A geometric description of the sets of palindromic and alternating pencils with bounded rank

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Outline

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Basic definitions

We consider matrix pencils $A_0 + \lambda A_1$, with $A_0, A_1 \in \mathbb{C}^{n\times n}$.

Structure set $\mathbb{S}$	Definition $A_0 + \lambda A_1 \in \mathbb{S}$	Notation $\{A_0 + \lambda A_1 \in \mathbb{S} : \operatorname{rank}(A_0 + \lambda A_1) \leq r\}$
$\top$-palindromic	$A_1^\top = A_0$	Pal_r
$\top$-anti-palindromic	$A_1^\top = -A_0$	Apal_r
$\top$-even	$A_0^\top = A_0,\ A_1^\top = -A_1$	Even_r
$\top$-odd	$A_0^\top = -A_0,\ A_1^\top = A_1$	Odd_r

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($P(\lambda)$ is $\top$-pal $\Leftrightarrow \mathrm{rev}\, P(\lambda) = P(\lambda)^\top$).

Kronecker canonical form

Any pencil $A_0+\lambda A_1$ is strictly equivalent ($P(A_0+\lambda A_1)Q$, P,Q nonsingular) to a direct sum of blocks:

Right singular block of order α	**Left singular** block of order α	**Jordan block** of order k associated with $\lambda_0\in\mathbb{C}$	**Jordan block** of order k associated with the **infinite** eigenvalue
$L_\alpha(\lambda)$	$L_\alpha(\lambda)^\top$	$J_k(\lambda-\lambda_0)$	$J_k^\infty(\lambda)$
$\begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \end{bmatrix}$	$\begin{bmatrix} \lambda & & \\ 1 & \ddots & \\ & \ddots & \lambda \\ & & 1 \end{bmatrix}$	$\begin{bmatrix} \lambda-\lambda_0 & 1 & & \\ & \ddots & \ddots & \\ & & \lambda-\lambda_0 & 1 \\ & & & \lambda-\lambda_0 \end{bmatrix}$	$\begin{bmatrix} 1 & \lambda & & \\ & \ddots & \ddots & \\ & & 1 & \lambda \\ & & & 1 \end{bmatrix}$
$\alpha\times(\alpha+1)$	$(\alpha+1)\times\alpha$	$k\times k$	$k\times k$

The number and sizes of blocks of each type is **uniquely determined** (KCF).

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$\alpha \times (\alpha + 1)$	$(\alpha + 1) \times \alpha$	$k \times k$	$k \times k$

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Strict equivalence destroys the ($\top$-pal, $\top$-anti-pal, $\top$-alternating) structure.

$\top$-palindromic canonical form (KCF-like)

Any $\top$-palindromic pencil $A_0+\lambda A_1$ is congruent ($P(A_0+\lambda A_1)P^\top$) to a direct sum of blocks:

Pair of **left-right singular** blocks

$$M_\alpha^\sharp(\lambda) := \begin{bmatrix} 0 & L_\alpha(\lambda) \\ \operatorname{rev} L_\alpha(\lambda)^\top & 0 \end{bmatrix} \qquad (2\alpha+1)\times(2\alpha+1)$$

Pair of **Jordan-like** blocks with **even size** ($\lambda_0=-1$)

$$\begin{bmatrix} 0 & J_\beta(\lambda+1) \\ \operatorname{rev} J_\beta(\lambda+1)^\top & 0 \end{bmatrix} \qquad (2\beta)\times(2\beta),\ \beta \textbf{ even}$$

Pair of **Jordan-like** blocks with **odd size** ($\lambda_0=1$)

$$\begin{bmatrix} 0 & J_\gamma(\lambda-1) \\ \operatorname{rev} J_\gamma(\lambda-1)^\top & 0 \end{bmatrix} \qquad (2\gamma)\times(2\gamma),\ \gamma \textbf{ odd}$$

Pairs of Jordan-like blocks ($\lambda_0, 1/\lambda_0$, $\lambda_0\neq\pm1$)

$$\begin{bmatrix} 0 & J_\delta(\lambda-\lambda_0) \\ \operatorname{rev} J_\delta(\lambda-\lambda_0)^\top & 0 \end{bmatrix} \qquad (2\delta)\times(2\delta)$$

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$$\left[\begin{array}{cccc|cccc} & & & & & & & \lambda-1 \\ & & & & & & \lambda-1 & 1 \\ & & & & & \iddots & \iddots & \\ & & & & \lambda-1 & 1 & & \\ \hline & & & 1-\lambda & \lambda+1 & & & \\ & & 1-\lambda & \lambda & & & & \\ & \iddots & \iddots & & & & & \\ 1-\lambda & \lambda & & & & & & \end{array}\right] \qquad (2\varepsilon)\times(2\varepsilon)$$

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$\uparrow$ **Size**

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Def'n: A property $\mathfrak{p}$ is **generic** in a set A if $\mathfrak{p}$ holds in A_i, with $A = \cup_{i=1}^{k} \overline{A}_i$ and A_i open.

