Binary Black Hole Merger: Characteristics of the Extreme Mass-Ratio Limit

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1. Abstract

We study binary black hole (BH) mergers in the extreme mass-ratio limit. The **energy**, **angular momentum** and **linear momentum** of the **remnant BH** are determined based on the notion that toward the merger, small mass-ratio binary systems follow a **geodesic universal infall (GUI)** trajectory. Therefore, our analysis is performed **directly in the test-particle limit** by solving the Regge-Wheeler-Zerilli (RWZ) wave equation with a source that moves along a geodesic. This formalism captures well the final inspiral stages of small mass-ratio binaries, and thus provides a straightforward universal description in a region inaccessible to numerical relativity simulations.

3. A Numerical Note - Extrapolation to $\mathcal{I}^+$

The GWs flux is formally evaluated at null infinity, $\mathcal{I}^+$. Yet, we can numerically extract them only at finite distances. We overcome this obstacle by analytically calculating the GWs propagation from a given distance to $\mathcal{I}^+$ by solving the homogeneous RWZ equation in the frequency domain.

As can be seen in Figure 2, after the extrapolation to $\mathcal{I}^+$, the alignment of the GW amplitude that was numerically extracted at different radii improves by more than an order of magnitude.

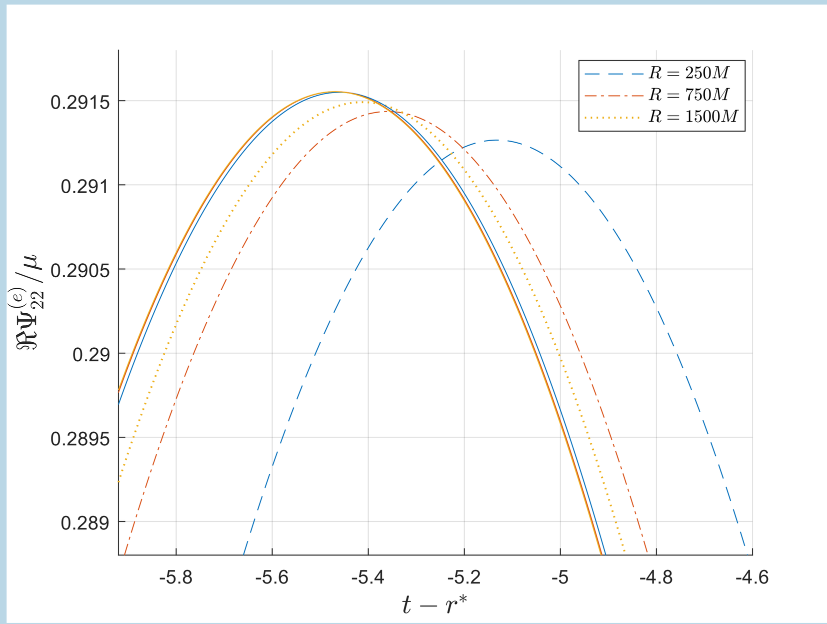

Figure 2: GWs extracted at $R = 250M$ (blue dashed line), $750M$ (red dash-dotted line), $1500M$ (yellow dotted line), and their extrapolation to $\mathcal{I}^+$ (solid lines, respective colors).

2. GUI trajectory & Gravitational Waves

The merger scenario can be qualitatively divided into three stages:

(i) Quasicircular inspiral, during which the orbit evolves by the emission of gravitational waves (GWs).

(ii) Universal plunge, where the infall path tends to the GUI trajectory.

(iii) Quasinormal modes ringdown.

First, we solve analytically the equations of motion to get the GUI - the geodesic infall of a test particle, initially in a circular orbit at the ISCO ($R = 6M$). The GUI is universal in the sense that any small mass-ratio binary, that evolves due to GWs emission, tends to this infall trajectory after it crosses the ISCO. Then, we calculate the GWs emitted by a test particle that moves along this geodesic by numerically solving the RWZ equation[1, 2]. The latter is a wave equation with a potential, induced by the spacetime curvature, and a source term, derived from the stress-energy tensor:

$$\partial_t^2 \psi_{\ell,m}^{(\lambda)} - \partial_{r^*}^2 \psi_{\ell,m}^{(\lambda)} + V_\ell^{(\lambda)} \psi_{\ell,m}^{(\lambda)} = S_{\ell,m}^{(\lambda)},$$

