left(\frac{3}{4}\right)^{5-r}$$

(i) P (all the five cards are spades)

$$P(X = 5) = {}^5C_5 P^5 q^0 = \left(\frac{1}{4}\right)^5 = \frac{1}{1024}$$

(ii) P (only 3 cards are spades) = P (X=3)

$${}^5C_3 P^3 q^2 = \frac{5 \times 4 \times 3}{3!} \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^2 = \frac{60}{1 \times 2 \times 3} \times \frac{3^2}{4} = \frac{90}{1024} = \frac{45}{512}$$

(iii) P (none is a spade) = P (X=0)

$$= {}^5C_0 P^0 q^5 = \left(\frac{3}{4}\right)^5 = \frac{243}{1024}$$

PART – E

Answer any one question:

10×1=10

49. a) Prove that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ and hence evaluate $\int_0^{\pi/2} \frac{\cos^3 x}{\sin^5 x + \cos^5 x} dx$

Sol.

Prove the expression,

Consider

$$I = \int_0^a f(x)dx$$

Put $a - x = t$

$dx = -dt$. when $x=0$, $t=a$ and when $x=a$, $t=0$

$$\begin{aligned} \int_0^a f(a-x)dx &= \int_0^a f(t)(-dt) = \int_0^a f(t)dt \\ &= \int_0^a f(x)dx = \therefore \int_0^a f(x)dx = \int_0^a f(a-x)dx \end{aligned}$$

Let $\int_0^{\pi/2} \frac{\cos^3 x}{\sin^5 x + \cos^5 x} dx \dots\dots\dots (i)$

$$I = \int_0^{\pi/2} \frac{\cos^5 x}{\sin^5 \left(\frac{\pi}{2} - x\right) + \cos^5 \left(\frac{\pi}{2} - x\right)} dx$$

$$= \int_0^{\pi/2} \frac{\sin^5 x}{\cos^5 x + \sin^5 x} dx \dots\dots\dots (2)$$

Equation (i) and (ii)

$$2I = \int_0^{\pi/2} \frac{\sin^5 x + \cos^5 x}{\sin^5 x + \cos^5 x} dx = \int_0^{\pi/2} dx = \frac{\pi}{2} \quad \therefore I = \frac{\pi}{4}$$

b) Show that
$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc \left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right)$$

Sol.

$$\Delta = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = \begin{vmatrix} a & -b & 1 \\ 0 & b & -c \\ 1 & 1 & 1+c \end{vmatrix} \quad \begin{matrix} [R_1 = R_1 - R_2] \\ [R_2 = R_2 - R_3] \end{matrix}$$

$$\Delta = a(a+bc+c) - (-b)(0+c) + 0$$

50. a) Minimize and Maximise $z = 5x + 10y$

Subject to constraints:

$$x + 2y \leq 120$$

$$x + y \geq 60$$

$$x - 2y \geq 0$$

$$x \geq 0 \text{ and } y \geq 0$$

By graphical method.

Sol.

Solution lies in region CEDA

Sl.No	Corner point	Value of $Z = 5x + 10y$
1	C(60,0)	300 ← <i>minimum</i>
2	E(40,20)	400
3	D(60,30)	600 ← <i>maximum</i>
4	A (120,0)	600 ← <i>maximum</i>

Minimum value of Z is 300 which occurs at (60,0) and maximum value of Z is 600 which

Occurs at all point on the line segment joining the point A (120,0) and D (60,30).

b) Find the value of K , if f(x)

$$\begin{cases} kx+1, & \text{if } x \leq 5 \\ 3x-5, & \text{if } x > 5 \end{cases} \text{ is continuous at } x=5$$

Sol. Given function is continuous at $x=5$ and $f(5) = 5k + 1$,

$$\lim_{x \rightarrow 5} f(x) = f(5) = \lim_{x \rightarrow 5} f(x)$$

$$\Rightarrow 5k + 1 = (3 \times 5) - 5$$

$$\Rightarrow 5k + 1 = 10$$

$$\Rightarrow 5k = 9$$

$$\Rightarrow k = \frac{9}{5}$$