

Subject Code: 35 (NS)

MATHEMATICS

(English Version)

Instructions:

1. The question paper has five parts namely A, B, C, D, and E. Answer all the parts.
2. Use the Graph Sheet for the question on Linear Programming Problem on Part –E.

Answer all the ten questions:

(10 x

1 = 10)

PART -A

1. Let * be a operation defined on the set of rational numbers by $a * b = \frac{ab}{4}$, Find the identity element.

Sol.

Simplify the equation,

Suppose 'e' is the identity element

Then,

$$a * e = a$$

$$\frac{ae}{4} = a$$

$$\therefore e = 4$$

2. Write the value of x for which $2 \tan^{-1} x = \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right)$, holds,

Sol.

Simplify the expression,

The value of x for which

$$2 \tan^{-1} x = \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right), \text{ holds good is } x \geq 0$$

3. Construct a 2×2 matrix $A = [a_{ij}]$ whose elements are given by $a_{ij} = \frac{1}{2}|-3i + j|$

Sol.

Simplify the expression,

$$a_{11} = \frac{1}{2}|-3+1| = \frac{1}{2}|-2| = 1$$

$$a_{12} = \frac{1}{2}|-3+2| = \frac{1}{2}|-1| = \frac{1}{2}$$

$$a_{21} = \frac{1}{2}|-6+1| = \frac{1}{2}|-5| = \frac{5}{2}$$

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$$a_{22} = \frac{1}{2}|-6+2| = \frac{1}{2}|-4| = 2$$

$$\therefore A = \begin{bmatrix} 1 & 1/2 \\ 5/2 & 2 \end{bmatrix}$$

4. Find the values of x for which $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 4 & 1 \end{vmatrix}$.

Sol.

Simplify the expression,

$$\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 4 & 1 \end{vmatrix}$$

$$x^2 = 3 - 5$$

$$x^2 = -2$$

$$x = \pm\sqrt{-2}$$

5. Find $\frac{dy}{dx}$, if $y = \sin(x^2 + 5)$

Sol.

Simplify the expression,

$$y = \sin(x^2 + 5)$$

Diff w.r.t 'x'

$$y = \sin(x^2 + 5)$$

$$\frac{dy}{dx} = \cos(x^2) \times \frac{d}{dx}(x^2 + 5)$$

$$\therefore \frac{dy}{dx} = \cos(x^2 + 5) \times 2x$$

$$\frac{dy}{dx} = 2x(\cos(x^2 + 5))$$

6. Evaluate $\int e^x \left(\frac{x-1}{x^2} \right) dx$

Sol.

Simplify the expression,

$$\int e^x \left(\frac{x-1}{x^2} \right) dx$$

$$= \int e^x \left(\frac{1}{x} - \frac{1}{x^2} \right)$$

$$= e^x \left(\frac{1}{x} \right) + c$$

7. Define negative of a vector.

Sol.

A vector which has the same magnitude as that of a given vector and direction opposite to it is called negative of a given vector.

8. Write the direction cosines of x-axis.

Sol.

The x-axis makes angle $0^\circ, 90^\circ$ with x, y and z axes, respectively.

Therefore, the direction cosines of x- axis are $\cos 0^\circ, \cos 90^\circ, \cos 90^\circ$ 1,0,0.

9. Define feasible region in LPP.

Sol.

A feasible region is the space where all candidates are used in mathematical optimization. This is where all sets of possible solutions are placed within constraints. The feasible region is also known as the search space, solution space or feasible set.

10. If $P(A) = \frac{3}{5}, P(B) = \frac{1}{5}$, Find $P(A \cap B)$ if A and B are independent event.

Sol.

$$P(A \cap B) = P(A).P(B)$$

$$= \frac{3}{5} \times \frac{1}{5} = \frac{3}{25}$$

$$P(A \cap B) = \frac{3}{25}$$

PART – B

Answer all the ten questions:

(10 x 2 = 20)

11. Show that $f : A \rightarrow B$ and $g : B \rightarrow C$ are one-one, then $gof : A \rightarrow C$ is also one-one.

Sol:

Let us consider an arbitrary element, $z \in C$

So, there will be a pre-image y of z under g,

Such that $g(y) = z$ since g: is onto.

For $y \in B$, there exists a pre-image x of y under f, such that $g(y) = z$. since g: is onto.

Therefore,

$$gof(x) = g(f(x)) = g(y) = z.$$

12. Show that $\sin^{-1}(2x\sqrt{1-x^2}) = 2\sin^{-1}x$ for $\frac{-1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}$.

Sol.

Prove the expression,

$$\text{Taking L.H.S } \sin^{-1}(2x\sqrt{1-x^2})$$

$$\text{Substitute } x = \sin \theta = \sin^{-1} x \text{ for } \frac{-1}{\sqrt{x}} = \frac{1}{\sqrt{x}}$$

$$= \sin^{-1}(2\sin \theta \sqrt{1-\sin^2 \theta})$$

$$= \sin^{-1}(2\sin \theta \cos \theta)$$

$$\sin^{-1}(\sin 2\theta)$$

13. Show that $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$

Sol.

We know that,

$$2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

$$2\tan^{-1}\left(\frac{2 \times \frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}\right)$$

$$= 2\tan^{-1}\left(\frac{1}{3/4}\right)$$

$$= \tan^{-1}\left(\frac{4}{3}\right)$$

Again consider L.H.S

$$\begin{aligned}
&= 2 \tan^{-1} \left(\frac{1}{2} \right) + \tan^{-1} \left(\frac{1}{7} \right) \\
&= \tan^{-1} \left(\frac{4}{3} \right) + \tan^{-1} \left(\frac{1}{7} \right) \\
&= \tan^{-1} x + \tan^{-1} y = \tan^{-1} \left(\frac{x+y}{1-xy} \right) \\
&= \tan^{-1} \left(\frac{4/3 + 1/7}{1 - 4/3 \times 1/7} \right) \\
&= \tan^{-1} \left(\frac{31/21}{17/21} \right) \\
&= \tan^{-1} \left(\frac{31}{17} \right) = R.H.S
\end{aligned}$$

14. If the area of the triangle with vertices $(-2,0)$, $(0,4)$ and $(0,k)$ is 4 square units, find the values of k using determinants.

Sol.

Suppose

$$(x_1, y_1) = (-2, 0)$$

$$(x_2, y_2) = (0, 4)$$

$$(x_3, y_3) = (0, k)$$

$$\text{Area of triangle} = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$4 = \frac{1}{2} \begin{vmatrix} -2 & 0 & 1 \\ 0 & 4 & 1 \\ 0 & k & 1 \end{vmatrix}$$

$$\pm 4 = \frac{1}{2} [-2(4-k) + 0 + 1(0)]$$

$$\pm 4 = \frac{1}{2} [-8 + 2k]$$

$$\pm 4 = -4 + k$$

$$k = 4 + 4 = 8$$

or,

$$k = -4 + 4 = 0$$

$$\therefore k = 8, 0$$

15. Differential $\left(x + \frac{1}{x}\right)^x$ with respect to x.

Sol.

Simplify the expression,

Taking log on both side

$$\log y = \log \left(x + \frac{1}{x}\right)^x$$

$$\log y = x \log \left(x + \frac{1}{x}\right)$$

Diff w.r.t 'x' on both sides.

