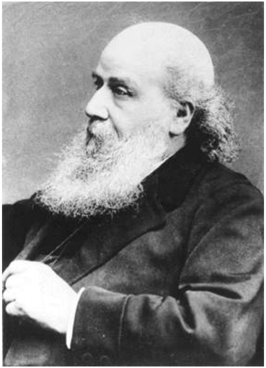
uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

An overview on the solvability of Sylvester-like equations

Fernando De Terán

The Sylvester equation: $AX + XD = 0$



James Joseph Sylvester
(London, 1814–1897)

ANALYSE MATHÉMATIQUE. — *Sur l'équation en matrices* $px = xq$; par M. SYLVESTER.

« Soient p et q deux matrices de l'ordre ω.

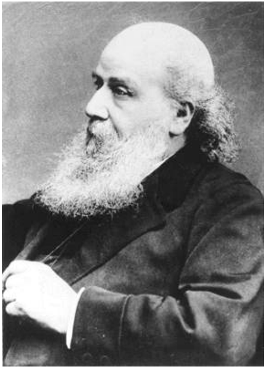
» Pour résoudre l'équation $px = xq$, on obtiendra ω^2 équations homogènes linéaires entre les ω^2 éléments de l'inconnue x et les éléments de p et de q, de sorte que, afin que l'équation donnée soit résoluble, les éléments de p et de q doivent être liés ensemble par une et une seule équation.

» Mais, si l'*équation identique* en p est écrite sous la forme

$$p^\omega + Bp^{\omega-1} + Cp^{\omega-2} + \ldots + L = 0,$$

Comptes Rendus Acad. Sci. 99 (1884)

The Sylvester equation: $AX + XD = 0$



James Joseph Sylvester
(London, 1814–1897)

ANALYSE MATHÉMATIQUE. — *Sur l'équation en matrices* $px = xq$;
par M. SYLVESTER.

« Soient p et q deux matrices de l'ordre ω.

» Pour résoudre l'équation $px = xq$, on obtiendra ω^2 équations homogènes linéaires entre les ω^2 éléments de l'inconnue x et les éléments de p et de q, de sorte que, afin que l'équation donnée soit résoluble, les éléments de p et de q doivent être liés ensemble par une et une seule équation.

» Mais, si l'*équation identique* en p est écrite sous la forme

$$p^\omega + Bp^{\omega-1} + Cp^{\omega-2} + \ldots + L = o,$$

Comptes Rendus Acad. Sci. 99 (1884)

$$X \in \mathbb{C}^{m\times n} \rightsquigarrow A \in \mathbb{C}^{m\times m}, D \in \mathbb{C}^{n\times n}$$

Sylvester-like equations considered in this talk

$AX + XD = E$		(Sylvester equation)
$AXB + CXD = E$		(Generalized Sylvester equation)
$AX + X^\star D = E$	$\star = \top, *$	($\star$-Sylvester equation)
$AXB + CX^\star D = E$	$\star = \top, *$	(Generalized $\star$-Sylvester equation)

Sylvester-like equations considered in this talk

$AX + XD = E$		(Sylvester equation)
$AXB + CXD = E$		(Generalized Sylvester equation)
$AX + X^{\star}D = E$	$\star = \top, *$	($\star$-Sylvester equation)
$AXB + CX^{\star}D = E$	$\star = \top, *$	(Generalized $\star$-Sylvester equation)

We are interested in:

- the solvability, and
- the uniqueness of solution.

Sylvester-like equations considered in this talk

$AX + XD = E$		(Sylvester equation)
$AXB + CXD = E$		(Generalized Sylvester equation)
$AX + X^\star D = E$	$\star = \top, *$	($\star$-Sylvester equation)
$AXB + CX^\star D = E$	$\star = \top, *$	(Generalized $\star$-Sylvester equation)

We are interested in:

- the solvability, and
- the uniqueness of solution.

☞ In all cases, the equation is a linear system (over $\mathbb{C}$ or $\mathbb{R}$).

Sylvester-like equations considered in this talk

$AX + XD = E$		(Sylvester equation)
$AXB + CXD = E$		(Generalized Sylvester equation)
$AX + X^\star D = E$	$\star = \top, *$	($\star$-Sylvester equation)
$AXB + CX^\star D = E$	$\star = \top, *$	(Generalized $\star$-Sylvester equation)

We are interested in:

- the solvability, and
- the uniqueness of solution.

☞ In all cases, the equation is a linear system (over $\mathbb{C}$ or $\mathbb{R}$).

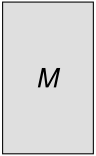
☞ We want to obtain conditions on the coefficient matrices A, B, C, D, E.

Uniqueness and the linear system

$M = B^\top \otimes A + (D^\top \otimes C)\Pi$: the matrix of the linear system (over $\mathbb{C}$).

(Π : an appropriate permutation matrix).

Uniqueness for the **homogeneous** equation $\Leftrightarrow$ **uniqueness** for **any** right-hand side.



Uniqueness for the **homogeneous** equation $\Leftrightarrow$ **at most** one solution for **any** right-hand side.

Transformations

- **Sylvester**:
$AX - XD = E \Leftrightarrow (PAP^{-1})(PXQ^{-1}) - (PXQ^{-1})(QDQ^{-1}) = PEQ^{-1} \Leftrightarrow \widetilde{A}Y - Y\widetilde{D} = \widetilde{E}$.

Transformations

- **Sylvester**:
  $AX - XD = E \Leftrightarrow (PAP^{-1})(PXQ^{-1}) - (PXQ^{-1})(QDQ^{-1}) = PEQ^{-1} \Leftrightarrow \widetilde{A}Y - Y\widetilde{D} = \widetilde{E}$.

  $\boxed{(A, D) \leftrightarrow (PAP^{-1}, QDQ^{-1})}$ (two similarities).

Transformations

- **Sylvester**:
  $AX - XD = E \Leftrightarrow (PAP^{-1})(PXQ^{-1}) - (PXQ^{-1})(QDQ^{-1}) = PEQ^{-1} \Leftrightarrow \widetilde{A}Y - Y\widetilde{D} = \widetilde{E}$.

  $\boxed{(A, D) \leftrightarrow (PAP^{-1}, QDQ^{-1})}$ (two similarities).

- **Generalized Sylvester**:
  $AXB - CXD = E \Leftrightarrow (PAQ)(Q^{-1}XR^{-1})(RBS) - (PCQ)(Q^{-1}XR^{-1})(RDS) = PES \Leftrightarrow \widetilde{A}Y\widetilde{B} - \widetilde{C}Y\widetilde{D} = \widetilde{E}$.

Transformations

- **Sylvester**:
  $AX - XD = E \Leftrightarrow (PAP^{-1})(PXQ^{-1}) - (PXQ^{-1})(QDQ^{-1}) = PEQ^{-1} \Leftrightarrow \widetilde{A}Y - Y\widetilde{D} = \widetilde{E}$.

  $\boxed{(A, D) \leftrightarrow (PAP^{-1}, QDQ^{-1})}$ (two similarities).

- **Generalized Sylvester**:
  $AXB - CXD = E \Leftrightarrow (PAQ)(Q^{-1}XR^{-1})(RBS) - (PCQ)(Q^{-1}XR^{-1})(RDS) = PES \Leftrightarrow \widetilde{A}Y\widetilde{B} - \widetilde{C}Y\widetilde{D} = \widetilde{E}$.

  $\boxed{(A, B, C, D) \leftrightarrow (PAQ, RBS, PCQ, RDS)}$ (two strict equivalences on $A + \lambda C$ and $B + \lambda D$).

Transformations

- **Sylvester**:
  $AX - XD = E \Leftrightarrow (PAP^{-1})(PXQ^{-1}) - (PXQ^{-1})(QDQ^{-1}) = PEQ^{-1} \Leftrightarrow \widetilde{A}Y - Y\widetilde{D} = \widetilde{E}$.

  $\boxed{(A,D) \leftrightarrow (PAP^{-1}, QDQ^{-1})}$ (two similarities).

- **Generalized Sylvester**:
  $AXB - CXD = E \Leftrightarrow (PAQ)(Q^{-1}XR^{-1})(RBS) - (PCQ)(Q^{-1}XR^{-1})(RDS) = PES \Leftrightarrow \widetilde{A}Y\widetilde{B} - \widetilde{C}Y\widetilde{D} = \widetilde{E}$.

  $\boxed{(A,B,C,D) \leftrightarrow (PAQ, RBS, PCQ, RDS)}$ (two strict equivalences on $A + \lambda C$ and $B + \lambda D$).

- $\star$**-Sylvester**:
  $AX + X^\star D = E \Leftrightarrow (PAQ)(Q^{-1}XP^\star) + (Q^{-1}XP^\star)^\star(Q^\star DP^\star) = PEP^\star \Leftrightarrow \widetilde{A}Y + Y^\star\widetilde{D} = \widetilde{E}$.

Transformations

- **Sylvester**:
  $AX - XD = E \Leftrightarrow (PAP^{-1})(PXQ^{-1}) - (PXQ^{-1})(QDQ^{-1}) = PEQ^{-1} \Leftrightarrow \widetilde{A}Y - Y\widetilde{D} = \widetilde{E}$.

  $\boxed{(A, D) \leftrightarrow (PAP^{-1}, QDQ^{-1})}$ (two similarities).

- **Generalized Sylvester**:
  $AXB - CXD = E \Leftrightarrow (PAQ)(Q^{-1}XR^{-1})(RBS) - (PCQ)(Q^{-1}XR^{-1})(RDS) = PES \Leftrightarrow \widetilde{A}Y\widetilde{B} - \widetilde{C}Y\widetilde{D} = \widetilde{E}$.

  $\boxed{(A, B, C, D) \leftrightarrow (PAQ, RBS, PCQ, RDS)}$ (two strict equivalences on $A + \lambda C$ and $B + \lambda D$).

- **$\star$-Sylvester**:
  $AX + X^\star D = E \Leftrightarrow (PAQ)(Q^{-1}XP^\star) + (Q^{-1}XP^\star)^\star(Q^\star DP^\star) = PEP^\star \Leftrightarrow \widetilde{A}Y + Y^\star\widetilde{D} = \widetilde{E}$.

  $\boxed{(A, D^\star) \leftrightarrow (PAQ, PD^\star Q)}$ (strict equivalence on $A + \lambda D^\star$).

Transformations

- **Sylvester**:
  $AX - XD = E \Leftrightarrow (PAP^{-1})(PXQ^{-1}) - (PXQ^{-1})(QDQ^{-1}) = PEQ^{-1} \Leftrightarrow \widetilde{A}Y - Y\widetilde{D} = \widetilde{E}$.

  $\boxed{(A,D) \leftrightarrow (PAP^{-1}, QDQ^{-1})}$ (two similarities).

- **Generalized Sylvester**:
  $AXB - CXD = E \Leftrightarrow (PAQ)(Q^{-1}XR^{-1})(RBS) - (PCQ)(Q^{-1}XR^{-1})(RDS) = PES \Leftrightarrow \widetilde{A}Y\widetilde{B} - \widetilde{C}Y\widetilde{D} = \widetilde{E}$.

  $\boxed{(A,B,C,D) \leftrightarrow (PAQ, RBS, PCQ, RDS)}$ (two strict equivalences on $A + \lambda C$ and $B + \lambda D$).

- **$\star$-Sylvester**:
  $AX + X^\star D = E \Leftrightarrow (PAQ)(Q^{-1}XP^\star) + (Q^{-1}XP^\star)^\star(Q^\star DP^\star) = PEP^\star \Leftrightarrow \widetilde{A}Y + Y^\star\widetilde{D} = \widetilde{E}$.

  $\boxed{(A,D^\star) \leftrightarrow (PAQ, PD^\star Q)}$ (strict equivalence on $A + \lambda D^\star$).

- **Generalized $\star$-Sylvester**: ???

The Sylvester equation

☞ Solvability:

Theorem [Roth'52]

$AX - XD = E$ is consistent iff

$$\begin{bmatrix} A & E \\ 0 & D \end{bmatrix} = P \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} P^{-1} \qquad \text{(Roth's criterion)}$$

for some invertible P.

The Sylvester equation

☞ Solvability:

Theorem [Roth'52]

$AX - XD = E$ is consistent iff

$$\begin{bmatrix} A & E \\ 0 & D \end{bmatrix} = P \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} P^{-1} \qquad \text{(Roth's criterion)}$$

for some invertible P.

(If $AX - XD = E$ then $P = \left[\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right]$).

The Sylvester equation

☞ Solvability:

Theorem [Roth'52]

$AX - XD = E$ is consistent iff

$$\begin{bmatrix} A & E \\ 0 & D \end{bmatrix} = P \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} P^{-1} \qquad \text{(Roth's criterion)}$$

for some invertible P.

(If $AX - XD = E$ then $P = \left[\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right]$).

☞ Uniqueness:

Theorem [Gantmacher'59]

$AX - XD = E$ has a unique solution, for every right-hand side E iff A and D have disjoint spectra.

The Sylvester equation

☞ Solvability:

Theorem [Roth'52]

$AX - XD = E$ is consistent iff

$$\begin{bmatrix} A & E \\ 0 & D \end{bmatrix} = P \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} P^{-1} \qquad \text{(Roth's criterion)}$$

for some invertible P.

(If $AX - XD = E$ then $P = \left[\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right]$).

☞ Uniqueness:

Theorem [Gantmacher'59]

$AX - XD = E$ has a unique solution, for every right-hand side E iff A and D have disjoint spectra.

☞ The coefficient matrix of the associated linear system is square ($mn \times mn$).

The generalized Sylvester equation: consistency

Theorem [Dmytryshyn-Kågström'15]

$AXB - CXD = E$ is consistent iff

$$P_1^{-1}\begin{bmatrix} A & E \\ 0 & D \end{bmatrix} P_2 = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}, \quad P_2^{-1}\begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} P_3 = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}, \quad P_3^{-1}\begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix} P_1 = \begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix},$$

for some P_1, P_2, P_3 invertible.

uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

The generalized Sylvester equation: consistency

Theorem [Dmytryshyn-Kågström'15]

$AXB - CXD = E$ is consistent iff

$$P_1^{-1}\begin{bmatrix} A & E \\ 0 & D \end{bmatrix} P_2 = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}, \quad P_2^{-1}\begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} P_3 = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}, \quad P_3^{-1}\begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix} P_1 = \begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix},$$

for some P_1, P_2, P_3 invertible.

Comes from a more general result on systems of Sylvester equations:

$$AXB - CXD = E \Leftrightarrow \left\{ \begin{array}{ccc} AZ - YD & = & E, \\ CX - Y & = & 0, \\ Z - XD & = & 0. \end{array} \right.$$

The generalized Sylvester equation: consistency

Theorem [Dmytryshyn-Kågström'15]

$AXB - CXD = E$ is consistent iff

$$P_1^{-1}\begin{bmatrix} A & E \\ 0 & D \end{bmatrix} P_2 = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}, \quad P_2^{-1}\begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} P_3 = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}, \quad P_3^{-1}\begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix} P_1 = \begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix},$$

for some P_1, P_2, P_3 invertible.

Comes from a more general result on systems of Sylvester equations:

$$AXB - CXD = E \Leftrightarrow \left\{ \begin{array}{rcl} AZ - YD & = & E, \\ CX - Y & = & 0, \\ Z - XD & = & 0. \end{array} \right.$$

☞ Another characterization in terms of the Kronecker canonical form of $A + \lambda C$ and $B + \lambda D$ in:

T. Kosir.
The matrix equation $AXD^T - BXC^T = E$.
U. Calgary, Dept. Math. & Stat., Res. paper no. 737 (1992)

The generalized Sylvester equation: uniqueness

Theorem [Chu'87]

$AXB - CXD = E$ has a unique solution (for any E) iff:
$A + \lambda C$ and $D + \lambda B$ are regular and have disjoint spectra.

uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

The generalized Sylvester equation: uniqueness

☞ $A, C \in \mathbb{R}^{m\times m}, B, D \in \mathbb{R}^{n\times n}$.

Theorem [Chu'87]

$AXB - CXD = E$ has a unique solution (for any E) iff:
$A + \lambda C$ and $D + \lambda B$ are regular and have disjoint spectra.

The generalized Sylvester equation: uniqueness

☞ $A, C \in \mathbb{R}^{m \times m}, B, D \in \mathbb{R}^{n \times n}$. Not the most general situation: $A, C \in \mathbb{R}^{p \times m}, B, D \in \mathbb{R}^{n \times q}$

Theorem [Chu'87]

$AXB - CXD = E$ has a unique solution (for any E) iff: $A + \lambda C$ and $D + \lambda B$ are regular and have disjoint spectra.

uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

The generalized Sylvester equation: uniqueness

☞ $A, C \in \mathbb{R}^{m\times m}, B, D \in \mathbb{R}^{n\times n}$. Not the most general situation:
$A, C \in \mathbb{R}^{p\times m}, B, D \in \mathbb{R}^{n\times q}$

Theorem [Chu'87]

$AXB - CXD = E$ has a unique solution (for any E) iff:
$A + \lambda C$ and $D + \lambda B$ are regular and have disjoint spectra.

☞ The coefficient matrix of the linear system is $pq \times mn$.

The generalized Sylvester equation: uniqueness

☞ $A, C \in \mathbb{R}^{m\times m}, B, D \in \mathbb{R}^{n\times n}$. Not the most general situation:
$A, C \in \mathbb{R}^{p\times m}, B, D \in \mathbb{R}^{n\times q}$

Theorem [Chu'87]

$AXB - CXD = E$ has a unique solution (for any E) iff:
$A + \lambda C$ and $D + \lambda B$ are regular and have disjoint spectra.

☞ The coefficient matrix of the linear system is $pq \times mn$.

Open problem: Characterize:

- the **uniqueness** of solution when $pq = mn$ and, more in general,
- the existence of **at most** one solution.

The generalized Sylvester equation: uniqueness

☞ $A, C \in \mathbb{R}^{m\times m}, B, D \in \mathbb{R}^{n\times n}$. Not the most general situation:
$A, C \in \mathbb{R}^{p\times m}, B, D \in \mathbb{R}^{n\times q}$

Theorem [Chu'87]

$AXB - CXD = E$ has a unique solution (for any E) iff:
$A + \lambda C$ and $D + \lambda B$ are regular and have disjoint spectra.

☞ The coefficient matrix of the linear system is $pq \times mn$.

~~**Open problem**~~: Characterize:

- the **uniqueness** of solution when $pq = mn$ and, more in general,
- the existence of **at most** one solution.

Answer: Analyze the solution of the homogeneous equation given in [Kosir'92].

DT-Iannazzo-Poloni-Robol.
Solvability and uniqueness criteria for generalized Sylvester-type equations.
LAA 542 (2018) 501–521.

The $\star$-Sylvester equation: consistency

$\mathbb{F}$ a field with $\mathrm{char}\,\mathbb{F} \neq 2$, $A \in \mathbb{F}^{n\times m}, B \in \mathbb{F}^{m\times n}, C \in \mathbb{F}^{m\times m}$

Theorem [Wimmer'94], [DT-Dopico'11]

$AX + X^\star D = E$ is consistent iff

$$P^\star \begin{bmatrix} E & A \\ D & 0 \end{bmatrix} P = \begin{bmatrix} 0 & A \\ D & 0 \end{bmatrix},$$

for some nonsingular P.

The $\star$-Sylvester equation: consistency

$\mathbb{F}$ a field with $\operatorname{char}\mathbb{F} \neq 2$, $A \in \mathbb{F}^{n\times m}, B \in \mathbb{F}^{m\times n}, C \in \mathbb{F}^{m\times m}$

Theorem [Wimmer'94], [DT-Dopico'11]

$AX + X^\star D = E$ is consistent iff

$$P^\star \begin{bmatrix} E & A \\ D & 0 \end{bmatrix} P = \begin{bmatrix} 0 & A \\ D & 0 \end{bmatrix},$$

for some nonsingular P.

$X \in \mathbb{F}^{m\times n}$

uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

The $\star$-Sylvester equation: consistency

$\mathbb{F}$ a field with $\operatorname{char}\mathbb{F} \neq 2$, $A \in \mathbb{F}^{n\times m}, B \in \mathbb{F}^{m\times n}, C \in \mathbb{F}^{m\times m}$

Theorem [Wimmer'94], [DT-Dopico'11]

$AX + X^\star D = E$ is consistent iff

$$P^\star \begin{bmatrix} E & A \\ D & 0 \end{bmatrix} P = \begin{bmatrix} 0 & A \\ D & 0 \end{bmatrix},$$

for some nonsingular P.

$X \in \mathbb{F}^{m\times n}$

☞ **Size**: Most general setting.

The $\star$-Sylvester equation: consistency

$\mathbb{F}$ a field with $\mathrm{char}\,\mathbb{F} \neq 2$, $A \in \mathbb{F}^{n\times m}, B \in \mathbb{F}^{m\times n}, C \in \mathbb{F}^{m\times m}$

Theorem [Wimmer'94], [DT-Dopico'11]

$AX + X^\star D = E$ is consistent iff

$$P^\star \begin{bmatrix} E & A \\ D & 0 \end{bmatrix} P = \begin{bmatrix} 0 & A \\ D & 0 \end{bmatrix},$$

for some nonsingular P.

$X \in \mathbb{F}^{m\times n}$

☞ **Size**: Most general setting.

Open problem: What happens for $\mathrm{char}\,\mathbb{F} = 2$?

uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

The $\star$-Sylvester equation: uniqueness

$A, D \in \mathbb{C}^{n\times m}$

Theorem [DT-Iannazzo'24]

$AX + X^\star D = 0$ has a unique solution iff $A + \lambda D^\star$ has full column rank, and

- $\star = \top$: $\Lambda(A+\lambda D^\star) \setminus \{-1\}$ is reciprocal free and $m_a(-1, A+\lambda D^\star) \leq 1$.
- $\star = *$: $\Lambda(A+\lambda D^\star)$ is $*$-reciprocal free.

Definition: $\mathscr{S} \in \mathbb{C} \cup \{\infty\}$ is:

(a) reciprocal free if $\lambda\mu \neq 1$, for any $\lambda, \mu \in \mathscr{S}$.

(b) $*$-reciprocal free if $\lambda\overline{\mu} \neq 1$, for any $\lambda, \mu \in \mathscr{S}$.

$m_a(\mu, A+\lambda D^\star)$: algebraic multiplicity of μ in $A+\lambda D^\star$.

The $\star$-Sylvester equation: uniqueness

$A, D \in \mathbb{C}^{n\times m}$

Theorem [DT-Iannazzo'24]

$AX + X^\star D = 0$ has a unique solution iff $A + \lambda D^\star$ has full column rank, and

- $\star = \top$: $\Lambda(A+\lambda D^\star) \setminus \{-1\}$ is reciprocal free and $m_a(-1, A+\lambda D^\star) \leq 1$.
- $\star = *$: $\Lambda(A+\lambda D^\star)$ is $*$-reciprocal free.

Definition: $\mathscr{S} \in \mathbb{C} \cup \{\infty\}$ is:

(a) reciprocal free if $\lambda\mu \neq 1$, for any $\lambda, \mu \in \mathscr{S}$.

(b) $*$-reciprocal free if $\lambda\overline{\mu} \neq 1$, for any $\lambda, \mu \in \mathscr{S}$.

$m_a(\mu, A+\lambda D^\star)$: algebraic multiplicity of μ in $A+\lambda D^\star$.

☞ Generalizes the characterization in [Byers-Kressner'06], [Kressner-Schröder-Watkins'09] for $m = n$ ($A+\lambda D^\star$ must be regular).

Generalized $\star$-Sylvester: $n \times n$ coefficients (uniqueness)

The “magic" pencil: $\mathscr{Q}(\lambda) := \begin{bmatrix} \lambda D^\star & B^\star \\ A & \lambda C \end{bmatrix}$.

Generalized $\star$-Sylvester: $n \times n$ coefficients (uniqueness)

The "magic" pencil: $\mathscr{Q}(\lambda) := \begin{bmatrix} \lambda D^\star & B^\star \\ A & \lambda C \end{bmatrix}$.

Theorem [DT-Iannazzo'16]

The equation $AXB + CX^\star D = E$, with $A, B, C, D \in \mathbb{C}^{n \times n}$ has a **unique solution**, for every E, if and only if:

(i) $\mathscr{Q}(\lambda)$ is **regular**;

(ii-a) if $\star = \top$, $\Lambda(\mathscr{Q}) \setminus \{\pm 1\}$ is **reciprocal free** and $m_a(\pm 1, \mathscr{Q}) \leq 1$;

(ii-b) if $\star = *$, $\Lambda(\mathscr{Q})$ is $*$**-reciprocal free.**

FDT, B. Iannazzo.
Uniqueness of solution of a generalized $\star$-Sylvester matrix equation.
LAA, 493 (2016) 323-335.

Generalized $\star$-Sylvester: $n \times n$ coefficients (uniqueness)

The "magic" pencil: $\mathscr{Q}(\lambda) := \begin{bmatrix} \lambda D^\star & B^\star \\ A & \lambda C \end{bmatrix}$.

Theorem [DT-Iannazzo'16]

The equation $AXB + CX^\star D = E$, with $A, B, C, D \in \mathbb{C}^{n \times n}$ has a **unique solution**, for every E, if and only if:

(i) $\mathscr{Q}(\lambda)$ is **regular**;

(ii-a) if $\star = \top$, $\Lambda(\mathscr{Q}) \setminus \{\pm 1\}$ is **reciprocal free** and $m_a(\pm 1, \mathscr{Q}) \leq 1$;

(ii-b) if $\star = *$, $\Lambda(\mathscr{Q})$ is $*$**-reciprocal free**.

FDT, B. Iannazzo.
Uniqueness of solution of a generalized $\star$-Sylvester matrix equation.
LAA, 493 (2016) 323-335.

☞ The matrix of the linear system **is square**.

Generalized $\star$-Sylvester: coefficients with any size

$$\mathscr{Q}(\lambda) := \begin{bmatrix} \lambda D^\star & B^\star \\ A & \lambda C \end{bmatrix}.$$

Theorem [DT-Iannazzo'24, in preparation]

The equation $AXB + CX^\star D = 0$, with $A \in \mathbb{C}^{p\times m}, B \in \mathbb{C}^{n\times q}, C \in \mathbb{C}^{p\times n}, D \in \mathbb{C}^{m\times q}$ has **only the trivial solution** if and only if:

(a) $\mathscr{Q}(\lambda)$ has **full column rank**.

(b-$\top$) If $\star = \top$, the set $\Lambda(\mathscr{Q}) \setminus \{0, \infty, \pm 1\}$ is **reciprocal free** and $m_a(1, \mathscr{Q}) = m_a(-1, \mathscr{Q}) \leq 1$.

(b-$*$) If $\star = *$, the set $\Lambda(\mathscr{Q}) \setminus \{0, \infty\}$ is $*$**-reciprocal free**.

(c) If $0, \infty$ are e-vals of $\mathscr{Q}$, at least one of them is **semisimple**.

(d) At least one of A, C has **full column rank**.

(e) At least one of $B^\star, D^\star$ has **full column rank**.

uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

Generalized $\star$-Sylvester: coefficients with any size

$$\mathscr{Q}(\lambda) := \begin{bmatrix} \lambda D^\star & B^\star \\ A & \lambda C \end{bmatrix}.$$

Theorem [DT-Iannazzo'24, in preparation]

The equation $AXB + CX^\star D = 0$, with $A \in \mathbb{C}^{p\times m}, B \in \mathbb{C}^{n\times q}, C \in \mathbb{C}^{p\times n}, D \in \mathbb{C}^{m\times q}$ has **only the trivial solution** if and only if:

(a) $\mathscr{Q}(\lambda)$ has **full column rank.**

(b-$\top$) If $\star = \top$, the set $\Lambda(\mathscr{Q}) \setminus \{0, \infty, \pm 1\}$ is **reciprocal free** and $m_a(1, \mathscr{Q}) = m_a(-1, \mathscr{Q}) \leq 1$.

(b-$*$) If $\star = *$, the set $\Lambda(\mathscr{Q}) \setminus \{0, \infty\}$ is $*$**-reciprocal free.**

(c) If $0, \infty$ are e-vals of $\mathscr{Q}$, at least one of them is **semisimple.**

(d) At least one of A, C has **full column rank.**

(e) At least one of $B^\star, D^\star$ has **full column rank.**

☞ Equivalently: $AXB + CX^\star D = E$ has **at most one solution** $\forall E$.

Generalized $\star$-Sylvester: coefficients with any size

$$\mathscr{Q}(\lambda) := \begin{bmatrix} \lambda D^\star & B^\star \\ A & \lambda C \end{bmatrix}.$$

Theorem [DT-Iannazzo'24, in preparation]

The equation $AXB + CX^\star D = 0$, with $A \in \mathbb{C}^{p\times m}, B \in \mathbb{C}^{n\times q}, C \in \mathbb{C}^{p\times n}, D \in \mathbb{C}^{m\times q}$ has **only the trivial solution** if and only if:

(a) $\mathscr{Q}(\lambda)$ has **full column rank.**

(b-$\top$) If $\star = \top$, the set $\Lambda(\mathscr{Q}) \setminus \{0, \infty, \pm 1\}$ is **reciprocal free** and $m_a(1, \mathscr{Q}) = m_a(-1, \mathscr{Q}) \leq 1$.

(b-$*$) If $\star = *$, the set $\Lambda(\mathscr{Q}) \setminus \{0, \infty\}$ is $*$**-reciprocal free.**

(c) If $0, \infty$ are e-vals of $\mathscr{Q}$, at least one of them is **semisimple.**

(d) At least one of A, C has **full column rank.**

(e) At least one of $B^\star, D^\star$ has **full column rank.**

☞ Equivalently: $AXB + CX^\star D = E$ has **at most one solution** $\forall E$.

☞ For **at least one solution**, replace $\mathscr{Q}(\lambda)$ by $\mathscr{Q}^\sharp(\lambda) = \begin{bmatrix} \lambda C^\star & B \\ A^\star & \lambda D \end{bmatrix}$.

Transformation on $AXB + CX^{\star}D = E$

$AXB + CX^{\star}D = E$ is equivalent to:

$$(R_2AS_1)Y(S_2^{\star}BR_1^{\star}) + (R_2CS_2)Y^{\star}(S_1^{\star}DR_1^{\star}) = R_2ER_1^{\star},$$

where $Y = S_1^{-1}XS_2^{-\star}$.

Transformation on $AXB + CX^\star D = E$

$AXB + CX^\star D = E$ is equivalent to:

$$(R_2 A S_1) Y (S_2^\star B R_1^\star) + (R_2 C S_2) Y^\star (S_1^\star D R_1^\star) = R_2 E R_1^\star,$$

where $Y = S_1^{-1} X S_2^{-\star}$.

☞ Corresponds to a block diagonal strict equivalence of $\mathscr{Q}(\lambda)$:

$$\begin{bmatrix} \lambda (S_1^\star D R_1^\star)^\star & (S_2^\star B R_1^\star)^\star \\ R_2 A S_1 & \lambda (R_2 C S_2) \end{bmatrix} = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \begin{bmatrix} \lambda D^\star & B^\star \\ A & \lambda C \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}.$$

A summary of known results

S: Solvable, for given A, B, C, D, E.
US: Unique solution, for given A, B, C, D, E.
SR: Solvable, for any E.
OR: At most one solution, for any E (i.e.: the homogeneous eq. has only the trivial solution).
UR: Unique solution, for any E.

	$AXB - CXD = E$ square coefficients	$AXB - CXD = E$ general coefficients	$AXB + CX^\star D = E$ square coefficients	$AXB + CX^\star D = E$ general coefficients
S	[DK16]	[K92],[DK16]	[DK16]	[DK16]
US	[Chu87]	[K92]	[DI16]	open
SR	same as **US**	[DIPR18] (using [K92])	same as **US**	[DI24]
OR	same as **US**	[K96]	same as **US**	[DI24]
UR	same as **US**	[DIPR18] (using [K92])	same as **US**	[DIPR18]

[DI16]: DT-Iannazzo'16; [DI24]: DT-Iannazzo'24; [DIPR18]: DT-Iannazzo-Poloni-Robol'18; [DK16]: Dmytryshyn-Kågström'16; [K92]: Kosir'92; [K96]: Kosir'96.

A summary of known results

S: Solvable, for given A, B, C, D, E.
US: Unique solution, for given A, B, C, D, E.
SR: Solvable, for any E.
OR: At most one solution, for any E (i.e.: the homogeneous eq. has only the trivial solution).
UR: Unique solution, for any E.

	$AXB - CXD = E$ square coefficients	$AXB - CXD = E$ general coefficients	$AXB + CX^{\star}D = E$ square coefficients	$AXB + CX^{\star}D = E$ general coefficients
S	[DK16]	[K92],[DK16]	[DK16]	[DK16]
US	[Chu87]	[K92]	[DI16]	open
SR	same as **US**	[DIPR18] (using [K92])	same as **US**	[DI24]
OR	same as **US**	[K96]	same as **US**	[DI24]
UR	same as **US**	[DIPR18] (using [K92])	same as **US**	[DIPR18]

[DI16]: DT-Iannazzo'16; [DI24]: DT-Iannazzo'24; [DIPR18]: DT-Iannazzo-Poloni-Robol'18; [DK16]: Dmytryshyn-Kågström'16; [K92]: Kosir'92; [K96]: Kosir'96.

Explicit expression for the solution

(Up to the change matrices leading to the canonical form).

Equation	Reference
$AX + XD = 0$	[Gantmacher'59]
$AXB + CXD = 0$	[Kosir'92]
$AX + X^{\star}D = 0$	[DT-Dopico-Guillery-Montealegre-Reyes'13]
$AXB + CX^{\star}D = 0$	open


Thank you!
Gracias!