

Subject Code: 35 (NS)

MATHEMATICS

(English Version)

Instructions:

1. The question paper has five parts namely A, B, C, D, and E. Answer all the parts.
2. Use the Graph Sheet for the question on Linear Programming Problem on Part –E.

PART-A

Answer all the ten question.

($10 \times 2 = 20$)

1. **Define bijective function**

Sol:

A function which is both one-one and onto is called Bijective function.

A function $f : A \rightarrow B$ is a bisection if it is one-one as well as onto.

2. **Write the principal value branch of $\cos^{-1} x$**

Sol:

$$[0, \pi]$$

3. **Constant a 2×2 Matrix $A = [a_{ij}]$ whose elements are given by $a_{ij} = \frac{i}{j}$**

Sol:

$$A = [a_{ij}]_{2 \times 2}$$

$$a_{ij} = \frac{i}{j}$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$a_{11} = \frac{i}{j} = \frac{1}{1} = 1, a_{12} = \frac{i}{j} = \frac{1}{2}$$

$$a_{21} = \frac{i}{j} = \frac{2}{1} = 2, a_{22} = \frac{i}{j} = \frac{2}{2} = 1$$

$$\begin{pmatrix} 1 & \frac{1}{2} \\ 2 & 1 \end{pmatrix}$$

4. If **A** is an invertible matrix of order 2 then find $[A^{-1}]$

Sol:

A is an invertible matrix of order 2

Therefore,

$$AA^{-1} = I = A^{-1}A \text{ and } |A| \neq 0$$

$$|AA^{-1}| = |I|$$

$$|A||A^{-1}| = 1$$

$$|A^{-1}| = \frac{1}{|A|}$$

5. If $y = e^{x^3}$, find $\frac{dy}{dx}$

Sol:

$$y = e^{x^3}$$

Taking log on both sides

$$\log y = \log_e e^{x^3}$$

$$\log y = x^3 \log_e e$$

$$\log y = x^3$$

$$\frac{d}{dx} \log y = \frac{d}{dx} x^3$$

$$\frac{1}{y} \frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} = y \cdot 3x^2$$

$$\frac{dy}{dx} = 3x^2 e^{x^3}$$

6. Find $\int \frac{x^2-1}{x^2} dx$

Sol:

$$\begin{aligned} & \int \frac{x^2-1}{x^2} dx \\ &= \int \frac{x^3}{x^2} dx - \int \frac{1}{x^2} dx \\ &= \int x dx - \int x^{-2} dx \\ &= \frac{x^2}{2} - \frac{x^{-2+1}}{-2+1} + C \\ &= \frac{x^2}{2} + \frac{1}{x} + C \end{aligned}$$

7. Find the unit vector in the direction of the vector $\vec{a} = \hat{i} + \hat{j} + \hat{k}$.

Sol:

$$\begin{aligned} \vec{a} &= \hat{i} + \hat{j} + \hat{k} \\ |\vec{a}| &= \sqrt{1^2 + 1^2 + 1^2} \\ &= \sqrt{3} \\ \hat{a} &= \frac{\vec{a}}{|\vec{a}|} \\ &= \frac{1}{\sqrt{3}} (\hat{i} + \hat{j} + \hat{k}) \\ &= \frac{1}{\sqrt{3}} \hat{i} + \frac{1}{\sqrt{3}} \hat{j} + \frac{1}{\sqrt{3}} \hat{k} \end{aligned}$$

8. If a line makes angle $90^\circ, 60^\circ$, and 30° with the positive direction of x, y and z-axis respectively. Find the direction cosines.

Sol.

Given that,

$$90^\circ, 60^\circ, \text{ and } 30^\circ$$

$$\cos 90^\circ = 0, m = \cos 60^\circ = \frac{1}{2}, n = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

9. **Define optimal solution in a linear programming problem.**

Sol. The any feasible solution of LPP which is maximized or minimized are called optimal solution.

10. **If $P(A) = \frac{7}{13}, P(B) = \frac{9}{13}$ and $P(A \cup B) = \frac{4}{13}$, find $P(A/B)$**

Sol:

$$P(A) = \frac{7}{13}, P(B) = \frac{9}{13}$$

and

$$P(A \cap B) = \frac{4}{13}$$

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{4/13}{9/13}$$

$$P(A/B) = 4/9$$

Section-B

**Answer any ten question:
= 20)**

(10 x 2

11. **Let $*$ be a binary operation on Q , defined by $a * b = \frac{ab}{2}, \forall a, b \in Q$. Determine whether $*$ is associative or not.**

Sol:

Getting

$$(a * b) * c = \left(\frac{ab}{2} \right) * c$$

$$= \frac{\left(\frac{ab}{2} \right) * c}{2} = \frac{abc}{4}$$

12. **Show that** $\left(\sin^{-1} \frac{1}{5} + \cos^{-1} x\right) = 1$ **then find the value of x.**

Sol:

$$\sin\left(\sin^{-1} \frac{1}{5} + \cos^{-1} x\right) = 1$$

$$\sin^{-1} \frac{1}{5} + \cos^{-1} x = \sin^{-1} 1$$

$$\sin^{-1} \frac{1}{5} + \cos^{-1} x = \frac{\pi}{2}$$

$$\sin^{-1} \frac{1}{5} = \frac{\pi}{2} - \cos^{-1} x$$

$$\sin^{-1} \frac{1}{5} = \sin^{-1} x + \cos^{-1} x - \cos^{-1} x$$

$$\sin^{-1} x \frac{1}{5} = \sin^{-1} x$$

$$x = \sin^{-1} \frac{1}{5} =$$

13. **Write the simplest form of** $\tan^{-1} \left(\frac{\cos x - \sin x}{\cos x + \sin x} \right), 0 < x < \frac{\pi}{2}$

Sol:

$$\tan^{-1} \left(\frac{\cos x - \sin x}{\cos x + \sin x} \right), \quad \left(0 < x < \frac{\pi}{2} \right)$$

Cosx Divided by numerator and denominator.

$$\tan^{-1} \left(\frac{\frac{\cos x}{\cos x} - \frac{\sin x}{\cos x}}{\frac{\cos x}{\cos x} + \frac{\sin x}{\cos x}} \right) = \tan^{-1} \left(\frac{1 - \tan x}{1 + \tan x} \right)$$

$$= \tan^{-1} \tan \left(\frac{\pi}{4} - x \right)$$

$$\tan^{-1} \left(\frac{\cos x - \sin x}{\cos x + \sin x} \right) = \frac{\pi}{4} - x$$

14. Find the area of the triangle whose vertices are (-2,-3) (3,2) and (-1,-8) by using determinant method.

Sol.

Area of $\Delta ABC =$

$$\begin{aligned} & \frac{1}{2} [x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] \\ &= \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \\ &= \frac{1}{2} \begin{vmatrix} 3 & 2 & 1 \\ -2 & -3 & 1 \\ -1 & -8 & 1 \end{vmatrix} \\ &= \frac{1}{2} [3(-3 \times 1 - 1(-8)) - 2(-2 \times 1 - 1 \times (-1))] + 1[-2(-8) - (-1)(-3)] \\ &= \frac{1}{2} [3(-3 + 8) - 2(-2 + 1) + 1(16 - 3)] \\ &\Delta ABC = \frac{1}{2} [15 + 2 + 13] \\ &\Delta ABC = \frac{1}{2} \times 30 = 15 \text{ unit}^2 \end{aligned}$$

15. Differentiate $x^{\sin x}$, $x > 0$ with respect to x .

Sol.

$$\log y = \log x^{\sin x} = \sin x \cdot \log x$$

$$\text{Or taking } \frac{dy}{dx} = x^{\sin x} \frac{d}{dx} (\sin x \log x)$$

$$\text{getting } \Rightarrow \frac{dy}{dx} = y \left[\sin x \times \frac{1}{x} + \log x \times \cos x \right]$$

$$\left[\sin x \times \frac{1}{x} + \log x \times \cos x \right]$$

16. Find $\frac{dy}{dx}$, if $x^2 + xy + y^2 = 100$

Sol.

Given the value is

$$x^2 + xy + y^2 = 100$$

Using the dy/dx

$$\frac{dy}{dx}(x^2) + \frac{dy}{dx}(xy) + \frac{dy}{dx}(y^2) = 0$$

$$2x + \left(x \frac{dy}{dx} + y\right) + 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-(2x + y)}{x + 2y}$$

17. Find the slope of the tangent to curve $y = x^3 - x$ at $x = 2$

Sol.

Given the value is slope of the tangent to curve $y = x^3 - x$ at $x = 2$

taking dy/dx

$$\frac{dy}{dx}(x^3 - x) = 3x^2 - 1$$

putting the value of $x = 2$

$$3x^2 - 1 = 3(2)^2 - 1 = 3 \times 4 - 1 = 11$$

$$\left(\frac{dy}{dx}\right)_{x=2} = 11$$

18. Integrate $\frac{e^{\tan^{-1}}}{1+x^2}$ with respect to x.

Sol.

let

$$\tan^{-1} x = t$$

$$\frac{1}{1+x^2} = \frac{dt}{dx}$$

$$\frac{dx}{1+x^2} = dt$$

$$\int e^t dt = e^t + c = \tan^{-1} x + c$$

$$\int \frac{e^{\tan^{-1}}}{1+x^2} dx = e^{\tan^{-1} x} + c$$

19. Evaluate : $\int_2^3 \frac{x dx}{x^2+1}$

Sol.

$$\text{Suppose } t = x^2 + 1 \Rightarrow x dx = \frac{dt}{2}$$

Then $x=2$, $t=5$, when $x=3$, $t=10$

$$\text{let } x^2 + 1 = t$$

$$\frac{d}{dx}(x^2 + 1) = \frac{dt}{dx}$$

$$2x + 0 = \frac{dt}{dx}$$

$$x dx = \frac{1}{2} dt$$

$$\text{then } x = 2 \quad t = 2^2 + 1 = 5$$

$$\text{then } x = 3 \quad t = 3^2 + 1 = 10$$

$$\begin{aligned}
 \int \frac{x}{x^2+1} dx &= \int_5^{10} \frac{1}{2} \cdot \frac{1}{t} dt \\
 &= \frac{1}{2} [\log t]_5^{10} \\
 &= \frac{1}{2} [\log 10 - \log 5] \\
 &= \frac{1}{2} \left[\log \frac{10}{5} \right] \\
 &= \frac{1}{2} \log_e 2
 \end{aligned}$$

20. Find the order and degree of the differential equation:

$$\left(\frac{d^3 y}{dx^3} \right)^2 + \left(\frac{d^2 y}{dx^2} \right)^2 + \left(\frac{dy}{dx} \right)^4 + y^5 = 0$$

Sol. Given the equation $\left(\frac{d^3 y}{dx^3} \right)^2 + \left(\frac{d^2 y}{dx^2} \right)^2 + \left(\frac{dy}{dx} \right)^4 + y^5 = 0$

Order = 3

Degree = 2

21. Find the projection of the vector $\hat{i} + \hat{j} + 7\hat{k}$ on the vector $7\hat{i} - \hat{j} + 8\hat{k}$.

Sol.

$$\text{Projection of } \vec{a} \text{ on } \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$$

$$\text{Here, } \vec{a} = \hat{i} + \hat{j} + 7\hat{k}.$$

And

$$\vec{b} = 7\hat{i} - \hat{j} + 8\hat{k}.$$

$$\begin{aligned}
|\vec{b}| &= \sqrt{7^2 + (-1)^2 + 8^2} \\
&= \sqrt{49 + 1 + 64} = \sqrt{114} \text{ unit} \\
\vec{a} \cdot \vec{b} &= (\hat{i} + \hat{j} + 7\hat{k}) (7\hat{i} - \hat{j} + 8\hat{k}) \\
&= 1 \cdot 7 + 3(-3) + 56 \\
&= 56 \text{ unit}
\end{aligned}$$

$$\text{Projection of } \vec{a} \text{ on } \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \frac{60}{\sqrt{114}}$$

22. Find the area of the parallelogram whose adjacent sides are determined by the vectors $\vec{a} = 3\hat{i} + \hat{j} + 4\hat{k}$, and $\vec{b} = \hat{i} - \hat{j} + \hat{k}$.

Sol.

Given the vector $\vec{a} = 3\hat{i} + \hat{j} + 4\hat{k}$, and $\vec{b} = \hat{i} - \hat{j} + \hat{k}$.

$$\begin{aligned}
\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 4 \\ 1 & -1 & 1 \end{vmatrix} = \hat{i} (1 \times 1 - 4(-1)) + \hat{j} (1 \times 4 - 1 \times 3) + \hat{k} ((-1) \times 3 - 1 \times 1) \\
&= 5\hat{i} + \hat{j} - 4\hat{k} \\
&= \sqrt{(5)^2 + (1)^2 + (-4)^2} = \sqrt{25 + 1 + 16} = \sqrt{42}
\end{aligned}$$

23. Find the angle between the planes whose vector equation are $\vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) = 5$ and $\vec{r} \cdot (3\hat{i} - 3\hat{j} + 5\hat{k}) = 3$.

Sol.

$$\text{Plane A} = \vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) = 5$$

$$\text{Plane B} = \vec{r} \cdot (3\hat{i} - 3\hat{j} + 5\hat{k}) = 3$$

The angle between two planes is defined as the angle between their normals.

If $m_1 = d_1$ and $m_2 = d_2$ are two planes then angle between two plane is

$$\cos \theta = \frac{\left| \vec{n_1} \cdot \vec{n_2} \right|}{\left| \vec{n_1} \right| \cdot \left| \vec{n_2} \right|}$$

$$\vec{n_1} = (2\hat{i} + 2\hat{j} - 3\hat{k}) \cdot (3\hat{i} - 3\hat{j} + 5\hat{k})$$

$$= 2 \cdot 3 + 2 \cdot (-3) + (-3)(5)$$

$$= 6 - 6 - 15 = -15$$

$$\left| \vec{n_1} \right| = \sqrt{2^2 + 2^2 + (-3)^2}$$

$$= \sqrt{4 + 4 + 9} = \sqrt{17}$$

$$\left| \vec{n_2} \right| = \sqrt{3^2 + (-3)^2 + 5^2}$$

$$\sqrt{9 + 9 + 25} = \sqrt{43}$$

$$\cos \theta = \frac{\left| \vec{n_1} \cdot \vec{n_2} \right|}{\left| \vec{n_1} \right| \cdot \left| \vec{n_2} \right|}$$

$$\cos \theta = \frac{-15}{\sqrt{17} \sqrt{43}}$$

$$\theta = \cos^{-1} \left(\frac{-15}{\sqrt{731}} \right)$$

$$\theta = \pi - \cos^{-1} \left(\frac{-15}{\sqrt{731}} \right)$$

24. A random variable X has the following probability distribution :

X	0	1	2	3	4
P(x)	0.1	k	2k	2k	k

Sol.

$$\sum p(x) = 1$$

$$(i) \quad 0.1 + k + 2k + 2k + k = 1$$

$$6k = 1 - 0.1$$

$$k = \frac{0.9}{6} = \frac{3}{20} = 0.15$$

$$(ii) \quad p(x \geq 2)$$

$$\begin{aligned}
 2k + 2k + k &= 5k \\
 &= 5 \times 0.15 \\
 P(x \geq 2) &= 0.75
 \end{aligned}$$

Section –C

Answer any ten question

(10 × 3 = 30)

25. Show that the relation R in the set A = {1,2,3,4,5} given by R = {a,b} |a-b| is even } is an equation relation.

Sol.

$$R = \{(1,1)(1,3)(1,5)(2,2)(2,4)(3,1)(3,3)(3,5)(4,2)(4,4)(5,1)(5,3)(5,5)\}$$

$$(a,a) \in R, \forall a \in A$$

R is Reflexive

$$(a,b) \in R \Rightarrow (b,a) \in R \quad \text{R is symmetric}$$

$$(a,b) \text{ and } (b,c) \in R \Rightarrow (a,c) \in R$$

R is transitive .

R is an equivalent relation.

26. **Prove that** $2 \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{7} = \tan^{-1} \frac{31}{17}$

Sol.

$$\text{Given the value is } 2 \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{7} = \tan^{-1} \frac{31}{17}$$

Prove the expression,

Taking L.H.S

$$\begin{aligned}
 & \tan^{-1} \left(\frac{2 \times \frac{1}{2}}{1 - \left(\frac{1}{2} \right)^2} \right) + \tan^{-1} \frac{1}{7} \\
 & 2 \tan^{-1} \frac{1}{2} = \tan^{-1} \left(\frac{2 \times \frac{1}{2}}{1 - \left(\frac{1}{2} \right)^2} \right) \\
 & = \tan^{-1} \left(\frac{1}{1 - \frac{1}{4}} \right) = \tan^{-1} \frac{4}{3} \\
 & \tan^{-1} \frac{4}{3} + \tan^{-1} \frac{1}{7} \\
 & \tan^{-1} \left(\frac{\frac{4}{3} + \frac{1}{7}}{1 - \frac{4}{3} \times \frac{1}{7}} \right) = \tan^{-1} \frac{31}{17}
 \end{aligned}$$

Hence Proved.

27. By using elementary transformation, find the inverse of the matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$

Sol.

Simplify the inverse matrix

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$$

$$\mathbf{A} = \mathbf{I} \mathbf{A}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} A$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A$$

$$R_1 \rightarrow R_1 - 3R_2$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} A$$

$$A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} A$$

28. If $x = \sin t$, $y = \cos 2t$ then prove that $\frac{dy}{dx} = -4 \sin t$.

Sol.

Let $x = \sin t$

$$\frac{d}{dt}(x) = \frac{d}{dt}(\sin t)$$

$$\frac{dx}{dt} = \cos t$$

$$y = \cos 2t$$

$$\frac{dy}{dt} = \frac{d}{dt}(\cos 2t)$$

$$\frac{dy}{dt} = -2 \sin 2t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2 \sin 2t}{\cos t}$$

$$\frac{dy}{dx} = \frac{-2 \cdot 2 \cdot \sin t \cos t}{\cos t}$$

$$\frac{dy}{dx} = -4 \sin t$$

Hence, proved it.

29. Verify Rolle's theorem for the function $f(x) = x^2 + 2$, $x \in [-2, 2]$.

Sol.

Given the value

$$f(x) = x^2 + 2, x \in [-2, 2]$$

We know that a polynomial function is everywhere continuous and differentiable in real number.

$F(x)$ is continuous on $[-2, 2]$ and differentiable on $(-2, 2)$

$$f(x) = x^2 + 2$$

$$f(-2) = (-2)^2 + 2 = 6$$

$$f(2) = (2)^2 + 2 = 6$$

$$f(-2) = f(2) = 6$$

Since all three conditions of Rolle's Theorem are satisfied.

$$f(x) = x^2 + 2$$

$$f'(x) = 2x$$

$$f'(c) = 2c$$

$$0 = 2c \quad \therefore c = 0$$

$$-2 < c < 2$$

Hence, Rolle's Theorem is verified.

30. Find two numbers whose sum is 24 and whose product is as large as possible.

Sol.

Suppose the two number be x and y

Given $x+y=24$

$P=xy$ (where P is the minimum value of xy)

$$P = x(24 - x)$$

$$P = 24x - x^2$$

$$\frac{dP}{dx} = \frac{d}{dx}(24x) - \frac{d}{dx}x^2$$

$$\frac{dP}{dx} = 0$$

$$24 - 2x = 0$$

$$2x = 24$$

$$x = 12$$

then,

$$x = 12, y = 24 - x = 24 - 12 = 12$$

$$\frac{d^2P}{dx^2} = -2$$

$$\frac{d^2P}{dx^2} < 0$$

Hence, number are 12 and 12 when product is large as possible.

31. **Find:** $\int \frac{x dx}{(x+1)(x+2)}.$

Sol.

Simplify the expression,

$$\int \frac{x dx}{(x+1)(x+2)}$$
$$\frac{x}{(x+1)(x+2)} \sim \frac{A}{x+1} + \frac{B}{x+2}$$
$$x \sim A(x+1) + B(x+2)$$

Putting $x=-2$

$$-2 \sim A(-2+1) + B(-2+1)$$

$$B = 2$$

$$x = -1$$

Then,

$$-1 \sim A(-1+1) + B(-1+1)$$

$$A = -1$$

$$\frac{x}{(x+1)(x+2)} = -\int \frac{1}{x+1} dx + 2 \int \frac{dx}{x+2}$$

$$= -\log|x+1| + 2\log|x+2| + C$$

$$= -\log|x+1| + \log|x+2|^2 + C$$

$$\int \frac{x dx}{(x+1)(x+2)} = \log \left| \frac{(x+2)^2}{x+1} \right| + C$$

32. **Find:** $\int e^x \sin x dx$.

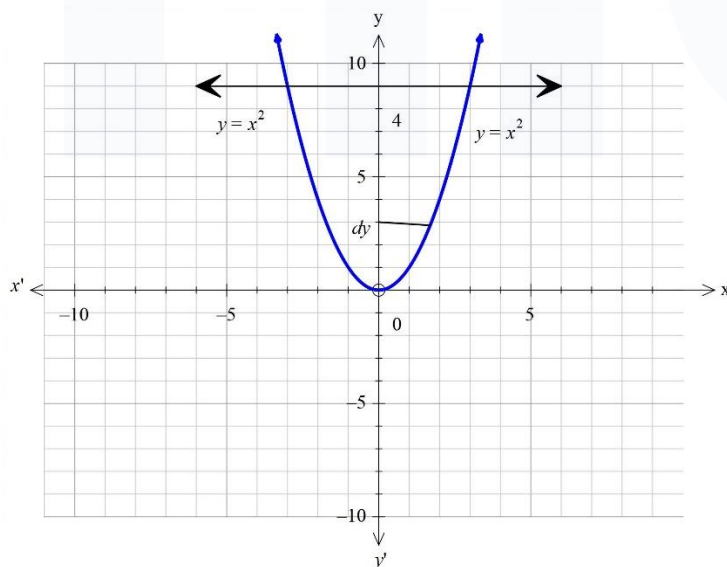
Sol.

Simplify the expression,

$$\begin{aligned}
 \int e^x \sin x dx &= \sin x \int e^x dx - \int \left[\frac{d}{dx} \sin x \int e^x dx \right] dx \\
 &= e^x \sin x - \int \cos x e^x dx \\
 &= e^x \sin x - \left[\cos x \int e^x dx - \int \left\{ \frac{d}{dx} \cos x \int e^x dx \right\} dx \right] \\
 &= e^x \sin x - \left[\cos e^x + \int \sin x e^x dx \right] \\
 &= \int e^x \sin x dx = e^x \sin x - e^x \cos x - \int e^x \sin x dx \\
 2 \int e^x \sin x dx &= e^x \sin x - e^x \cos x + C \\
 \int e^x \sin x dx &= \frac{e^x}{2} (\sin x - \cos x) + \frac{C}{2} \\
 \int e^x \sin x dx &= \frac{e^x}{2} (\sin x - \cos x) + C
 \end{aligned}$$

33. **Find the area of the region bounded by the curve $y = x^2$ and the line $y=4$**

Sol.



Area enclosed by

$$y = x^2 \quad \text{and } y=4 \text{ is BOAB}$$

$$\text{ar BOAB} = 2 \times \text{arOACO}$$

$$= 2 \int_0^4 x dy$$

$$= 2 \int_0^4 \sqrt{y} dy$$

$$= 2 \left[\frac{y^{3/2}}{3/2} \right]_0^4$$

$$= \frac{4}{3} \left[y^{3/2} \right]_0^4$$

$$= \frac{4}{3} \left[4^{3/2} - 0^{3/2} \right]$$

$$= \frac{4}{3} [8 - 0]$$

$$\text{arBOAB} = \frac{32}{3} \text{unit}^2$$

34. From the differential equation representing the family of curves $y = a \sin(x + b)$, where a, b are arbitrary constants.

Sol.

Simplify the equation,

$$\frac{dy}{dx} a \sin(x + b) = a \cos(x + b)$$

$$\frac{d^2y}{dx^2} a \cos(x + b) = -a \sin(x + b)$$

$$\frac{d^2y}{dx^2} + y = 0$$

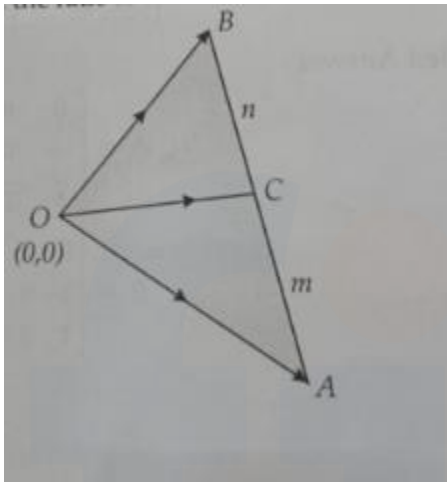
35. Show that the position vector of the point P, which divides the line joining the points A and B having position vectors $\vec{a}$ and $\vec{b}$ internally in the ratio m:n is $\frac{m\vec{b} + n\vec{a}}{m+n}$.

Sol.

Let O be the origin

Then $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$

Suppose c be the position vector of C which divides AB internally in the ratio m:n then,



$$\begin{aligned}\frac{AC}{CB} &= \frac{m}{n} \\ nAC &= mCB \\ n(\vec{c} - \vec{a}) &= m(\vec{b} - \vec{c}) \\ &= n\vec{c} - n\vec{a} = m\vec{b} - m\vec{c} \\ &= n\vec{c} + m\vec{c} = m\vec{b} + n\vec{a} \\ \vec{c}(m+n) &= m\vec{b} + n\vec{a} \\ \vec{c} &= \frac{m\vec{b} + n\vec{a}}{m+n}\end{aligned}$$

Hence, the position vector of C is $\frac{m\vec{b} + n\vec{a}}{m+n}$ proved.

36. Find x such that the four points A (3,2,1), B (4,X,5), C(4,2,-2) and D(6,5,-1) are coplanar.

Sol.

Getting $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} \equiv (1, x-2, 4)$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} \equiv (1, 0, -3), \overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} \equiv (3, 3, 2)$$

(For any two correct vectors, award 1 mark)

Writing,

$$\begin{vmatrix} \overrightarrow{AB} & \overrightarrow{AC} & \overrightarrow{AD} \end{vmatrix} = 0$$

Or,

$$\begin{vmatrix} 1 & x-2 & 4 \\ 1 & 0 & -3 \\ 3 & 3 & -2 \end{vmatrix} = 0$$

We get $x = 5$

37. Find the equation of the plane through the intersection of the planes $3x-y+2z-4=0$ and the point (2, 2, 1).

Sol.

Given: Equation of the planes $3x-y+2z-4=0$ and $x+y+z-2=0$

Equation of the plane which is passing through intersection of the plane is

$$3x - y + 2z - 4 + \lambda(x + y + z - 2) = 0 \dots (i)$$

plane passes through (2,2,1)

$$(3 + \lambda)x + (\lambda - 1)y + (2 + \lambda)z - 4 - 2\lambda = 0$$

$$6 + 2\lambda + 2\lambda - 2 + 2 + \lambda - 4 - 2\lambda = 0$$

$$\lambda = -2/3$$

Equation of the plane

$$\left(3 - \frac{2}{3}\right)x + \left(\frac{-2}{3} - 1\right)y + \left(2 - \frac{2}{3}\right)z - 4 - 2\left(\frac{-2}{3}\right) = 0$$

$$\frac{7}{3}x - \frac{5}{3}y + \frac{4}{3}z - \frac{8}{3}$$

$$7x - 5y + 4z - 8 = 0$$

38. A bag contains 4 red and 4 black balls, another bag contains 2 red and 6 black balls. One of the two bags is selected at random and a ball is drawn from the bag which is found to be red. Find the probability that the ball is drawn from the first bag.

Sol.

Simplify the probability equation,

$$P(E_1) = \frac{1}{2} \text{ and } P(E_2) = \frac{1}{2}$$

$$P(A/E_1) = \frac{1}{2} \text{ and } P(A/E_2) = \frac{1}{4}$$

$$P(E_1/A) = \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2)}$$

$$P(E_1/A) = \frac{\frac{1}{2} \times \frac{1}{2}}{\frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{4}}$$

$$P(E_1/A) = \frac{2}{3}$$

PART - D

Answer any six questions:

(6×5=30)

39. Let R_+ be the set of all-negative real number. Show that the function. Show that $f : R_+ \rightarrow [4, \infty]$ given by $f(x) = x^2 + 4$ invertible and write the inverse of F.

Sol.

Simplify the expression,

$$f : R^+ \rightarrow [4, \infty] \text{ such that } f(x) = x^2 + 4$$

Suppose $x, y \in R^+$ such that

$$f(x) = f(y)$$

$$x^2 + 4 = y^2 + 4$$

$$x^2 = y^2$$

$$x = y$$

So, f is an injective

suppose $y \in [4, \infty]$, then

$$f(x) = x^2 + 4$$

$$y = x^2 + 4$$

$$x = \sqrt{y - 4}$$

Therefore, $y \in [4, \infty]$, then exists

$$x = \sqrt{y - 4}$$

$$f(x) = f(\sqrt{y - 4})$$

$$f(\sqrt{y - 4}) = (\sqrt{y - 4})^2 + 4$$

$$= y - 4 + 4$$

$$= y$$

So, $\mathbb{R}^+ \rightarrow [4, \infty]$ is onto.

$f: \mathbb{R}^+ \rightarrow [4, \infty]$ is a bijection

Since it invertible

Let f^{-1} denote the inverse of $f(x)$

Then $f \circ f^{-1}(x) = x$, for all $x \in [4, \infty]$

$$f\{f^{-1}(x)\} = x \text{ for all } x \in [4, \infty]$$

$$\{f^{-1}(x)\}^2 = x - 4$$

$$f^{-1}(x) = \sqrt{x - 4} \text{ for all } x \in [4, \infty]$$

40. If $A = \begin{bmatrix} 0 & 6 & 7 \\ -6 & 0 & 8 \\ 7 & -8 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 6 & 7 \\ -6 & 0 & 8 \\ 7 & -8 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$ calculate AC, BC and (A+B)C. Also,

verify that $(A+B)C = AC + BC$.

Sol:

$$A = \begin{bmatrix} 0 & 6 & 7 \\ -6 & 0 & 8 \\ 7 & -8 & 0 \end{bmatrix} B = \begin{bmatrix} 0 & 6 & 7 \\ -6 & 0 & 8 \\ 7 & -8 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$$

$$\therefore AC = \begin{bmatrix} 0 & 6 & 7 \\ -6 & 0 & 8 \\ 7 & -8 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \times 2 + 6(-2) + 7 \times 3 \\ -6 \times 2 + 0(-2) + 8 \times 3 \\ 7 \times 2 + (-8)(-2) + 0 \times 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 - 12 + 21 \\ -12 + 0 + 24 \\ 14 + 16 + 0 \end{bmatrix}$$

$$= \begin{bmatrix} 9 \\ 12 \\ 30 \end{bmatrix}$$

$$BC = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \times 2 + 1(-2) + 1 \times 3 \\ 1 \times 2 + 0(-2) + 2 \times 3 \\ 1 \times 2 + 2 \times (-2) + 0 \times 3 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & 6 \\ 2 & -4 & 0 \end{bmatrix}$$

$$BC = \begin{bmatrix} 1 \\ 8 \\ -2 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 6 & 7 \\ -6 & 0 & 8 \\ 7 & -8 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$$

$$\begin{aligned}
 A+B &= \begin{bmatrix} 0 & 6 & 7 \\ 6 & 0 & 8 \\ 7 & -8 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 7 & 8 \\ -5 & 0 & 10 \\ 8 & -6 & 0 \end{bmatrix} \\
 (A+B)C &= \begin{bmatrix} 0 & 7 & 8 \\ -5 & 0 & 10 \\ 8 & -6 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} \\
 &= \begin{bmatrix} 0 \times 2 + 7(-2) + 8 \times 3 \\ -5 \times 2 + 0(-2) + 10 \times 3 \\ 8 \times 2 - 6 \times (-2) + 0 \times 3 \end{bmatrix} = \begin{bmatrix} 0 & -14 & 24 \\ -10 & 0 & 30 \\ 16 & 12 & 0 \end{bmatrix} \\
 &= \begin{bmatrix} 10 \\ 20 \\ 28 \end{bmatrix} \\
 AC+BC &= \begin{bmatrix} 9 \\ 12 \\ 30 \end{bmatrix} + \begin{bmatrix} 1 \\ 8 \\ -2 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \\ 28 \end{bmatrix} \\
 AC+BC &= (A+B)C
 \end{aligned}$$

41. Solve the following system of linear equation by matrix method.

$$\begin{aligned}
 x - y + 2z &= 7 \\
 3x + 4y - 5z &= -5 \\
 2x - y + 3z &= 12
 \end{aligned}$$

Sol.

Simplify the matrix

The given system of equation can be written as

$$\begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & 5 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$$

$$\text{where } A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$$

$$AX=B$$

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B$$

$$|A| = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 4 & 5 \\ 2 & -1 & 3 \end{vmatrix}$$

$$|A| = 1 \begin{vmatrix} 4 & -5 \\ -1 & 3 \end{vmatrix} + 1 \begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix}$$

$$= 1(12-5) + 1(9+10) + 2(-3-8)$$

$$= 7 + 19 - 22$$

$$|A| = 4$$

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 4 & -5 \\ -1 & 3 \end{vmatrix}$$

$$= 12 - 5 = 7$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 3 & -5 \\ 2 & 3 \end{vmatrix}$$

$$= -(9+10) = -19$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix}$$

$$= -3 - 8 = -11$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} -1 & 2 \\ -1 & 3 \end{vmatrix}$$

$$= -(-3+2) = 1$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}$$

$$= 3-4 = -1$$

$$C_{23} = (-1)^{1+3} \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix}$$

$$= -(-1+2) = -1$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 1 & 2 \\ 4 & -5 \end{vmatrix}$$

$$= 5-8 = -3$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 2 \\ 3 & -5 \end{vmatrix}$$

$$= -(-5-6) = 11$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 1 & -1 \\ 3 & 4 \end{vmatrix}$$

$$= 4+3 = 7$$

Ad joint matrix of

$$|A| = \begin{vmatrix} 7 & -19 & -11 \\ 3 & -1 & -1 \\ 2 & 11 & 7 \end{vmatrix}$$

$$= \begin{vmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{vmatrix}$$

$$A^{-1} = \frac{\text{Adjoint of A}}{|A|}$$

$$= \frac{1}{4} \begin{vmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & 1 & 7 \end{vmatrix}$$

$$\begin{aligned}
 X &= A^{-1}B \\
 \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 7 & -1 & -3 \\ -19 & -1 & 11 \\ -11 & 1 & 7 \end{bmatrix} \begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix} \\
 &= \frac{1}{4} \begin{vmatrix} 7 \times 7 + 1 \times (-5) + (-3)(12) \\ -19(7) + (-1)(-5) + (11)(12) \\ -11(7) + (-1)(-5) + 7(12) \end{vmatrix} \\
 &= \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 49 - 5 - 36 \\ -133 + 5 + 132 \\ -77 + 5 + 84 \end{bmatrix} \\
 &= \frac{1}{4} \begin{bmatrix} 8 \\ 4 \\ 12 \end{bmatrix} \\
 \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}
 \end{aligned}$$

$x = 2, y = 2$ and $z = 3$

42.If $(\tan^{-1} x)^2$, Show that $(x^2 + 1)^2 y_2 + 2x(x^2 + 1) y_1 = 2$

Sol.

$$y = (\tan^{-1} x)^2$$

$$\frac{dy}{dx} = \frac{d}{dx} (\tan^{-1} x)^2$$

$$\frac{dy}{dx} = 2 \tan^{-1} x \frac{d}{dx} \tan^{-1} x$$

$$\frac{dy}{dx} = 2 \tan^{-1} x \frac{1}{1 + x^2}$$

$$(1+x^2) \frac{dy}{dx} = 2 \tan^{-1} x$$

$$(1+x^2) \frac{d}{dx} \frac{dy}{dx} + \frac{dy}{dx} \frac{d}{dx} (1+x^2) = 2 \tan^{-1} x$$

$$(1+x^2) \frac{d^2 y}{dx^2} + \frac{dy}{dx} (0+2x) = \frac{2}{1+x^2}$$

$$(1+x^2) \frac{d^2 y}{dx^2} + 2x(1+x^2) \frac{dy}{dx} = 2$$

$$(1+x^2) y_2 + 2x(1+x^2) y_1 = 2$$

Hence, proved.

43. Sand is pouring from a pipe at the rate of $12 \text{ cm}^3/\text{s}$. The falling sand forms a cone on a ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?

Sol.

Simplify the expression,

$$\frac{dV}{dt} = 12 \text{ cm}^3 / \text{sec}$$

And height of the cone = $1/6$ of radius of the base of the cone.

$$\text{Volume of cone} = \frac{1}{3} \pi r^2 h$$

$$V = \frac{1}{3}\pi(6h)^2 h \quad \left[h = \frac{r}{6} \right]$$

$$V = 12\pi h^3$$

$$\frac{dV}{dt} = \frac{d}{dt} 12\pi h^3$$

$$12 = 12\pi \cdot 3h^2 \frac{dh}{dt}$$

$$1 = \pi \times 3(4)^2 \frac{dh}{dt}$$

$$\frac{1}{48\pi} = \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{48\pi}$$

$$= \frac{1}{48 \times \frac{22}{7}}$$

$$\frac{dh}{dt} = \frac{7}{48 \times 22} = 0.0066 \text{ cm / sec}$$

44. Find the integral of $\frac{1}{x^2 + a^2}$ with respect to x and hence find $\int \frac{1}{x^2 - 6x + 13} dx$

Sol.

Suppose $x = a \tan \theta$

$$\frac{dx}{d\theta} = \frac{d}{d\theta} a \tan \theta = a \sec^2 \theta$$

$$\int \frac{1}{x^2 + a^2} dx = \int \frac{1}{a^2 \tan^2 \theta + a^2} a \sec^2 \theta d\theta$$

$$= \int \frac{1}{a^2 (\tan^2 \theta + 1)} a \sec^2 \theta d\theta$$

$$= \int \frac{\sec^2 \theta}{a \sec^2 \theta} d\theta$$

$$= \frac{1}{a} \int d\theta$$

$$= \frac{1}{a} \theta + C$$

$$= \frac{1}{a} \tan^{-1} \frac{x}{a} + C$$

$$\begin{aligned}
 & \int \frac{1}{x^2 - 6x + 13} dx \\
 &= \int \frac{1}{(x-3)^2 + 2^2} dx \\
 &= \frac{1}{2} \tan^{-1} \left| \frac{x-3}{2} \right| + C
 \end{aligned}$$

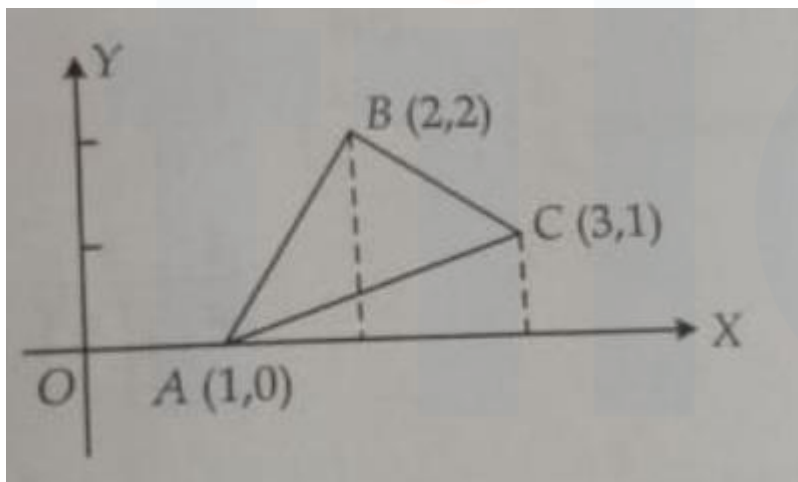
45. Using integration find the area of the region bounded by triangle whose vertices are (1,0). (2,2) and (3,1).

Sol.

Simplify the equation,

$$AB: y = 2x - 2, BC: y = -x + 4$$

$$AC: y = \frac{1}{2}(x-1)$$



The area A =

$$\int_1^2 (2x-2) dx + \int_2^3 (-x+4) dx - \frac{1}{2} \int_1^3 (x-1) dx$$

getting area =

$$\left[x^2 - 2x \right]_1^2 + \left[-\frac{x^2}{2} + 4x \right]_2^3 - \frac{1}{2} \left[-\frac{x^2}{2} - x \right]_1^3$$

Getting area = $\frac{3}{2}$ square units.

46. Find the general solution of the differential equation $x \frac{dy}{dx} + 2y = x^2 \log x$

Sol.

$$x \frac{dy}{dx} + 2y = x^2 \log x$$

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{x^2}{x} \log x$$

$$\frac{dy}{dx} + \frac{2}{x}y = x \log x$$

Now, compare the differential equation.

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$P(x) = \frac{2}{x} \text{ and } Q(x) = x \log x$$

Integrating factor

$$(I.F) = e^{\int p dx}$$

$$I.F = e^{2 \log x}$$

$$= \log x^2$$

$$= x^2$$

Solution of the differential equation is

$$y \times IF = \int IF \times Q(x) dx + C$$

$$y \times x^2 = \int x^2 \cdot x \log x dx + C$$

$$x^2 y = \log x \int x^3 dx - \int \left[\frac{d}{dx} (\log x) \int x^3 dx \right] dx + C$$

$$x^2 y = \frac{x^4}{4} \log x - \int \frac{1}{x} \cdot \frac{x^4 4}{\log x} - \frac{x^4}{16} + C$$

$$y = \frac{x^2}{4} \log x - \frac{x^2}{16} + \frac{C}{x^2}$$

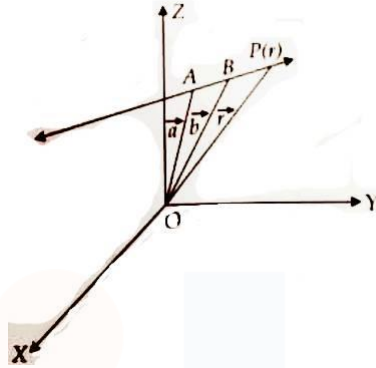
$$16y = 4x^2 \log x - x^2 + 16Cx^2$$

47. Derive the equation of the line in space passing through two given points., both in vector and Cartesian form.

Sol.

Suppose O be the origin and A and B be the given points with position vector $\vec{a}$ and $\vec{b}$ the line passing through the points A and B.

Then,



$$\vec{OP} = \vec{r}, \quad \vec{OA} = \vec{a} \text{ and } \vec{OB} = \vec{b}$$

$\vec{AP}$ is collinear with $\vec{AB}$

$$\vec{AP} = \lambda \vec{AB}$$

$$\vec{OP} - \vec{OA} = \lambda (\vec{OB} - \vec{OA})$$

$$\vec{r} - \vec{a} = \lambda (\vec{b} - \vec{a})$$

$$\vec{r} = \vec{a} + \lambda (\vec{b} - \vec{a})$$

Since every point on the line satisfies this equation for each value of λ

Since $\vec{r}$ is the position vectors of any point P (x,y,z) on the line.

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Similar $\vec{a}$ and $\vec{b}$ are the position vectors of point A (x_1, y_1, z_1) and B (x_2, y_2, z_2)

$$\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$$

$$\text{and } \vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$$

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$$

$$x\hat{i} + y\hat{j} + z\hat{k} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k} + \lambda(x_2\hat{i} + y_2\hat{j} + z_2\hat{k} - x_1\hat{i} - y_1\hat{j} - z_1\hat{k})$$

$$(x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k} = \lambda\{(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}\}$$

On equation coefficient of $\hat{i}$, $\hat{j}$, and $\hat{k}$

$$(x - x_1) = \lambda(x_2 - x_1), (y - y_1) = \lambda(y_2 - y_1)$$

$$\text{and } z - z_1 = \lambda(z_2 - z_1)$$

$$\frac{x - x_1}{x_2 - x_1} = \lambda, \frac{y - y_1}{y_2 - y_1} = \lambda$$

$$\text{and } \frac{z - z_1}{z_2 - z_1} = \lambda$$

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = \lambda$$

Hence, the equation of the line passing through the point is.

48. If a fair coin is tossed 10 times, find the probability of

(i) Exactly six heads

(ii) Atleast six heads.

Sol.

Simplify the expression,

$$q = 1 - p$$

$$\text{N=10 and } P = \frac{1}{2} \text{ then } = 1 - \frac{1}{2} = \frac{1}{2}$$

$$P(X = x) = {}^{10}C_x \left(\frac{1}{2}\right)^{10-x} \left(\frac{1}{2}\right)^x$$

$$= {}^{10}C_x \left(\frac{1}{2}\right)^{10}$$

$$P(X = 6) = {}^{10}C_x \left(\frac{1}{2}\right)^{10-x}$$

$$= \frac{10!}{4!(10-4)!} \times \frac{1}{2^{10}}$$

Therefore

$$P(X = 6) = \frac{105}{512}$$

$$P(X \geq 6) = P(X = 6) + P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10)$$

we get

$$\left[\frac{10!}{6! \times 4!} + \frac{10!}{7! \times 3!} + \frac{10!}{8! \times 2!} + \frac{10!}{9! \times 1!} + \frac{10!}{10!} \right] \frac{1}{2^{10}} = \frac{193}{512}$$

PART – E

10×1=10

Answer any one off the following question:

49. Prove that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ **and hence evaluate** $\int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} dx$

(b) Prove that:

$$\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3$$

Sol.

(a) If $f(x)$ is continuous function defined on $[0, a]$, then prove that.

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

Suppose

$$a - x = t \Rightarrow x = a - t$$

$$\frac{d}{dx}(a - x) = \frac{dt}{dx}$$

$$-1 = \frac{dt}{dx}$$

$$dx = -dt$$

then,

$$x = 0$$

when,

$$a - 0 = t \Rightarrow t = a$$

$$a - a = t \Rightarrow t = 0$$

$$\int_a^b f(x) dx = -\int_a^0 f(a-t) dt \quad \left[\int_a^b f(x) dt = -\int_a^b f(x) dx \right]$$

$$\int_a^b f(x) dx = \int_a^a f(a-t) dt$$

$$\int_a^b f(x) dx = \int_0^a f(a-x) dx$$

Hence, proved

$$I = \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} dx$$

$$I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x} + \sqrt{a-a-x}} dx$$

$$2I = \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} dx + \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x} + \sqrt{x}} dx$$

$$2I = \int_0^a \frac{\sqrt{x} + \sqrt{a-x}}{\sqrt{x} + \sqrt{a-x}} dx$$

$$2I = \int_0^a dx$$

$$2I = [x]_0^a$$

$$2I = a - 0$$

$$I = \frac{a}{2}$$

$$\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = C_1 \rightarrow C_1 + C_2 + C_3$$

(b)

$$\begin{vmatrix} 2x+2y+2z & x & y \\ 2x+2y+2z & y+z+2x & y \\ 1 & x & z+x+2y \end{vmatrix}$$

Taking $2x+2y+2z$ common from C_1

$$= 2(x+y+z) \begin{vmatrix} 1 & x & y \\ 1 & y+z+2x & y \\ 1 & x & z+x+2y \end{vmatrix}$$

$R_2 \rightarrow R_2 - R_1$

$$2(x+y+z) \begin{vmatrix} 1 & x & y \\ 0 & x+y+z & 0 \\ 1 & x & z+x+2y \end{vmatrix}$$

taking $(x+y+z)$ common from R_2

$$= 2(x+y+z) \begin{vmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 1 & x & z+x+2y \end{vmatrix}$$

Expand along R_2

$$= 2(x+y+z)^2 \begin{vmatrix} 1 & y \\ 1 & z+x+2y \end{vmatrix}$$

$$= 2(x+y+z)^2 (z+x+2y-y)$$

$$= 2(x+y+z)^3$$

50. (a) Solve the following problem graphically :

Minimise and Maximize:

$$Z = 3x + 9y$$

Subject to the constraints:

$$x + 3y \leq 60$$

$$x + y \geq 10$$

$$x \leq y$$

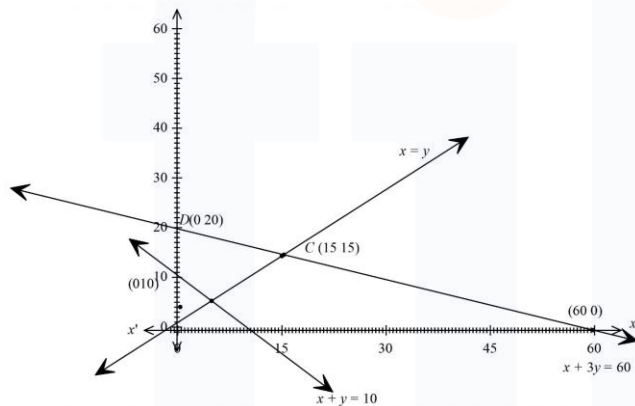
$$x \geq 0, y \geq 0$$

(c) Find the relationship between a and b so that the function f defined by:

$$f(x) = \begin{cases} ax+1, & \text{if } x \leq 3 \\ bx+3, & \text{if } x > 3 \end{cases}$$

Is continuous at $x=3$

Sol.



Writing corner A(0,10), B(5,5), C(15,15) and D(0,20)

We get

Sl. No.	Corner point	Value of $Z=3x+9y$
1	A(0,10)	90
2	B(5,5)	60 ← Minimum
3	C(15,15)	180 ← Maximum
4	D(0,20)	180 ← Maximum

Getting Minimum value of $Z=60$ at the point b(5,5)

Getting maximum Value of $z=180$ at the points D(0,20) and C(15,15). All the points on line segment joining the point D(0,20) and (15,15).

(b).

$$f(x) = ax + 1 \quad \text{if } x \leq 3$$

$$= bx + 3 \quad \text{if } x > 3$$

$f(x)$ is continuous at $x=3$

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} f(x) = f(3)$$

$$\begin{aligned} \lim_{x \rightarrow 3^-} f(x) &= \lim_{x \rightarrow 3^-} (bx + 3) \\ &= \lim_{x \rightarrow 3^-} [b(3 + h) + 3] \\ &= b(3 + 0) + 3 \\ &= 3b + 3 \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 3^+} f(x) &= \lim_{x \rightarrow 3^+} (ax + 1) \\ &= \lim_{x \rightarrow 3^+} (a(3 - h) + 1) \end{aligned}$$

$$\lim_{x \rightarrow 3^+} a(3 - 0) + 1$$

$$f(x) = \lim_{x \rightarrow 3^+} f(x)$$

$$3b + 3 = 3a + 1$$

$$3 - 1 = 3a - 3b$$

$$2 = 3(a - b)$$

$$a - b = \frac{2}{3}$$