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Companion pencils for scalar polynomials in the monomial basis

Fernando De Terán

Joint work with **C. Hernando**

Outline

Basic definitions

The Smith form of companion pencils

On the sparsity

Companion matrix vs companion pencil

$q(z) = a_0 + za_1 + \cdots + z^{n-1}a_{n-1} + z^n$ $\qquad (a_0, \ldots, a_{n-1} \in \mathbb{F})$

a (scalar) monic polynomial. $\qquad \mathbb{F}$ an **arbitrary field**.

Companion matrix

$A(a_0, \ldots, a_{n-1}) \in \mathbb{F}[a_0, \ldots, a_{n-1}]^{n \times n}$ such that

$$p_A(z) = \det(zI - A) = a_0 + za_1 + \cdots + z^{n-1}a_{n-1} + z^n = q(z)$$

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For a general (non-necessarily monic) polynomial:

$$p(z) = a_0 + za_1 + \cdots + z^{n-1}a_{n-1} + z^n a_n \qquad (a_0, \ldots, a_n \in \mathbb{F}, a_n \neq 0)$$

Companion pencil

$A + zB$, with $A, B \in \mathbb{F}[a_0, \ldots, a_{n-1}, a_n]^{n \times n}$ such that

$$\det(A + zB) = \alpha(a_0 + za_1 + \cdots + z^{n-1}a_{n-1} + z^n a_n) = \alpha p(z)$$

$(\alpha \in \mathbb{F})$

Example

Frobenius companion matrix

$$C_1 = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_0 \\ 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & 1 & 0 \end{bmatrix}, \qquad C_2 = C_1^\top$$

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MATLAB's command `roots`: QR algorithm on C_2

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$$zI - C_1 = \begin{bmatrix} a_{n-1}+z & a_{n-2} & \cdots & a_0 \\ -1 & z & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & -1 & z \end{bmatrix}, \qquad zI - C_2 = zI - C_1^\top$$

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$$F_i(z) = z\,\mathrm{diag}(a_n, 1, \ldots, 1) - C_i \qquad i = 1, 2$$

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- Strict equivalence: $P, Q \in \mathbb{F}^{n\times n}$ invertible:

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- Unimodular equivalence (u. e.): $U(z), V(z)$ with constant nonzero determinant (i. e., **unimodular**):

  $U(z)(A+zB)V(z)$: u. e. of $A+zB$.

Two basic transformations (II)

- **Strict equivalence:**

$A + zB$ a companion pencil for $p(z)$ $\Big\} \Rightarrow$ $P(A+zB)Q$ is a companion pencil for $p(z)$

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☞ Provides insight on some other “intrinsic information" of $p(z)$.

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The Smith canonical form

Theorem (Smith canonical form)

Let $A + zB$ be an $n \times n$ matrix pencil (A, B have entries in a field).
There are two unimodular matrices $U(z), V(z)$ such that

$$U(z)(A+zB)V(z) = \begin{bmatrix} d_1(z) & & \\ & \ddots & \\ & & d_n(z) \end{bmatrix} := \Delta(z),$$

with $d_i(z) \in \mathbb{F}[z]$ and $d_i(z) | d_{i+1}(z)$, for $i = 1, \ldots, n-1$.

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Q: Which is the Smith form of a companion pencil?

The Smith form of a companion pencil

Lemma

$p(z) = a_0 + za_1 + \cdots + z^n a_n$ is irreducible over $\mathbb{F}(a_0, \ldots, a_n)$.

It's a generalization of the same result for monic polynomials in

C. Ma, X. Zhan.
Extremal sparsity of the companion matrix of a polynomial.
LAA 438 (2013) 621-625

The Smith form of a companion pencil

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If $A + zB$ is a companion pencil, and $\Delta(z) = \mathrm{diag}(d_1(z), \ldots, d_n(z))$ its Smith form, then:

$$\det(A + zB) \quad = \quad p(z) = d_1(z) \cdots d_n(z) \quad \text{(up to constants)}$$

$$\underset{\text{Lemma}}{\Rightarrow} \quad d_1(z) = \cdots = d_{n-1}(z) = 1, d_n(z) = p(z)$$

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$p(z) = a_0 + za_1 + \cdots + z^n a_n$ is irreducible over $\mathbb{F}(a_0, \ldots, a_n)$.

If $A + zB$ is a companion pencil, and $\Delta(z) = \mathrm{diag}(d_1(z), \ldots, d_n(z))$ its Smith form, then:

$$\begin{aligned} \det(A + zB) &= p(z) = d_1(z) \cdots d_n(z) \quad \text{(up to constants)} \\ &\overset{\text{Lemma}}{\Rightarrow} d_1(z) = \cdots = d_{n-1}(z) = 1, d_n(z) = p(z) \end{aligned}$$

Theorem

The Smith form of any companion $n \times n$ pencil is $\begin{bmatrix} I_{n-1} & 0 \\ & p(z) \end{bmatrix}$.

(The base field is $\mathbb{F}(a_0, \ldots, a_n)$).

Non-derogatory

Corollary

Every companion pencil is non-derogatory.

(The geometric multiplicity of every eigenvalue is equal to 1).

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Every companion pencil is non-derogatory.

This is immediate for the **Frobenius** companion pencil:

$$\operatorname{rank} F_1(z) = \operatorname{rank} \begin{bmatrix} a_{n-1} + z a_n & a_{n-2} & \cdots & a_0 \\ -1 & z & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & -1 & z \end{bmatrix} \geq n-1$$

for all $z \in \mathbb{F}$.

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Every companion pencil is non-derogatory.

Proof?: If

$$U(z)(A+zB)V(z) = \begin{bmatrix} I_{n-1} & 0 \\ & p(z) \end{bmatrix},$$

then $\operatorname{rank}(A+z_0B) = \operatorname{rank} U(z_0)(A+z_0B)V(z_0) \geq n-1$, for all $z_0 \in \mathbb{F}$.$\square$

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However, $U(z), V(z) \in \mathbb{F}(a_0,\ldots,a_n)[z]$, so they are not necessarily defined for all $a_0,\ldots,a_n$!!!

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However, $U(z), V(z) \in \mathbb{F}(a_0, \ldots, a_n)[z]$, so they are not necessarily defined for all $a_0, \ldots, a_n$!!!

Non-derogatory

Theorem

Every companion pencil is non-derogatory when $\mathbb{F}$ is infinite.

Proof (sketch): By contradiction, we can prove that
$\text{rank}(A + z_0 B) \geq n - 1$, for all $z_0 \in \mathbb{F}$.
Uses polynomial identities and ends up with a nonzero polynomial that vanishes over all values in $\mathbb{F}$. □

FDT, C. Hernando.
A note on generalized companion pencils in the monomial basis.
RACSAM 114 (2020), Article 8.

U. e. over $\mathbb{F}[a_0, \ldots, a_n, z]$ and $\mathbb{F}(a_0, \ldots, a_n)[z]$

u. e. over $\mathbb{F}(a_0, \ldots, a_n)[z] \not\Rightarrow$ u. e. over $\mathbb{F}[a_0, \ldots, a_n, z]$:

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Counterexample: $\left[\begin{smallmatrix} y & z \\ 0 & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ are:

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All transformations belong to $\mathbb{F}[a_0,\ldots,a_n,z]$!

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Q: When is a companion pencil u. e. over $\mathbb{F}[a_0, \ldots, a_n, z]$ to its Smith form?

The result by Li, Liu & Chu

Theorem [Li, Liu & Chu'2020]

Let $P(z_1, \ldots, z_m) \in \mathbb{F}[z_1, \ldots, z_m]^{n \times n}$. If

$$\det P(z_1, \ldots, z_m) = z_1 - f(z_2, \ldots, z_m)$$

then $P(z_1, \ldots, z_m)$ is u. e. over $\mathbb{F}[z_1, \ldots, z_m]$ to its Smith form.

D. Li, J. Liu, D. Chu.
The Smith form of a multivariate polynomial matrix over an arbitrary coefficient field.
LAMA 70 (2020) 366–379.

Relies on the Quillen-Suslin Theorem in

Quillen D.
Projective modules over polynomial rings.
Invent. Math. 36 (1976) 167–171.

The result by Li, Liu & Chu

Theorem [Li, Liu & Chu'2020]

Let $P(z_1,\ldots,z_m) \in \mathbb{F}[z_1,\ldots,z_m]^{n\times n}$. If

$$\det P(z_1,\ldots,z_m) = z_1 - f(z_2,\ldots,z_m)$$

then $P(z_1,\ldots,z_m)$ is u. e. over $\mathbb{F}[z_1,\ldots,z_m]$ to its Smith form.

For companion pencils, we can set: $z_1 = a_0,\ldots,z_{m-1} = a_n, z_m = z$, so that $P(z_1,\ldots,z_m) = A(a_0,\ldots,a_n) + zB(a_0,\ldots,a_n)$ and

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Theorem

Every companion pencil is u. e. over $\mathbb{F}[a_0, \ldots, a_n, z]$ to $\begin{bmatrix} I_{n-1} & \\ & p(z) \end{bmatrix}$.

Outline

The main question

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For companion matrices:

Theorem [Ma & Zhan'2013]

If $A(a_0, \ldots, a_{n-1})$ is a companion matrix, then it has, at least, $2n-1$ nonzero entries.

C. Ma, X. Zhan.
Extremal sparsity of the companion matrix of a polynomial.
LAA 438 (2013) 621–625.

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This is the case of the Frobenius companion matrices

$$C_1 = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_0 \\ 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & 1 & 0 \end{bmatrix}, \qquad (C_2 = C_1^\top).$$

Companion matrix vs companion pencil

The companion pencil includes also the leading entries:

$$F_i(z) = z\,\mathrm{diag}(a_n, 1, \ldots, 1) - C_i \qquad\qquad i = 1, 2$$

$$F_1(a_0, \ldots, a_n, z) = \begin{bmatrix} a_{n-1} + za_n & a_{n-2} & \cdots & a_0 \\ -1 & z & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & -1 & z \end{bmatrix}, \qquad\qquad (F_2 = F_1^\top).$$

In this case, $n-1$ additional entries are added: $3n-2$ in total.

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But we can do it better:

$$\begin{bmatrix} a_{n-1} + za_n & 0 & a_{n-3} + za_{n-2} & \cdots & a_0 + za_1 \\ -1 & z & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ & & -1 & z & 0 \\ 0 & & & -1 & z \end{bmatrix} \qquad (n \text{ odd}).$$

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Number of nonzero entries: $\boxed{2n - 1 + \lfloor \frac{n}{2} \rfloor}$

When each coefficient appears once

Theorem

If $A + zB$ is a companion pencil where each coefficient $a_0, \dots, a_n$ appears in just one entry, then $A + zB$ has, at least, $2n - 1 + \lfloor \frac{n}{2} \rfloor$ nonzero entries.

FDT, C. Hernando.
A note on generalized companion pencils in the monomial basis.
RACSAM 114 (2020), Article 8.

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Conjecture: $2n-1+\lfloor\frac{n}{2}\rfloor$


Thank you!!
Go raibh maith agat!!