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☞ We will see that Pal_r is an irreducible algebraic set with just **one generic eigenstructure** (different for r odd/even).

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Constructing low-rank $\top$-palindromic pencils

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- **Attempt 1**: Fix A with low rank. Then set: $A+\lambda A^\top$.

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Wrong approach: For a fixed r, this does not provide all pencils $A+\lambda A^\top$ with $\operatorname{rank}(A+\lambda A^\top)\leq r$: r must be even and, if $\operatorname{rank}A\leq r_A$, this construction does not provide all $A+\lambda A^\top$ with $\operatorname{rank}(A+\lambda A^\top)\leq 2r_A$.

Example: $\begin{bmatrix} 0 & 1+\lambda \\ 1+\lambda & 0 \end{bmatrix}$ is not of the form $A+\lambda A^\top$, with $\operatorname{rank}A\leq 1$.

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When $B=A^\top$, either

- $v_i=w_i$ ($\deg v_i=0$) $\rightsquigarrow (1+\lambda)vv^\top$, or
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Example: $M_1^\sharp(\lambda) = \left[\begin{array}{cc|c} & & \lambda \\ & & 1 \\ \hline 1 & \lambda & \end{array}\right] = \underbrace{\begin{bmatrix} 0\\0\\1\end{bmatrix}}_{v}\underbrace{\begin{bmatrix} 1 & \lambda & 0\end{bmatrix}}_{w^\top} + \underbrace{\begin{bmatrix} \lambda\\1\\0\end{bmatrix}}_{\mathrm{rev}\, w}\underbrace{\begin{bmatrix} 0 & 0 & 1\end{bmatrix}}_{v^\top}.$

Rank-one expression of $\top$-palindromic pencils

Theorem (rank-1 decomposition of $\top$-pal pencils)

$A \in \mathbb{C}^{n\times n}$. If $\operatorname{rank}(A+\lambda A^\top) = r \leq n$, then

$$A+\lambda A^\top = \begin{cases} v_1 w_1^\top + \cdots + v_{r/2} w_{r/2}^\top \\ +(\operatorname{rev} w_1) v_1^\top + \cdots + (\operatorname{rev} w_{r/2}) v_{r/2}^\top, & \text{if } r \text{ is even,} \\ \\ (1+\lambda) u u^\top + v_1 w_1^\top + \cdots + v_{(r-1)/2} w_{(r-1)/2}^\top \\ +(\operatorname{rev} w_1) v_1^\top + \cdots + (\operatorname{rev} w_{(r-1)/2}) v_{(r-1)/2}^\top, & \text{if } r \text{ is odd,} \end{cases}$$

where

- $u, v_1, \ldots, v_{\lfloor r/2 \rfloor} \in \mathbb{C}^n$,
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Proof. Through the $\top$-pal canonical form. □

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Congruence orbits and their closures

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Irreducible components

Pal_r is an algebraic set in $\mathbb{C}^{n^2}$

Is the set of common zeroes of $p_{k\ell}(x_{ij})$, $1 \leq i,j \leq n$, with

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Q: Which are the irreducible components of Pal_r?

Outline

Geometric description of Pal_r

Theorem

Pal_r is an irreducible algebraic set with dimension

$$\dim \mathrm{Pal}_r = \begin{cases} (3n-r)r/2, & \text{if } r \text{ is even,} \\ (3n-r)(r-1)/2+n, & \text{if } r \text{ is odd.} \end{cases}$$

- If r is **even**, then $\mathrm{Pal}_r = \overline{\mathscr{O}}_c(K_P^e(\lambda))$, with

$$K_P^e(\lambda) := \mathrm{diag}(\overbrace{M^\sharp_{\alpha+1}(\lambda),\dots,M^\sharp_{\alpha+1}(\lambda)}^{s},\overbrace{M^\sharp_{\alpha}(\lambda),\dots,M^\sharp_{\alpha}(\lambda)}^{n-r-s}),$$

where $r/2=(n-r)\alpha+s$ is the Euclidean division of $r/2$ by $n-r$.

- If r is **odd**, then $\mathrm{Pal}_r = \overline{\mathscr{O}}_c(K_P^o(\lambda))$, with

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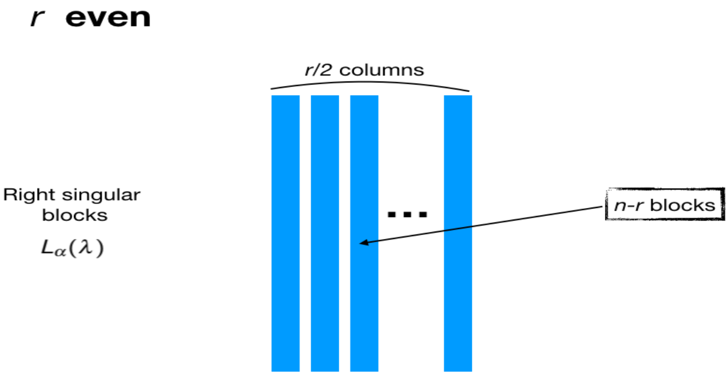

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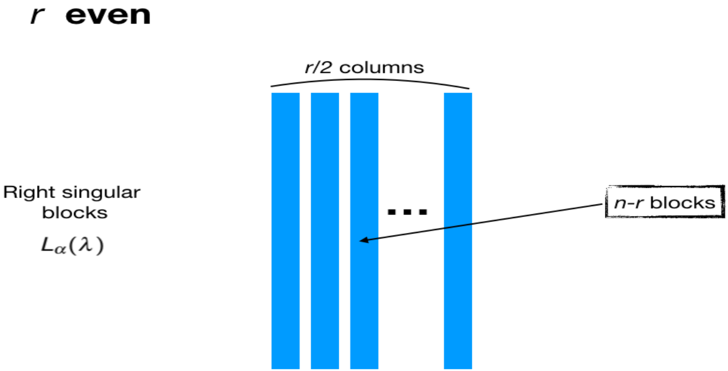

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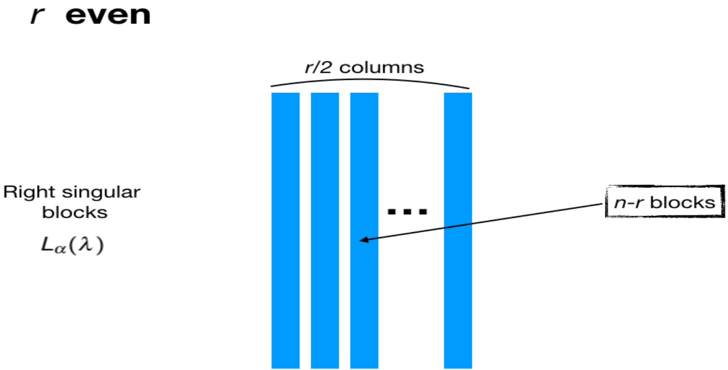

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☞ Divide: $r/2=(n-r)\alpha+s$. Then:

$$\frac{r}{2}=s\cdot(\alpha+1)+(n-r-s)\cdot\alpha$$

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Recall that $A+\lambda A^\top = \begin{cases} v_1 w_1^\top + \cdots + v_{r/2} w_{r/2}^\top \\ +(\operatorname{rev} w_1) v_1^\top + \cdots + (\operatorname{rev} w_{r/2}) v_{r/2}^\top, & \text{if } r \text{ is even,} \\ (1+\lambda) u u^\top + v_1 w_1^\top + \cdots + v_{(r-1)/2} w_{(r-1)/2}^\top \\ +(\operatorname{rev} w_1) v_1^\top + \cdots + (\operatorname{rev} w_{(r-1)/2}) v_{(r-1)/2}^\top, & \text{if } r \text{ is odd.} \end{cases}$

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To prove claim ②:

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- The set of $n \times n$ skew-symmetric pencils with rank at most r has just one generic eigenstructure (i. e., it is irreducible) **[Dmytryshyn-Dopico'18].**
- The set of $n \times n$ symmetric pencils with rank at most r has $\lfloor \frac{r}{2} \rfloor + 1$ generic eigenstructures (i. e., $\lfloor \frac{r}{2} \rfloor + 1$ irreducible components) **[DT.-Dopico-Dmytryshyn, in preparation].**

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