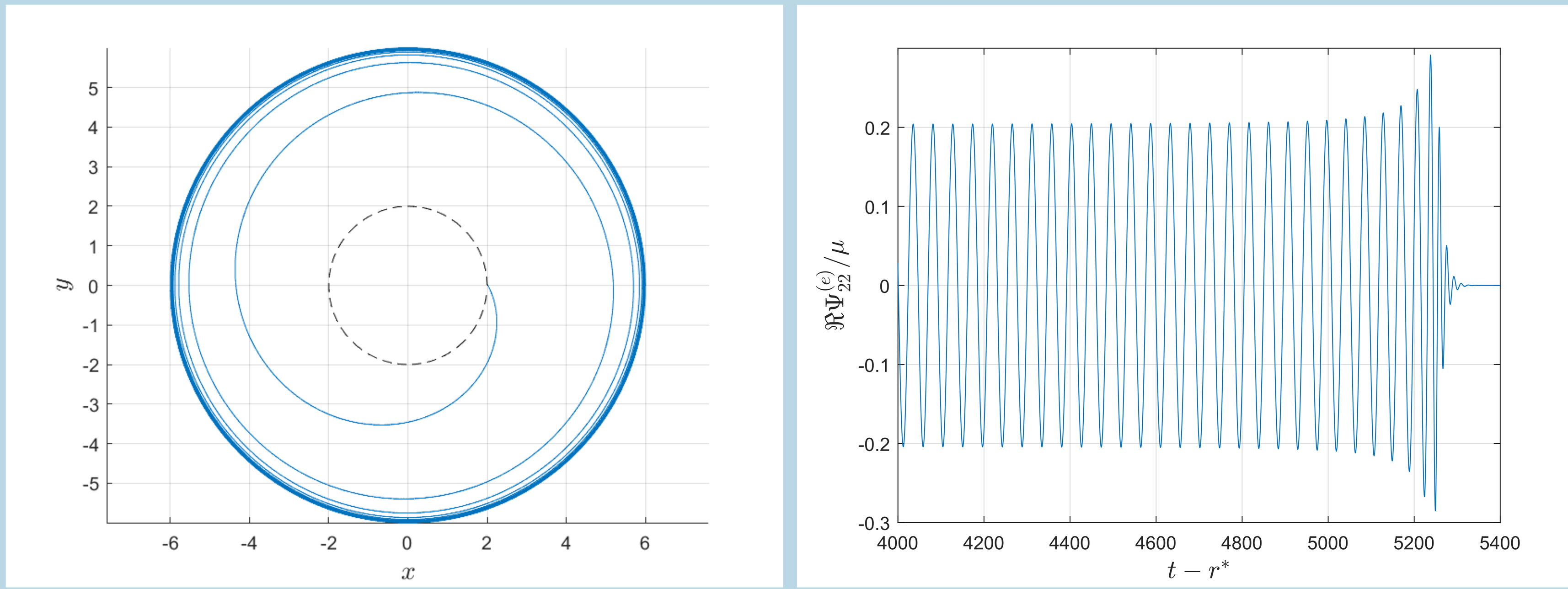

Figure 1: The left panel presents the GUI trajectory of a test particle that inspirals from the ISCO into the horizon (black dashed line). The right panel presents the corresponding GWs emission.

4. Results: Recoil Velocity

Using the above GWs calculation, we evaluate the linear momentum flux and the corresponding recoil velocity of the remnant BH. We calculate numerically the contribution up to $\ell = 10$ multipole. Assuming that the contributions of higher multipoles decay exponentially, as can be seen in Figure 3, we estimate the total recoil velocity, which has a quadratic dependence on the symmetric mass ratio, ν, and is given by:

$$\boldsymbol{V/c \approx 0.0467\nu^2}.$$

This result is greater than the known value in the literature[3, 4] by approximately 4%, mostly as a result of the high multipole contributions.

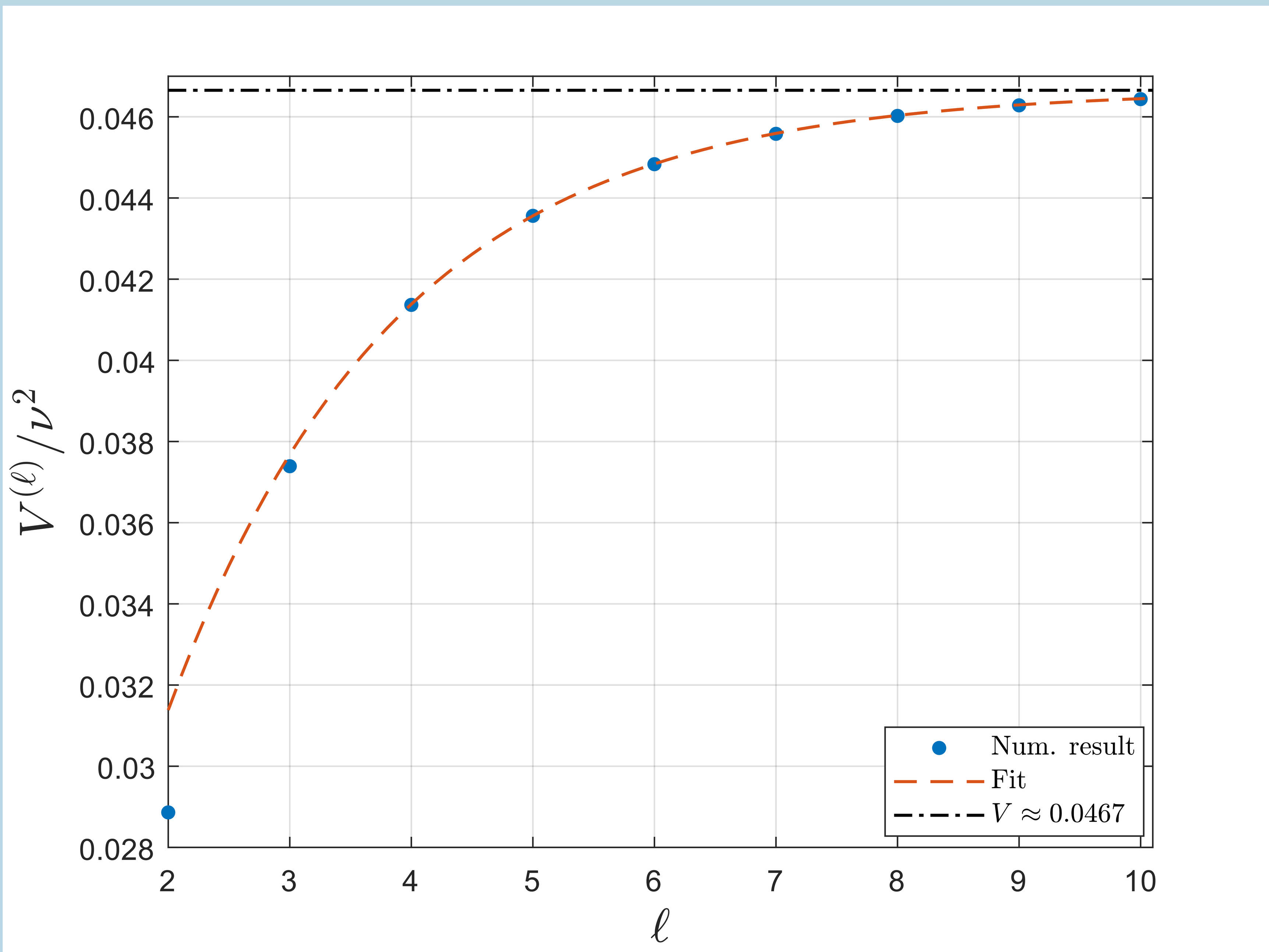

Figure 3: The final recoil velocity, up to a given multipole ℓ. The total velocity converges to $V/\nu^2 \approx 0.0467c$ (black dash-dotted line) and corresponds to an exponential decay trend (red dashed line)

5. Waveform

The test particle GUI waveform, shown in the right panel of Figure 1, has a linear dependence on the mass ratio and hence it can be easily scaled to capture the late stages of realistic small mass-ratio binary mergers. The dephasing time of the GUI and the small mass-ratio signal scales as:

$$\boldsymbol{t_{dp} \propto \nu^{-3/10}}.$$

Thus, for example, given a merger of a stellar mass object and a super-massive BH, with a mass of $10^6 - 10^9 M_\odot$, the GUI waveform faithfully describes between tens to hundreds of the final GWs cycles.

6. Final Mass & Spin

The final mass and spin of the remnant BH are given by:

$$\frac{M_f}{M} = 1 - \left(1 - \sqrt{\frac{8}{9}}\right)\nu + O\left(\nu^{9/5}\right),$$

$$a = \sqrt{12}\nu + O\left(\nu^{9/5}\right).$$

The first order term is derived analytically from the energy and angular momentum of the test particle at the ISCO:

$$E_{ISCO}/\mu = \sqrt{\frac{8}{9}},\ L_{ISCO}/\mu = \sqrt{12}M.$$

The next-order term corresponds to the GWs emission after the ISCO crossing. Its scaling can be deduced based on the scaling of the emitted fluxes, $\mathcal{F} \propto \nu^2$, and the infall time from the ISCO, $t \propto \nu^{-1/5}$[5, 6].

References

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