Low rank perturbation of regular matrix pencils with symmetry structures

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Co-authors

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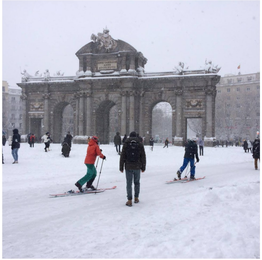
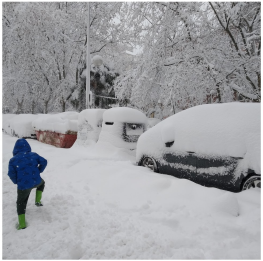
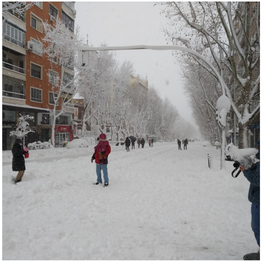
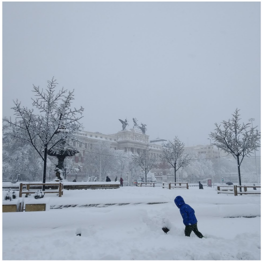
Some pictures (Madrid, Jan’21 !!!!)

Outline

1. Motivation and related work
2. The unstructured case for pencils
3. The case of structured pencils
   - The $\top$-palindromic case
   - Other symmetry structures

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Applications

- **Matrix pencils** $A+\lambda B$: Associated to the system $Bx'(t) = Ax(t)$.
- In practice, A, B present some symmetry structure (they are "structured").
- Low-rank perturbations: Arise in problems related to modification of structures depending on many parameters, but only a few of them are to be modified.
- The **size** (in norm) **doesn't matter**.
- We are interested in the (generic) change of the spectral structure.
- The **spectral structure is key** in the behavior of the solutions of the associated system.

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Given a matrix pencil: $A+\lambda B$ ($A, B \in \mathbb{C}^{m\times n}$) with known spectral info:

Q: Which is the **generic** spectral info of $(A+\widetilde{A})+\lambda(B+\widetilde{B})$?

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- Generic change vs ~~all possible changes~~.
- Conditions of **genericity** in terms of **algebraic geometry** (the complement is contained in a proper algebraic set).
- Revision of the unstructured case, and focus on structured pencils.

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Previous contributions (structured)

- Structured **matrices**:

J. H. Fourie, G. J. Groenewald, D. B. Janse van Rensburg, A. C. M. Ran. Linear Algebra Appl. 439 (2013) 653–674.

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- Rank-1 & rank-2 perturbations of structured **pencils**:
  - **L. Batzke**. Linear Algebra Appl., 458 (2014) 638–670.
  - **L. Batzke**. Oper. Matrices, 10 (2016) 83–112.

The matrix case: Jordan Canonical Form (JCF)

Theorem

Given $A \in \mathbb{C}^{n\times n}$ and λ_0 e-val of A with g associated Jordan blocks in the JCF of A. Let $E \in \mathbb{C}^{n\times n}$ such that $\mathrm{rank}(E) < g$.

Then, generically, the blocks associated to λ_0 in the JCF of $A+E$ are the $g - \mathrm{rank}(E)$ smallest blocks of A with e-val λ_0.

- **L. Hormander, A. Mellin**. Math. Scand., 75 (1994), pp. 255–262.
- **J. Moro, F. M. Dopico**. SIAM J. Matrix Anal. Appl., 25 (2003) 495–506.
- **S. V. