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Departamento de Matemáticas

# A class of quasi-sparse companion pencils of matrix polynomials over arbitrary fields

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ILAS2019 (July 8, 2019)

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# Companion matrix vs companion pencil

$q(z) = a_0 + za_1 + \cdots + z^{n-1}a_{n-1} + z^n$ $\quad (a_0, \ldots, a_{n-1} \in \mathbb{F})$
a (scalar) monic polynomial. $\mathbb{F}$ an **arbitrary field.**

## Companion matrix

$A(a_0, \ldots, a_{n-1}) \in \mathbb{F}[a_0, \ldots, a_{n-1}]^{n\times n}$ such that

$$p_A(z) = \det(zI - A) = a_0 + za_1 + \cdots + z^{n-1}a_{n-1} + z^n = q(z)$$

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For a general (non-necessarily monic) polynomial:

$$p(z) = a_0 + za_1 + \cdots + z^{n-1}a_{n-1} + z^n a_n \qquad (a_0, \ldots, a_n \in \mathbb{F})$$

## Companion pencil

$A + zB$, with $A, B \in \mathbb{F}[a_0, \ldots, a_{n-1}, a_n]^{n \times n}$ such that

$$\det(A + zB) = \alpha(a_0 + za_1 + \cdots + z^{n-1}a_{n-1} + z^n a_n) = \alpha p(z)$$

$(\alpha \in \mathbb{F})$

# Example

Frobenius companion matrix

$$C_1 = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_0 \\ 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & 1 & 0 \end{bmatrix}, \qquad C_2 = C_1^\top$$

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MATLAB's command `roots`: QR algorithm on $C_2$

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Frobenius companion matrix

$$zI - C_1 = \begin{bmatrix} a_{n-1} + z & a_{n-2} & \cdots & a_0 \\ -1 & z & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & -1 & z \end{bmatrix}, \qquad zI - C_2 = zI - C_1^\top$$

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# Basic transformations

From now on: $z \rightsquigarrow \lambda$

$$\left.\begin{array}{c} A+\lambda B \text{ a companion pencil for } p(\lambda) \\ P, Q \in \mathbb{F}^{n\times n} \text{ invertible} \end{array}\right\} \Rightarrow \begin{array}{c} P(A+\lambda B)Q \\ \text{is a companion pencil for } p(\lambda) \end{array}$$

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E.g.: $P,Q$ are permutation matrices.

$P(A+\lambda B)Q$: strict equivalence of $A+\lambda B$.

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(long time ago)

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☞ Valid for matrix polynomials.

# A block-Kronecker pencil (example)

$(p = a_0 + a_1\lambda + \cdots + a_5\lambda^5)$

$$A + \lambda B = \left[\begin{array}{ccc|cc} a_4 + \lambda a_5 & 0 & a_2 & -1 & 0 \\ 0 & 0 & a_1 & \lambda & -1 \\ \lambda a_3 & 0 & a_0 & 0 & \lambda \\ \hline -1 & \lambda & 0 & 0 & 0 \\ 0 & -1 & \lambda & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} M_0 + \lambda M_1 & L_2(\lambda) \\ \hline L_2(\lambda)^\top & 0 \end{array}\right]$$

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$$L_k(\lambda) = \begin{bmatrix} -1 & & & \\ \lambda & -1 & & \\ & \ddots & \ddots & \\ & & \lambda & -1 \\ & & & \lambda \end{bmatrix}_{(k+1)\times k}$$

(A “right (Kronecker) singular block" of size $(k+1) \times k$)

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$n-1$ entries equal to $\lambda$

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number of nonzero entries: $3n-2=13$:

- $n-1$ entries equal to $-1$
- $n-1$ entries equal to $\lambda$
- $n$ entries of the form $b_i + \lambda b_{i+1} = a_i, \lambda a_{i+1}, a_i + \lambda a_{i+1}$

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Number of nonzero entries: $2n-1+\lfloor \frac{n}{2} \rfloor = 11$

# A block-Kronecker pencil (example)

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Allows for more flexibility:

$$A+\lambda B = \left[\begin{array}{cc|ccc} \lambda a_5 & 0 & -1 & 0 & 0 \\ \lambda a_4 & a_2 & \lambda & -1 & 0 \\ \lambda a_3 & a_1 & 0 & \lambda & -1 \\ 0 & a_0 & 0 & 0 & \lambda \\ \hline -1 & \lambda & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} M_0+\lambda M_1 & L_3(\lambda) \\ \hline L_1(\lambda)^\top & 0 \end{array}\right]$$

# Common features of all these families

(Up to permutations)

- They all have $(n-1)$ entries equal to $\pm 1$, together with other $(n-1)$ entries equal to $\pm\lambda$.

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$$b_j + \lambda b_{j+1} \quad = \quad \begin{cases} 0, \\ a_i, \\ \lambda a_{i+1}, \\ a_i + \lambda a_{i+1}. \end{cases}$$

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- The smallest number of nonzero entries is $2n - 1 + \lfloor \frac{n}{2} \rfloor$.

# The work of Eastman et al.

[Eastman-Kim-Shader-Vander Meulen, LAA'14]

$A(a_0, \ldots, a_{n-1}) \in \mathbb{F}[a_0, \ldots, a_{n-1}]^{n \times n}$ is a companion matrix for $a_0 + a_1\lambda + \ldots + a_{n-1}\lambda^{n-1} + \lambda^n$ (**monic**) if it has

- $n-1$ nonzero entries in $\mathbb{F}$, and
- $n$ entries equal to $a_0, \ldots, a_{n-1}$.

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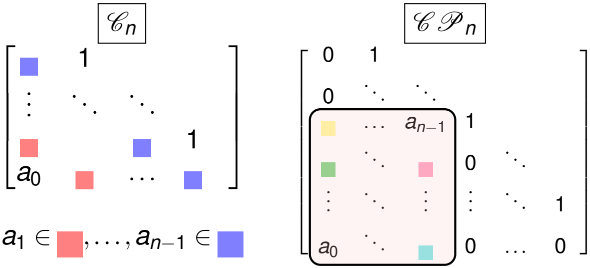
**Q**: Is there any “canonical form" under permutational similarity ($PAP^\top$, $P$ a permutation)?

# Canonical form under permutational similarity

We define the following (lower Hessenberg) classes of matrices:



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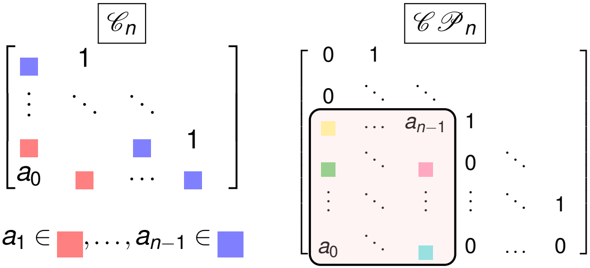


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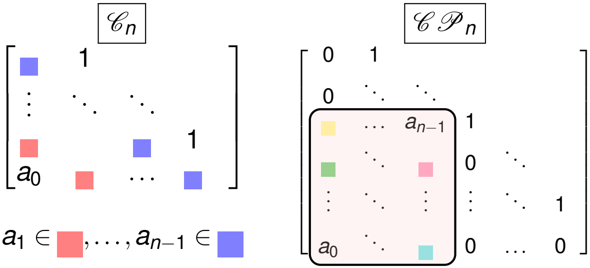


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## Theorem [Eastman-etal, LAA'14]

Any sparse companion matrix is permutationally similar to a matrix in $\mathscr{C}_n$.

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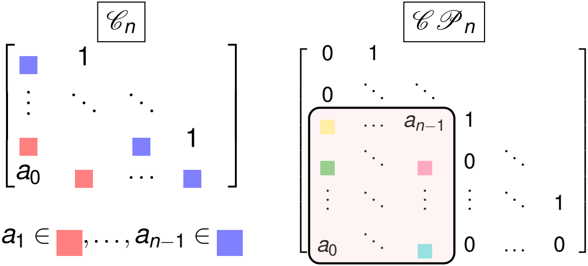


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## Theorem [Eastman-etal, LAA'14]

$A(a_0, \ldots, a_{n-1}) \in \mathscr{C}_n$ is a sparse companion matrix
$\Leftrightarrow A(a_0, \ldots, a_{n-1}) \in \mathscr{C}\mathscr{P}_n$.

# The family $\mathscr{R}_n$

## Definition ($\mathscr{R}_n$)

$A + \lambda B$ belongs to $\mathscr{R}_n$ if:

- $A, B \in \mathbb{F}[a_0, \ldots, a_n]^{n \times n}$.
- $A$ has $n - 1$ entries equal to $-1$.
- $B$ has $n - 1$ entries equal to $1$.
- (Quasi-sparsity) There are, at most, another $n$ nonzero entries, $x_0, \ldots, x_{k-1}$, which are of the form $0, a_i, \lambda a_{i+1}$, or $a_i + \lambda a_{i+1}$.

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**Q2**: Canonical form for those which are companion (up to **permutation**, as in [Eastman-etal, LAA'14])?

# The families $\mathscr{C}_n$ and $\mathscr{QC}_n$

## Definition ($\mathscr{C}_n$)

$A + \lambda B \in \mathscr{C}_n$ if:

(i) $(A + \lambda B)_{i,i+1} = -1$

$$\begin{bmatrix} * & -1 & & & \\ * & \ddots & \ddots & & \\ \vdots & \ddots & * & -1 & \\ * & \dots & * & * & \end{bmatrix}$$

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$A+\lambda B \in \mathscr{C}_n$ if:

(i) $(A+\lambda B)_{i,i+1} = -1$
(ii) $(A+\lambda B)_{ii} = \lambda$ or $x_{n-1}$ $(\neq 0)$.

$$\begin{bmatrix} \blacksquare & -1 & & & \\ * & \ddots & \ddots & & \\ \vdots & \ddots & \blacksquare & -1 \\ * & \cdots & * & \blacksquare \end{bmatrix}$$

$\blacksquare = \lambda, x_{n-1}$

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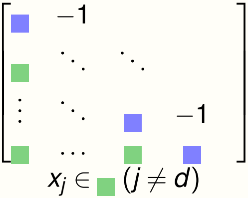

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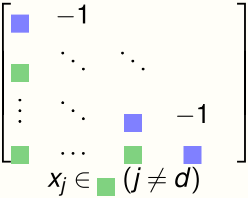



## Definition ($\mathscr{QC}_n$)

$A+\lambda B \in \mathscr{QC}_n$ if:

(i) $A+\lambda B \in \mathscr{C}_n$.

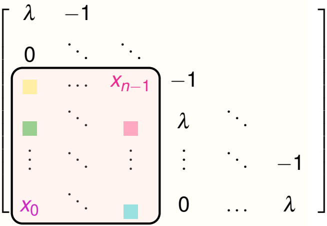
(ii) $x_0 := (A+\lambda B)_{n1} = a_0$ or $a_0 + \lambda a_1$.

(iii) $a_i \in S_{n-i-1}(A)$ or $a_i \in S_{n-i}(B)$

($\Rightarrow x_{n-1} = \lambda a_n$ or $a_{n-1} + \lambda a_n$).

(iv) Rectangle's condition.

($S_j(M) = j$th subdiagonal of $M$)



# $\mathscr{QC}$: Example

$$p(\lambda) = a_0 + \lambda a_1 + \lambda^2 a_2 + \lambda^3 a_3$$

$$\begin{bmatrix} * & -1 & 0 \\ * & * & -1 \\ a_0 & * & * \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} * & -1 & 0 \\ * & * & -1 \\ a_0 + \lambda a_1 & * & * \end{bmatrix}$$

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$\checkmark$ $\checkmark$

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✓ ✓

☞ Smallest number of nonzero entries: $2n - 1 + \lfloor \frac{n}{2} \rfloor = 6$

# Companion pencils in $\mathscr{R}_n$: canonical form

| Matrices ($A$) | Pencils ($A+\lambda B$) |
|---|---|
| $\det(\lambda I - A)$ | $\det(A+\lambda B)$ |
| Similarity: | Equivalence: |
| $PAP^{-1}$ | $P(A+\lambda B)Q$ |

($P, Q \in \mathbb{F}^{n\times n}$ invertible)

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**Theorem** [DT-Hernando, Springer'18]

(i) Any companion pencil in $\mathscr{R}_n$ is permutationally equivalent to a pencil in $\mathscr{Q}\mathscr{C}_n$.

(ii) The only pencils in $\mathscr{C}_n$ that are companion are those in $\mathscr{Q}\mathscr{C}_n$.

(iii) All pencils in $\mathscr{Q}\mathscr{C}_n$ are companion.

# Connection with block-Kronecker pencils

Any pencil in $\mathscr{Q}\mathscr{C}_n$ is permutationally equivalent to a block-Kronecker pencil:

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Any pencil in $\mathcal{QC}_n$ is permutationally equivalent to a block-Kronecker pencil:

$$A+\lambda B = \left[\begin{array}{cccc|cccc} \lambda & -1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & \lambda & -1 & & & & \\ \hline * & \cdots & * & x_{n-1} & -1 & & & \\ \vdots & \ddots & \ddots & * & \lambda & \ddots & & \\ \vdots & \ddots & \ddots & \vdots & & \ddots & -1 & \\ * & * & \cdots & * & & & \lambda & \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} p \\ \left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\} (q+1) \end{array},$$

with the first column block of width $(p+1)$ and the second of width $q$.

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with the left block having $(p+1)$ columns and the right block $q$ columns.

Set: $P_r : \begin{cases} p+i \to i & \text{for } i=1,\dots,q+1 \\ j \to n-j+1 & \text{for } j=1,\dots,p. \end{cases}$, $P_c : i \to p+2-i \quad \text{for } i=1,\dots,p+1.$

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$$P_r(A+\lambda B)P_c = \left[\begin{array}{cccc|cccc} x_{n-1} & * & \cdots & * & -1 & & & \\ * & \iddots & \iddots & \vdots & \lambda & \ddots & & \\ \vdots & \iddots & \iddots & * & & \ddots & -1 & \\ * & \cdots & * & * & & & \lambda & \\ \hline -1 & \lambda & & & & & & \\ & \ddots & \ddots & & & & & \\ & & -1 & \lambda & & & & \end{array}\right] \begin{array}{l} \} \; (q+1) \\ \\ \} \; p \end{array}.$$

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is a block-Kronecker pencil.

# $\mathscr{R}_n$: why these entries?

**Q**: Why entries of the form $0, a_i, \lambda a_{i+1}$, or $a_i + \lambda a_{i+1}$?

## Theorem [DT-Hernando, submitted]

If each coefficient $a_i$, for $i = 0, \ldots, k$, **appears in just one entry** of $A + \lambda B$, then the entry containing $a_i$ is either

$$\alpha_1 a_i + \alpha_2 + \lambda(\beta_1 a_{i+1} + \beta_2), \qquad \text{for} \quad 0 \le i \le k-1, \tag{1}$$

with $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{F}$, and $\alpha_1 \neq 0$, or

$$\alpha_1 a_{i-1} + \alpha_2 + \lambda(\beta_1 a_i + \beta_2), \qquad \text{for} \quad 0 \le i \le k-1, \tag{2}$$

with $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{F}$, and $\beta_1 \neq 0$.

# Other properties of companion matrices and pencils

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For general companion matrices $A(a_0,\ldots,a_{n-1})$ (only $p_{A(a_0,\ldots,a_{n-1})}(\lambda) = a_0 + a_1\lambda + \cdots + a_{n-1}\lambda^{n-1} + \lambda^n$ is imposed):

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Q1-Q2: What happens for general companion pencils?

# Answer...

(Companion pencil: $A + zB$, with $A, B \in \mathbb{F}[a_0, \ldots, a_{n-1}, a_n]$ such that $\det(A + zB) = a_0 + za_1 + \cdots + z^{n-1} a_{n-1} + z^n a_n = p(z)$)

The answer to Q2 is not known.

# Answer...

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## Conjecture

The smallest number of nonzero entries in a companion pencil for polynomials of degree $n$ is $2n-1+\lfloor \frac{n}{2} \rfloor$.

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As for Q1:

## Theorem [DT-Hernando, submitted]

If $\mathbb{F}$ is infinite, any companion pencil is non-derogatory.

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