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx} \left[x \log \left(x + \frac{1}{x}\right) \right]$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{d}{dx} \left(\log \left(x + \frac{1}{x}\right) + \log \left(x + \frac{1}{x}\right) \frac{d(x)}{dx} \right)$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{1}{\left(x + \frac{1}{x}\right)} \frac{d}{dx} \left(x + \frac{1}{x}\right) + \log \left(x + \frac{1}{x}\right) \cdot 1$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{x}{\left(x + \frac{1}{x}\right)} \cdot \left(1 - \frac{1}{x^2}\right) + \log \left(x + \frac{1}{x}\right)$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \left(\frac{x^2}{x^2 + 1}\right) \cdot \left(\frac{1 - x^2}{x^2}\right) + \log \left(x + \frac{1}{x}\right)$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \left(\frac{x^2 - 1}{x^2 + 1}\right) + \log \left(x + \frac{1}{x}\right)$$

$$\therefore \frac{dy}{dx} = \left(x + \frac{1}{x}\right) \left[\left(\frac{x^2 - 1}{x^2 + 1}\right) + \log \left(x + \frac{1}{x}\right) \right]$$

16. Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}$, $x \neq 2$ at $x = 10$

Sol:

The given curve is

$$y = \frac{x-1}{x-2}$$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{(x-2)(1) - (x-1)(1)}{(x-2)^2} \\ &= \frac{x-2-x+1}{(x-2)^2} = \frac{-1}{(x-2)^2}\end{aligned}$$

Thus, the slope of the tangent at $x = 10$ is given by,

$$\left. \frac{dy}{dx} \right|_{x=10} = \left. \frac{-1}{(x-2)^2} \right|_{x=10} = \frac{-1}{(10-2)^2} = \frac{-1}{64}$$

Hence, the slope of the tangent at $x = 10$ is $\frac{-1}{64}$

17. Find $\frac{dy}{dx}$, IF $x^2 + xy + y^2 = 100$.

Sol.

Diff. w.r.t 'x'

$$x^2 + xy + y^2 = 100$$

$$2x + x \cdot \frac{dy}{dx} + y + 2y \frac{dy}{dx} = 0$$

$$= \frac{dy}{dx}(x + 2y) = -2x - y$$

$$\frac{dy}{dx} = \frac{-(2x + y)}{x + 2y}$$

$$18. \int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} dx$$

Sol.

Simplify the expression,

$$\begin{aligned} & \int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} dx \\ &= \int \frac{2\cos^2 x - 1 - (2\cos^2 \alpha - 1)}{\cos x - \cos \alpha} dx \\ &= \int \frac{2\cos^2 x - 1 - 2\cos^2 \alpha + 1}{\cos x - \cos \alpha} dx \\ &= 2 \int \frac{(\cos^2 x - \cos^2 \alpha)}{(\cos x - \cos \alpha)} dx \\ &= 2 \int \frac{(\cos x - \cos \alpha)(\cos x + \cos \alpha)}{(\cos x - \cos \alpha)} dx \\ &= 2 \int (\cos x + \cos \alpha) dx \\ &= 2 \int \cos x dx + 2 \int \cos \alpha dx \\ &= 2 \sin x + 2x \cos \alpha \\ &= 2(\sin x + x \cos \alpha) + c \end{aligned}$$

$$19. \text{ Evaluate: } \int \frac{dx}{x - \sqrt{x}} dx$$

Sol.

Simplify the expression,

$$\int \frac{dx}{\sqrt{x}(\sqrt{x} - 1)} dx$$

Putting $\sqrt{x} - 1 = t$

Diff. w.r.t 'x'

$$\begin{aligned}
\frac{1}{2\sqrt{x}} dx &= dt \\
\frac{1}{\sqrt{x}} dx &= 2dt \\
&= \int \frac{2dt}{t} \\
&= 2 \int \frac{1}{t} dt \\
&= 2 \log|t| + c \\
&= 2 \log|\sqrt{x}-1| + c \\
&= \int \frac{1}{x-\sqrt{x}} = 2 \log|\sqrt{x}-1| + c
\end{aligned}$$

20. Find the order and degree, is defined of the differential equation.

$$\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 + \sin \frac{dy}{dx} + 1 = 0$$

Sol.

Order = 2

Degree = degree is not defined.

21. Find $|\vec{b}|$, if $(\vec{a} - \vec{b}) \cdot (\vec{a} + \vec{b}) = 8$ and $|\vec{a}| = 8|\vec{b}|$

Sol.

Simplify the expression,

$$\begin{aligned}
(\vec{a} - \vec{b}) \cdot (\vec{a} + \vec{b}) &= 8 \\
\Rightarrow |\vec{a}|^2 - |\vec{b}|^2 &= 8 \\
\Rightarrow (8|\vec{b}|)^2 - |\vec{b}|^2 &= 8 \\
\Rightarrow 64|\vec{b}|^2 - |\vec{b}|^2 &= 8 \\
\Rightarrow 63|\vec{b}|^2 &= 8 \\
\Rightarrow |\vec{b}|^2 &= \frac{8}{63} \\
\Rightarrow |\vec{a}|^2 &= 8|\vec{b}|^2 \\
&= 8 \times \frac{2\sqrt{2}}{3\sqrt{7}} \\
\Rightarrow |\vec{a}| &= \frac{16\sqrt{2}}{3\sqrt{7}} = \text{and } \frac{2\sqrt{2}}{3\sqrt{7}}
\end{aligned}$$

22. Find the area of the parallelogram whose adjacent sides are determined by the vector $|\vec{a}| = \hat{i} - \hat{j} + 3\hat{k}$ and $|\vec{b}| = 2\hat{i} - 7\hat{j} + \hat{k}$.

Sol.

$$\text{Area of the parallelogram} = |\vec{a} \times \vec{b}|$$

$$|\vec{a} \times \vec{b}| = (\hat{i} - \hat{j} + 3\hat{k})(2\hat{i} - 7\hat{j} + \hat{k})$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 2 & 7 & 1 \end{vmatrix}$$

$$\begin{aligned}
&= \hat{i}(-1 + 21) - \hat{j}(1 - 6) + \hat{k}(-7 + 2) \\
&= 20\hat{i} + 5\hat{j} - 5\hat{k}
\end{aligned}$$

$$|\vec{a} \times \vec{b}| = \sqrt{(20)^2 + (5)^2 + (-5)^2}$$

$$\begin{aligned}
\text{Area} &= \sqrt{400 + 25 + 25} \\
&= \sqrt{450} \\
&= 15\sqrt{2} \text{ sq units}
\end{aligned}$$

Area of parallelogram $15\sqrt{2}$.

23. Find the angle between the pair of lines given by

$$\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k}) \text{ and } \vec{r} = 5\hat{i} - 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k})$$

Sol.

$$\vec{b}_1 = \hat{i} + 2\hat{j} + 2\hat{k}$$

$$\vec{b}_2 = 3\hat{i} + 2\hat{j} + 6\hat{k}$$

Angle,

$$\begin{aligned} \cos \theta &= \frac{|\vec{b}_1 \cdot \vec{b}_2|}{|\vec{b}_1| |\vec{b}_2|} = \frac{(\hat{i} + 2\hat{j} + 2\hat{k}) \cdot (3\hat{i} + 2\hat{j} + 6\hat{k})}{\sqrt{1+4+4} \cdot \sqrt{9+4+36}} \\ &= \frac{3+4+12}{\sqrt{1+4+4} \cdot \sqrt{9+4+36}} = \frac{19}{3 \times 7} = \frac{19}{21} \\ \cos^{-1} \left(\frac{19}{21} \right) \end{aligned}$$

24. Let X denote the number of hours you study during a randomly selected school day. The probability that X can take the value of x, has the following form, where k is some constant.

$$P(X = x) = \begin{cases} 0, 1 & \text{if } x=0 \\ kx & \text{if } x=1 \text{ or } 2 \\ K(5-x), & \text{if } x=3 \text{ or } 4 \\ 0 & \text{otherwise} \end{cases}$$

Find the value of k.

Sol.

$$P(x=0)$$

$$\sum_{i=0}^n p(x_i) = 1$$

$$p(x=0) = 0$$

$$p(x=1) = k$$

$$p(k=2) = 2k$$

$$P(x=3) = 2k$$

$$P(x=4) = k$$

$$0 + k + 2k + 2k + k + 0 = 1$$

$$6k = 1$$

$$k = \frac{1}{6}$$

Part – C

Answer any ten question:

(10 x 3 = 30)

25. Determine whether the relation R in the set $A = \{1, 2, 3, \dots, 13, 14\}$ defined as

$R = \{(x, y) : 3x - y = 0\}$, is reflexive, symmetric and transitive.

Sol:

Given that,

$$A = \{1, 2, 3, \dots, 13, 14\}$$

$$R = \{(x, y) : 3x - y = 0\}$$

For $(x, x) \in R$

For $x \in A, (x, x) \notin R$

For $1 \in A, 3(1) - 1 = 0$

$\therefore R$ is not a reflexive.

Consider,

$$(1, 3) \in R$$

$$\Rightarrow 3(1) - 3 = 0$$

But,

$$3(3) - 1 \neq 0$$

$$\therefore (3, 1) \notin R$$

For,

$$(1, 3) \in R \quad (3, 1) \notin R$$

$\therefore R$ is not symmetric.

Consider,

$$(1, 3) \in R \text{ and } (3, 9) \in R$$

$$\Rightarrow 3(1) - 3 = 0 \text{ and } 3(3) - 9 = 0$$

But,

$$3(1) - 9 \neq 0$$

$$\therefore (1, 9) \notin R$$

For $(1, 3)$ and $(3, 9) \in R$ $(1, 9) \notin R$

$\therefore R$ is not transitive.

26. It $\tan^{-1} \frac{x-1}{x-2} + \tan^{-1} \frac{x+1}{x+2} = \frac{\pi}{4}$ then find the values of x .

Sol:

Given that,

$$\tan^{-1} \frac{x-1}{x-2} + \tan^{-1} \frac{x+1}{x+2} = \frac{\pi}{4}$$

$$\tan^{-1} \left(\frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1 - \left(\frac{x-1}{x-2} \right) \left(\frac{x+1}{x+2} \right)} \right) = \frac{\pi}{4}$$

$$\left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1} \left(\frac{x+y}{1-xy} \right) \right]$$

$$\Rightarrow \frac{(x-1)(x+2) + (x+1)(x-2)}{x^2 - 4 - (x^2 - 1)} = \frac{\pi}{4}$$

$$\Rightarrow \frac{x^2 - 4}{x^2 - 4 - (x^2 - 1)} = \frac{\pi}{4}$$

$$\Rightarrow \frac{x^2 - 4}{-3} = 1 \quad \left[\because \tan \frac{\pi}{4} = 1 \right]$$

$$\Rightarrow (2x^2 - 4) = -3$$

$$\Rightarrow 2x^2 = 1$$

$$\Rightarrow x^2 = \frac{1}{2}$$

$$\therefore x = \pm \frac{1}{\sqrt{2}}$$

But, $x = \frac{1}{\sqrt{2}}$ does not satisfy equation.

27. If A and B are invertible matrices of the same order, then prove that $(AB)^{-1} = B^{-1}A^{-1}$.

Sol:

Consider,

$$(AB)(AB)^{-1} = I$$

Multiplying both sides by A^{-1}

$$A^{-1}(AB)(AB)^{-1} = A^{-1}I$$

$$\Rightarrow (A^{-1}A)B(AB)^{-1} = A^{-1} \quad [\because A^{-1}I = A^{-1}]$$

$$\Rightarrow IB(AB)^{-1} = A^{-1} \quad [A^{-1}A = I]$$

$$\Rightarrow B(AB)^{-1} = A^{-1} \quad [IB = B]$$

Multiplying both sides by B^{-1}

$$\Rightarrow B^{-1}B(AB)^{-1} = B^{-1}A^{-1}$$

$$\Rightarrow I(AB)^{-1} = B^{-1}A^{-1} \left[\because B^{-1}B = I \right]$$

$$\Rightarrow (AB)^{-1} = B^{-1}A^{-1} \left[\because I(AB)^{-1} = (AB)^{-1} \right]$$

Hence, proved.

28. Verify Rolle's thermo for the function $f(x) = x^2 + 2x - 8, x \in [-4, 2]$

Sol:

Consider,

$$f(x) = x^2 + 2x - 8, x \in [-4, 2]$$

$$f'(x) = 2x + 2 \quad \text{---(1)}$$

$f(x)$ is a polynomial function which is continuous $\forall x \in R$

In particular for $x \in [-4, 2]$

From (1) $f'(x)$ is defined

$\therefore f(x)$ is differentiable $\forall x \in (-4, 2)$

$$f(-4) = (-4)^2 + 2(-4) - 8$$

$$= 16 - 8 - 8$$

$$= 0$$

$$f(2) = 2^2 + 2(2) - 8$$

$$= 4 + 4 - 8$$

$$= 0$$

Also, $f(-4) = f(2)$

$\therefore$ All conditions of Rolle's thermo are satisfied.

Hence exists same c is $(-4, 2)$ for which $f'(c) = 0$

$$\Rightarrow 2c + 2 = 0$$

$$\Rightarrow c = -1$$

Thus Rolle's thermo is verified.

29. If $x = \sqrt{a^{\sin^{-1}t}}$ and $y = \sqrt{a^{\cos^{-1}t}}$ then prove that $\frac{dy}{dx} = \frac{-y}{x}$.

Sol:

Consider,

$$x = \sqrt{a^{\sin^{-1}t}}$$

Diff. w.r.t “t”

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{2\sqrt{a^{\sin^{-1}t}}} \times \frac{d}{dt} \left(a^{\sin^{-1}t} \right) \\ &= \frac{1}{2\sqrt{a^{\sin^{-1}t}}} \times a^{\sin^{-1}t} \log_e a \times \frac{d}{dt} (\sin^{-1}t)\end{aligned}$$

$$\frac{dx}{dt} = \frac{1}{2\sqrt{a^{\sin^{-1}t}}} \times a^{\sin^{-1}t} \log_e a \times \frac{1}{\sqrt{1-t^2}}$$

$$y = \sqrt{a^{\cos^{-1}t}}$$

$$\begin{aligned}\frac{dy}{dt} &= \frac{1}{2\sqrt{a^{\cos^{-1}t}}} \times \frac{d}{dt} \left(a^{\cos^{-1}t} \right) \\ &= \frac{1}{2\sqrt{a^{\cos^{-1}t}}} \times a^{\cos^{-1}t} \log_e a \times \left(-\frac{1}{\sqrt{1-t^2}} \right)\end{aligned}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\begin{aligned}&= \frac{\frac{1}{2\sqrt{a^{\cos^{-1}t}}} \times a^{\cos^{-1}t} \log_e a \times \left(-\frac{1}{\sqrt{1-t^2}} \right)}{\frac{1}{2\sqrt{a^{\sin^{-1}t}}} \times a^{\sin^{-1}t} \log_e a \times \frac{1}{\sqrt{1-t^2}}}\end{aligned}$$

$$= -\frac{\sqrt{a^{\cos^{-1}t}}}{\sqrt{a^{\sin^{-1}t}}}$$

$$= -\frac{y}{x}$$

$$\therefore \frac{dy}{dx} = \frac{-y}{x}$$

30. Find the two positive number whose sum is 15 and sum of whose squares is minimum.

Sol:

Let one number be 'x', the other no is '-x'

$$\text{Sum of squares} = x^2 + (15-x)^2$$

$$y = x^2 + (15-x)^2$$

$$\frac{dy}{dx} = 2x + 2(15-x).(-1)$$

$$= 2x - 30 + 2x$$

$$\frac{dy}{dx} = 4x - 30 \text{ ----- (1)}$$

For maxima or minima $\frac{dy}{dx}$

$$\Rightarrow 4x - 30 = 0$$

$$\Rightarrow x = \frac{15}{2}$$

Diff. w.r.t 'x' on both sides of (1)

$$\frac{d^2y}{dx^2} = 4 > 0$$

$$\therefore y \text{ is minima at } x = \frac{15}{2}$$

$$\therefore \text{The required two numbers are } \frac{15}{2}, \frac{15}{2}.$$

31. **Evaluate:** $\int x \tan^{-1} x dx$

Sol:

Let,

$$I = \int x \tan^{-1} x dx$$

And,

$\tan^{-1} x = 1^{st}$ function

$x = 2^{nd}$ function

By the method of Integration by parts, we get

$$\begin{aligned}
I &= \tan^{-1} x \int x dx - \int \left[\frac{d}{dx} (\tan^{-1} x) \int x dx \right] dx \\
&= \tan^{-1} x - \frac{x^2}{2} - \int \left[\frac{1}{1+x^2} \cdot \frac{x^2}{2} \right] dx \\
&= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx \\
&= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \left[1 - \frac{1}{1+x^2} \right] dx \\
&= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int 1 dx + \frac{1}{2} \int \frac{1}{1+x^2} dx \\
&= \frac{x^2}{2} \tan^{-1} x - \frac{x}{2} + \frac{1}{2} \tan^{-1} x + c \\
&= \left(\frac{x^2+1}{2} \right) \tan^{-1} x - \frac{x}{2} + c
\end{aligned}$$

32. Evaluate $\int_0^2 e^x dx$ as the limit of a sum.

Sol:

Let,

$$\int_0^2 e^x dx$$

By definition,

$$\begin{aligned}
&\int_a^b f(x) dx \\
&= b-a \lim_{n \rightarrow \infty} \frac{1}{n} [f(a) + f(a+h) + \dots + f[a+(n-1)h]]
\end{aligned}$$

Where, $h = \frac{b-a}{n}$

Here $a=0, b=2, h = \frac{b-a}{n}$

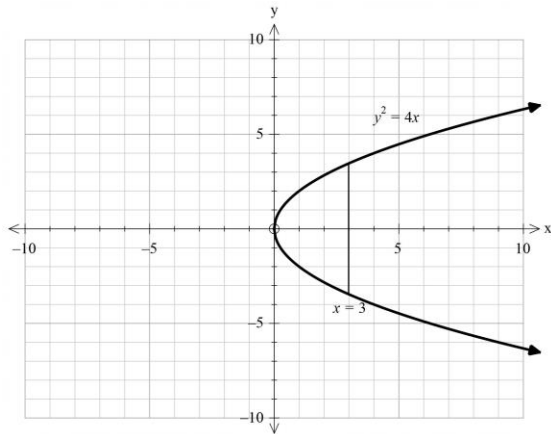
$$\begin{aligned}
\therefore \int_0^2 e^x dx &= 2 - 0 \lim_{n \rightarrow \infty} \frac{1}{n} \left[f\left(\frac{0}{n}\right) + f\left(\frac{2}{n}\right) + f\left(\frac{4}{n}\right) + \dots + f\left(\frac{2(n-1)}{n}\right) \right] \\
&= 2 \lim_{n \rightarrow \infty} \frac{1}{n} \left[e^0 + e^{\frac{2}{n}} + e^{\frac{4}{n}} + \dots + e^{\frac{2n}{n}} \right] \\
&= 2 \lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{1(e^{2n/n} - 1)}{(e^{2/n} - 1)} \right]
\end{aligned}$$

$$[\because \text{For GP sum up to } n \text{ comes } = \frac{a(r^n - 1)}{r - 1} \text{ there } r = e^{2/n}]$$

$$\begin{aligned}
&-2 \lim_{n \rightarrow \infty} \left[\frac{e^2 - 1}{(e^{2/n} - 1)} \right] \times \frac{1}{2} \\
&= \frac{2}{n} \\
&= p \lim_{n \rightarrow \infty} \frac{(e^2 - 1)}{\left(\frac{e^{2/n} - 1}{\frac{2}{n}} \right)} \\
&= \frac{\lim_{n \rightarrow \infty} (e^2 - 1)}{\lim_{n \rightarrow \infty} \left(\frac{e^{2/n} - 1}{\frac{2}{n}} \right)} \\
&= \frac{e^2 - 1}{1} \quad \because \lim_{n \rightarrow \infty} \frac{h - 1}{h} = 1 \\
&= e^2 - 1 \\
\therefore \int_0^2 e^x dx &= e^2 - 1
\end{aligned}$$

33. Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$

Sol:



Area bounded by $y^2 = 4x$ and $x = 3$

$$\begin{aligned}
 &= 2 \int_0^3 y dx \\
 &= 2 \int_0^3 \sqrt{4x} dx \\
 &= 2 \cdot 2 \int_0^3 \sqrt{x} dx \\
 &= 4 \times \frac{2}{3} \left[x^{3/2} \right]_0^3 \\
 &= \frac{8}{3} (3^{3/2} - 0^{3/2}) \\
 &= \frac{8 \times 3\sqrt{3}}{3} - 0 \\
 &= 8\sqrt{3} \text{ sq. units}
 \end{aligned}$$

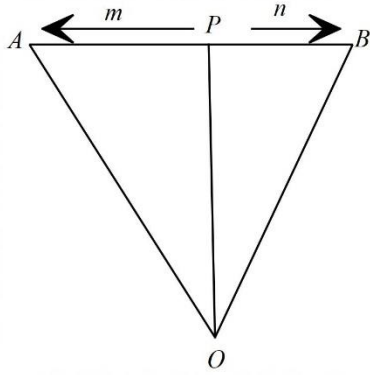
34. Show that the position vector of the point P, Which divides the line joining the points A and B having position vectors $\vec{a}$ and $\vec{b}$ internally in the ratio m:n is $\frac{m\vec{b} + n\vec{a}}{m+n}$.

Sol:

Consider line AB with position vector of point $A = \vec{a}$ and position vector of point $B = \vec{b}$

Let P be a point on AB. Which divides it internally is the ratio m : n.

Let the position vector of point $P = \vec{r}$.



$$\therefore \frac{\overrightarrow{AP}}{\overrightarrow{PB}} = \frac{m}{n}$$

$$n(\overrightarrow{AP}) = m(\overrightarrow{PB})$$

$$n(\overrightarrow{OP} - \overrightarrow{OA}) = m(\overrightarrow{OB} - \overrightarrow{OP})$$

$$\Rightarrow n(\vec{r} - \vec{a}) = m(\vec{b} - \vec{r})$$

$$\Rightarrow \vec{r}(m+n) = m\vec{b} + n\vec{a}$$

$$\Rightarrow \vec{r} = \frac{m\vec{b} + n\vec{a}}{m+n}$$

$\therefore$ Position vector of point P dividing the line joining AB in the ratio m:n is $\vec{r} = \frac{m\vec{b} + n\vec{a}}{m+n}$

35. Show that the four points with position vectors $4\hat{i} + 8\hat{j} + 12\hat{k}$, $2\hat{i} + 4\hat{j} + 6\hat{k}$, $3\hat{i} + 5\hat{j} + 4\hat{k}$ and $5\hat{i} + 8\hat{j} + 5\hat{k}$ are coplanar.

Sol:

Four vectors a,b,c,d to be coplanar, they should satisfy the condition of $xa + yb + cz + td = 0$

Here x,y,z,t are scalars, and they should satisfy $x + y + z + t = 0$

So, let $t = 1$

And on solving using the above conditions;

$$a = 4i + 8j + 12k$$

$$b = 2i + 4j + 6k$$

$$c = 3i + 5j + 4k$$

$$d = 5i + 8j + 5k$$

Let us take $xa + yb + zc + d = 0, x + y + z + 1 = 0$

Or,

$$x(4i + 8j + 12k) + y(2i + 4j + 6k) + z(3i + 5j + 4k) + (5i + 8j + 5k) = 0$$

Or,

$$(4x + 2y + 3z + 5)i + (8x + 4y + 5z + 8)j + (12x + 6y + 4z + 5)k = 0$$

Therefore, we have

$$4x + 2y + 3z + 5 = 0$$

$$8x + 4y + 5z + 8 = 0$$

$$12x + 6y + 4z + 5 = 0$$

$$x + y + z + 1 = 0$$

Solving, we get

$$x = -\frac{1}{2}, y = \frac{3}{2}, z = -2$$

Thus, there exist scalars $-\frac{1}{2}, \frac{3}{2}, -2, 1$ such that their sum is zero

$$\text{And } \frac{-1}{2}a + \frac{3}{2}b - 2c + 1d = 0$$

Hence the points are coplanar.

36. Find the equation of the plane passing through the intersection of the planes $3x - y + 2z - 4 = 0$ and $x + y + z + 2 = 0$ the point $(2, 2, 1)$.

Sol:

$$\pi_1 : 3x - y + 2z - 4 = 0$$

$$\pi_2 : x + y + z + 2 = 0$$

Equation of plane through intersection is

$$\pi_3 : \pi_1 + \lambda \pi_2$$

$$\pi_3 : 3x - y + 2z - 4 + \lambda(x + y + z + 2) = 0$$

$$\pi_3 : (3 + \lambda)x + (\lambda - 1)y + (2 + \lambda)z - 4 + 2\lambda = 0$$

Also, π_3 passes through $(2, 2, 1)$

$$\therefore (3+\lambda)2 + (\lambda-1)2 + (2+\lambda)1 - 4 + 2\lambda = 0$$

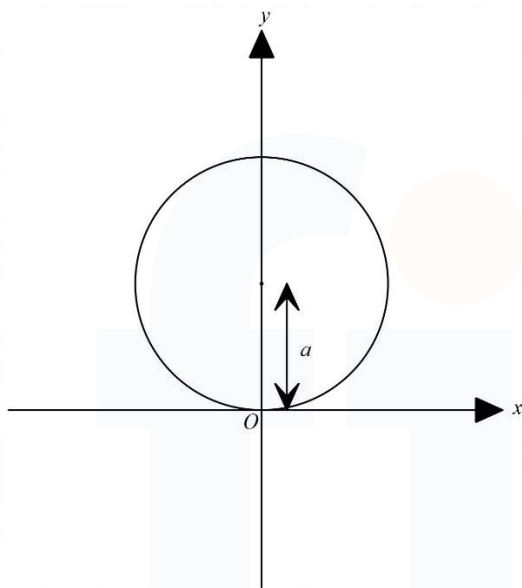
$$\Rightarrow 6 + 2\lambda + 2\lambda - 2 + 2 + \lambda - 4 + 2\lambda = 0$$

$$\Rightarrow 2 + 7\lambda = 0$$

$$\Rightarrow \lambda = \frac{-2}{7}$$

37. from the differential equation of the circles touching the x-axis origin.

Sol:



Equation of circle touching x axis at origin is given by

$$x^2 + (y-a)^2 = a^2 \text{ Where 'a' is radius of circle}$$

$$\Rightarrow x^2 + y^2 + a^2 - 2ay = a^2$$

$$\Rightarrow x^2 + y^2 = 2ay \text{ -----(1)}$$

Diff. both sides w.r.to 'x'.

$$2x + 2y \frac{dy}{dx} = 2a \frac{dy}{dx}$$

$$\frac{2x + 2y \frac{dy}{dx}}{2 \frac{dy}{dx}} = a$$

$$\Rightarrow a = \frac{x + y \frac{dy}{dx}}{\frac{dy}{dx}}$$

Substituting value of 'a' in eq. (1)

$$x^2 + y^2 = 2 \left[\frac{x + y \frac{dy}{dx}}{\frac{dy}{dx}} \right] y$$

$$(x^2 + y^2) \frac{dy}{dx} = 2xy + 2y^2 \frac{dy}{dx}$$

$$\Rightarrow x^2 \frac{dy}{dx} + y^2 \frac{dy}{dx} = 2xy + 2y^2 \frac{dy}{dx}$$

$$\Rightarrow x^2 \frac{dy}{dx} - y^2 \frac{dy}{dx} = 2xy$$

$$\Rightarrow (x^2 - y^2) \frac{dy}{dx} = 2xy$$

$$\Rightarrow \frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$$

The differential equation of circles touching x-axis at origin.

38. An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probabilities of an accident are 0.01, 0.03 and 0.15 respectively. One of the insured person meets with an accident. What is the probability that he is a scooter driver?

Sol;

Let E_1, E_2 and E_3 be the respective events that the driver is a scooter driver, a car driver and a truck driver.

Let A be the event that the person meets with an accident.

There are 2000 scooter drivers, 4000 car drivers, and 6000 truck drivers.

Total number of drivers = 2000 + 4000 + 6000 = 12000

$$P(E_1) = P(\text{driver is a scooter driver})$$

$$= \frac{2000}{12000} = \frac{1}{6}$$

$$P(E_2) = P(\text{driver is a scooter driver})$$

$$= \frac{4000}{12000} = \frac{1}{3}$$

$$P(E_3) = P(\text{driver is a scooter driver})$$

$$= \frac{6000}{12000} = \frac{1}{2}$$

$$P(A|E_1) = P(\text{scooter driver met with an accident}) = 0.01 = \frac{1}{100}$$

$$P(A|E_2) = P(\text{car driver met with an accident}) = 0.03 = \frac{3}{100}$$

$$P(A|E_3) = P(\text{truck driver met with an accident}) = 0.015 = \frac{15}{1000}$$

The probability that the driver is a scooter driver, given that he met with an accident, is given by $P(E_1|A)$.

By using Bayes' theorem, we obtain

$$\begin{aligned} P(A|E_1) &= \frac{P(E_1) \cdot P(A|E_1)}{P(E_1) \cdot P(A|E_1) + P(E_2) \cdot P(A|E_2) + P(E_3) \cdot P(A|E_3)} \\ &= \frac{\frac{1}{6} \cdot \frac{1}{100}}{\frac{1}{6} \cdot \frac{1}{100} + \frac{1}{3} \cdot \frac{3}{100} + \frac{1}{2} \cdot \frac{15}{1000}} \\ &= \frac{\frac{1}{6} \cdot \frac{1}{100}}{\frac{1}{100} \left(\frac{1}{6} + 1 + \frac{15}{2} \right)} \\ &= \frac{\frac{1}{6}}{\frac{104}{12}} \\ &= \frac{1}{6} \times \frac{12}{104} \\ &= \frac{1}{52} \end{aligned}$$

PART-D

Answer any ten question:

(6x5 = 30)

39. Let $\mathbb{R}_+$ be the set of all non-negative real numbers. Show that the function $\frac{dy}{dx}$ given by $\frac{dy}{dx}$ is invertible and write the inverse of f .

Sol:

Let,

$$y = x^2 + 4$$

$$\Rightarrow x = \sqrt{y-4}$$

$$f(x) : \mathbb{R}_+ \rightarrow [4, \infty]$$

Let us define $g(y) = \sqrt{y-4}$ $g(y) : [4, \infty]$

Consider,

$$(g \circ f)(x)$$

$$= g[f(x)]$$

$$= g(x^2 + 4)$$

$$= \sqrt{x^2 + 4 - 4}$$

$$= x$$

$$= I_A$$

Consider,

$$(f \circ g)(x)$$

$$= f[g(x)]$$

$$= f(\sqrt{x-4})$$

$$= (\sqrt{x-4})^2 + 4$$

$$= x - 4 + 4$$

$$= x$$

$$= I_B$$

$$\therefore (g \circ f)(x) = I_A \text{ and } (f \circ g)(x) = I_B$$

$$\therefore f \text{ is invertible with } f^{-1}(x) = \sqrt{x-4}$$

40. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$, then show that $A^3 - 23A - 40I = 0$

Sol:

Consider,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

$$A^2 = A.A$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+6+12 & 2-2+6 & 3+2+3 \\ 3-6+4 & 6+4+2 & 9-2+1 \\ 4+6+4 & 8-4+2 & 12+2+1 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix}$$

$$A^3 = A.A^2$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} 19+2+42 & 4+24+18 & 8+16+45 \\ 57-2+14 & 12-12+6 & 24-16+15 \\ 76+2+14 & 16+24+6 & 32+16+15 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 63 & 46 & 69 \\ 69 & 6 & 23 \\ 92 & 46 & 63 \end{bmatrix} \text{-----(1)}$$

$$23A = 23 \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

$$23A = \begin{bmatrix} 23 & 46 & 69 \\ 69 & -46 & 23 \\ 92 & 46 & 23 \end{bmatrix} \text{-----(2)}$$

$$40I = 40 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Adding 1,2 and 3

$$A^3 + 23A - 40I$$

$$= \begin{bmatrix} 63 & 46 & 69 \\ 69 & 6 & 23 \\ 92 & 46 & 63 \end{bmatrix} - \begin{bmatrix} 23 & 46 & 69 \\ 69 & -46 & 23 \\ 92 & 46 & 23 \end{bmatrix} - \begin{bmatrix} 40 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 40 \end{bmatrix}$$

$$= \begin{bmatrix} 63 & 46 & 69 \\ 69 & 6 & 23 \\ 92 & 46 & 63 \end{bmatrix} - \begin{bmatrix} 63 & 46 & 69 \\ 69 & 6 & 23 \\ 92 & 46 & 63 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$\therefore A^3 + 23A - 40I = 0$$

Hence proved.

41. Solve by matrix method:

$$2x + 3y + 3z = 5$$

$$x - 2y + z = -4$$

$$3x - y - 2z = 3$$

Sol:

Given that,

$$2x + 3y + 3z = 5$$

$$x - 2y + z = -4$$

$$3x - y - 2z = 3$$

Consider,

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

By matrix multiplication

$$AX = B$$

$$\Rightarrow x = A^{-1}B$$

Co-factors of

$$A = \begin{bmatrix} \begin{vmatrix} -2 & 1 \\ -1 & -2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} \\ -\begin{vmatrix} 3 & 3 \\ -1 & -2 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} \\ \begin{vmatrix} 3 & 3 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 5 & 5 \\ 3 & -13 & 11 \\ 9 & 1 & -7 \end{bmatrix}$$

$$|A| = 2(5) - 3(-5) + 3(5)$$

$$= 10 + 15 + 15$$

$$= 40$$

$$Adj A = \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj}A$$

$$= \frac{1}{40} \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

$$x = A^{-1}B$$

$$= \frac{1}{40} \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix} \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

$$= \frac{1}{40} \begin{bmatrix} 25 & -12 & 27 \\ 25 & 52 & 3 \\ 25 & -44 & -21 \end{bmatrix}$$

$$= \frac{1}{40} \begin{bmatrix} 40 \\ 80 \\ -40 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\therefore x = 1, y = 2, z = -1$$

42. If $y = Ae^{mx} + Be^{nx}$, show that $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$.

Sol:

Given that,

$$y = Ae^{mx} + Be^{nx}$$

Diff., w.r.to 'x'

$$\frac{dy}{dx} = mAe^{mx} + nBe^{nx}$$

Diff., w.r.to 'x'

$$\frac{d^2y}{dx^2} = m^2Ae^{mx} + n^2Be^{nx}$$

Consider,

$$\begin{aligned}
 \frac{d^2y}{dx^2} &= -(m+n) \frac{dy}{dx} + mny \\
 &= m^2 Ae^{mx} + n^2 Be^{nx} - (m+n) [mAe^{mx} + nBe^{nx}] + mn(Ae^{mx} + Be^{nx}) \\
 &= m^2 Ae^{mx} + n^2 Be^{nx} - m^2 Ae^{mx} - mnBe^{nx} - mnAe^{mx} - n^2 Be^{nx} + mnAe^{mx} + naBe^{nx} \\
 &= 0 \\
 LHS &= RHS
 \end{aligned}$$

Hence proved.

43. A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the y-coordinate is changing 8 times as fast as the x-coordinate.

Sol:

$$6y = x^3 + 2 \text{----- (1)}$$

Given,

$$\frac{dy}{dt} = 8 \cdot \frac{dx}{dt} \text{----- (2)}$$

Diff. (1) w.r.t 't'

$$6 \frac{dy}{dt} = 3x^2 \frac{dx}{dt} + 0$$

$$\Rightarrow 6 \cdot \frac{dy}{dx} = 3x^2 \frac{dx}{dt}$$

$$\Rightarrow 3x^2 = 48$$

$$x^2 = 16$$

$$x = \pm 4$$

When $x = 4$

$$\therefore 6y = (4)^3 + 2$$

$$6y = 64 + 2$$

$$6y = 66$$

$$y = 11$$

$\therefore A(4,11)$ is one point.

When $x = -4$

$$\therefore 6y = (-4)^3 + 2$$

$$6y = -64 + 2$$

$$6y = -62$$

$$y = \frac{-31}{3}$$

$\therefore B\left(-4, \frac{-31}{3}\right)$ is other points are $A(4, 11)$ and $B\left(-4, \frac{-31}{3}\right)$.

44. Find the integral of $\frac{1}{\sqrt{x^2 - a^2}}$ with respect to x and hence evaluate $\int \frac{dx}{\sqrt{x^2 + 6x - 7}}$.

Sol:

$$\frac{1}{\sqrt{x^2 - a^2}}$$

Integral,

$$\int \frac{1}{\sqrt{x^2 - a^2}} = x \sin^{-1} a$$

Simplify the further,

$$\begin{aligned} & \int \frac{dx}{\sqrt{x^2 + 6x - 7}} \\ &= \int \frac{dx}{\sqrt{(x+3)^2 - 4^2}} \end{aligned}$$

Put $x+3=t$

$dt=dx$

$$\int \frac{dt}{\sqrt{t^2 - (4)^2}}$$

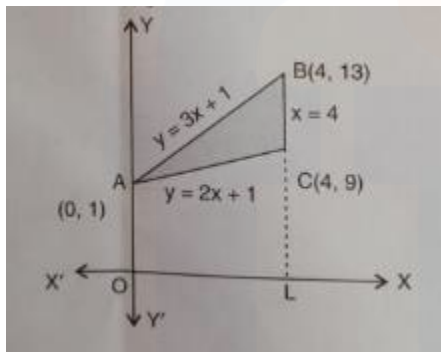
$$= \log \left| t + \sqrt{t^2 - (4)^2} \right| + C$$

45. Using integration find the area of the triangular region whose sides have the equation $y = 2x + 1$, $y = 3x + 1$ and $x = 4$.

Sol:

Given equation of sides of the triangle are $y = 2x + 1$, $y = 3x + 1$ and $x = 4$.

On solving these equations, we obtain the vertices of triangle as A (0, 1) B (4, 13) and C (4, 9).



Therefor,

Required area (shown in shaded region)

$$= \text{Area}(OLBAO) - \text{Area}(OLCAO)$$

$$= \int_0^4 (3x + 1) dx - \int_0^4 (2x + 1) dx$$

$$= \left[\frac{3x^2}{2} + x \right]_0^4 - \left[x^2 + x \right]_0^4$$

$$= \frac{3 \times 4^2}{2} - [42 + 4 - 0]$$

$$= 24 + 4 - 20$$

$$= 8 \text{ squnit.}$$

46. Solve the differential equation $\frac{dy}{dx} + y \sec x = \tan x, 0 \leq x < \frac{\pi}{2}$.

Sol;

$$\frac{dy}{dx} + y \sec x = \tan x, 0 \leq x < \frac{\pi}{2}.$$

This is of the form $\frac{dy}{dx} + py = Q$

Where $p = \sec x$ $Q = \tan x$

$\therefore$ Integrating factor,

$$\begin{aligned} &= e^{\int p dx} \\ &= e^{\int \sec x dx} \\ &= e^{\log_e |\sec x + \tan x|} \\ &= \sec x + \tan x \end{aligned}$$

The differential equation is given by

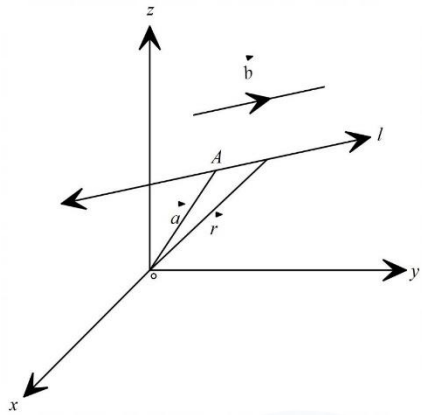
$$\begin{aligned} y(IF) &= \int Q.(IF) dx + c \\ \Rightarrow y(\sec x + \tan x) &= \int \tan x(\sec x + \tan x) dx \\ \Rightarrow y(\sec x + \tan x) &= \int (\sec x \tan x + \tan^2 x) dx \\ \Rightarrow y(\sec x + \tan x) &= \int \sec x \tan x dx + \int (\sec^2 x - 1) dx \\ \Rightarrow y(\sec x + \tan x) &= \sec x + \tan x - x + c \end{aligned}$$

The differential equation is

$$y(\sec x + \tan x) = \sec x + \tan x - x + c$$

47. Derive the equation of the line in space passing through a point and parallel to a vector both in vector and Cartesian from.

Sol:



From the above diagram it is clear that line 'l' passes through the point A and parallel to a vector $\vec{b}$

Then $\vec{AP}$ is parallel to $\vec{b}$.

Also,

$$\vec{AP} = \lambda \vec{b}$$

Where λ is a real number.

Now,

$$\vec{AP} = \vec{OP} - \vec{OA}$$

$$\lambda \vec{b} = \vec{r} - \vec{a}$$

or,

$$\vec{r} = \vec{a} + \lambda \vec{b}$$

Its Cartesian equation

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

48. A die is thrown 6 times. If getting an odd number is a success, what is probability of:

(a) 5 Successes?

(b) At least 5 successes?

(c) At most 5 successes?

Sol:

The given trials are Bernoulli trials $n = 6$

P = Probability that number is odd (success)

$$= \frac{3}{6}$$

$$\therefore P = \frac{1}{2}$$

$$q = 1 - P = 1 - \frac{1}{2} = \frac{1}{2} \text{ (Probability then no is even)}$$

$$p(X = x) = {}^nC_x P^x q^{n-x}$$

(a)

$$x = 5$$

$$\therefore P(x = 5) = {}^6C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^{6-5}$$

$$= {}^6C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)$$

$$= 6 \left(\frac{1}{2}\right)^6$$

$$= \frac{6}{64}$$

$$= \frac{3}{32}$$

$$\therefore P(5 \text{ successes}) = \frac{3}{32}$$

(b)

P (at least 5 successes)

$$\begin{aligned} &= P(x \geq 5) \\ &= P(x = 5) + P(x = 6) \\ &= {}^6C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^1 + {}^6C_6 \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^0 \\ &= 6 \left(\frac{1}{2}\right)^6 + 1 \left(\frac{1}{2}\right)^6 \\ &= 7 \left(\frac{1}{2}\right)^6 \\ &= \frac{7}{64} \\ \therefore P(\text{at least 5 successes}) &= \frac{7}{64} \end{aligned}$$

(c)

P (At most 5 successes)

$$\begin{aligned} &P(x \leq 5) \\ &= 1 - P(x > 5) \\ &= 1 - P(x = 6) \\ &= 1 - {}^6C_6 \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^0 \\ &= 1 - \left(\frac{1}{2}\right)^6 \\ &= 1 - \frac{1}{64} \\ &= \frac{63}{64} \\ \therefore P(\text{at most 5 successes}) &= \frac{63}{64} \end{aligned}$$

PART-E

Answer any one question:

(1x10 = 10)

49.(a) **Prove that** $\int_{-a}^a f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(x) \text{ is an even function} \\ 0, & \text{if } f(x) \text{ is an odd function} \end{cases}$ **and hence evaluate**

$$\int_{-1}^1 \sin^5 x \cos^4 x dx$$

(b) **Prove that**

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2.$$

Sol:

(a) Consider,

LHS,

$$\int_{-a}^b f(x)dx$$

$$= \int_{-a}^b f(x)dx + \int_{-a}^a f(x)dx$$

$$\left[\because \int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx \right]$$

$$= \int_0^{-a} f(x)dx + \int_0^a f(x)dx$$

$$\left[\because \text{prop, } 2\int_a^b f(x)dx = -\int_b^a f(x)dx \right]$$

Put, $x = -t$ is first integral.

$$\Rightarrow dx = -dt \quad \text{When, } x = 0, t = 0$$

$$x = -a, t = a$$

$$\int_0^a f(-t)dt + \int_0^a f(x)dx$$

$$= \int_0^a f(-x)dx + \int_0^a f(x)dx \text{-----(1)}$$

$$\left[\because \text{prop 1 } \int_a^b f(x)dx = \int_a^b f(t)dt \right]$$

If $f(x)$ is an even function, then

$$f(-x) = f(x)$$

$$\therefore \int_a^a f(-x)dx = \int_0^a f(x)dx \text{-----(2)}$$

If $f(x)$ is odd function, then

$$f(-x) = -f(x)$$

$$\int_0^a f(-x)dx = -\int_0^a f(x)dx \text{ --- (3)}$$

∴ Substituting 2 and 3 in 1

$$\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx \text{ if } f(x) \text{ is even function}$$

$$\int_{-a}^a f(x)dx = 0 \text{ if } f(x) \text{ is odd function}$$

$$f(x) = \sin^5 x \cos^4 x dx$$

$$f(-x) = \sin^5(-x) \cos^4(-x) dx$$

$$= -\sin^5 x \cos^4 x dx$$

$$f(-x) = -f(x)$$

∴ $f(x)$ is odd function

$$\therefore \int_{-1}^1 \sin^5 x \cos^4 x dx = 0$$

(b) Consider,

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$$

Multiplying $R_1 \times a, R_2 \times b, R_3 \times c$

$$= \frac{1}{abc} \begin{vmatrix} a(a^2+1) & a^2b & a^2c \\ ab^2 & b(b^2+1) & b^2c \\ c^2a & c^2b & c(c^2+1) \end{vmatrix}$$

Taking out a,b,c common from c_1, c_2, c_3 respectively.

$$= \frac{abc}{abc} \begin{vmatrix} a^2+1 & a^2 & a^2 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix}$$

Taking $R_1 \rightarrow R_1 + R_2 + R_3$

$$= 1 \begin{vmatrix} 1+a^2+b^2+c^2 & 1+a^2+b^2+c^2 & 1+a^2+b^2+c^2 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix}$$

Taking out $(1+a^2+b^2+c^2)$ common from R_1

$$= (1+a^2+b^2+c^2) \begin{vmatrix} 1 & 1 & 1 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix}$$

Applying $C_1 \rightarrow C_1 - C_3, C_2 \rightarrow C_2 - C_3$

$$= (1+a^2+b^2+c^2) \begin{vmatrix} 1-1 & 1 & 1 \\ b^2-b^2 & b^2+1-b^2 & b^2 \\ c^2-c^2-1 & c^2-c^2-1 & c^2+1 \end{vmatrix}$$

$$= (1+a^2+b^2+c^2) \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & b^2 \\ -1 & -1 & c^2+1 \end{vmatrix}$$

Expanding along R_1

$$= (1+a^2+b^2+c^2) [0+0+1(0+1)]$$

$$= (1+a^2+b^2+c^2)(1)$$

$$= (1+a^2+b^2+c^2)$$

Hence proved.

50. (a) A manufacturing company makes two models A and B of a product. Each piece of model A requires 9 labor hours for fabricating and 1 labour hour for finishing. Each piece of model B requires 12 labour hours available are 180 and 30 respectively. The company makes a profit of Rs. 8,000 on each piece of model A and Rs. 12,000 on each piece of model B. How many pieces of model A and B should be manufactured per week to realize a per week maximum profit? What is the maximum profit per week?

Sol:

Let x be the number of pieces of Model A and y be the number of pieces of Model B.

Let total profit be Z.

Then

$$\text{Total profit} = 8000x + 12000y$$

$$Z = 8000x + 12000y \text{ --- (1)}$$

Now we have the following mathematical model for the given problem Maximize

$$Z = 8000x + 12000y$$

Subject to the constraints:

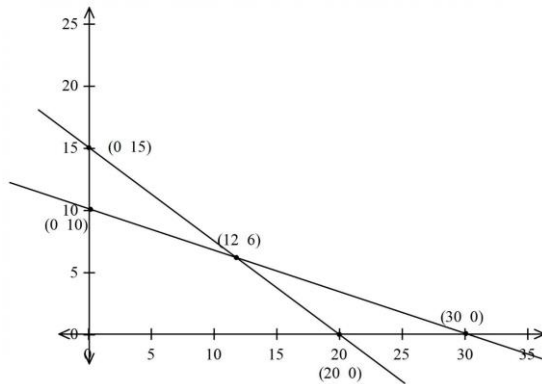
$$9x + 12y \leq 180 \text{ (fabricating constraint)}$$

$$3x + 4y \leq 60 \text{ --- (1)}$$

$$x + 3y \leq 30 \text{ (finishing constraint) --- (3)}$$

$$x \geq 0, y \geq 0 \text{ (non-negative constraint) --- (4)}$$

The feasible region (shaded) OABC determined by the linear inequalities (2) and (4) is shown below



Let us evaluate the objective function Z at each corner point as shown below:

Corner point	$Z = 8000x + 12000y$
$O(0,0)$	0
$A(20,0)$	160000
$B(12,6)$	168000
$C(0,10)$	120000

Maximum value of Z is 1, 68,000 at $B(12,6)$. Hence, the company should produce 12 Pieces of Model A and 6 pieces of Model B to realize maximum profit and maximum profit then will be Rs 1, 68,000.

(b) Find the value of K so that the function $f(x) = \begin{cases} Kx+1, & \text{if } x \leq 5 \\ 3x-5, & \text{if } x > 5 \end{cases}$ at $x = 5$ is a continuous function.

Sol:

Given,

$$f(x) = \begin{cases} Kx+1, & \text{if } x \leq 5 \\ 3x-5, & \text{if } x > 5 \end{cases}$$

Given function is continuous at $x = 5$

If $LHS = RHL = f(5)$

$$\lim_{x \rightarrow 5^-} f(x) = \lim_{x \rightarrow 5^+} f(x) = f(5)$$

LHL,

$$\lim_{x \rightarrow 5^-} f(x)$$

$$= \lim_{x \rightarrow 5^-} (kx + 1)$$

Putting $x = 5$

$$= k5 + 1$$

$$= 5k + 1$$

RHL,

$$\lim_{x \rightarrow 5^+} f(x)$$

$$= \lim_{x \rightarrow 5^+} (3x - 5)$$

putting $x = 5$

$$= 3(5) - 5$$

$$= 15 - 5$$

$$= 10$$

Now,

$$\text{LHL} = \text{RHL}$$

$$\Rightarrow 5k + 1 = 10$$

$$\Rightarrow 5k = 10 - 1$$

$$\Rightarrow 5k = 9$$

$$\Rightarrow k = \frac{9}{5}$$

$$\text{hence } k = \frac{9}{5}$$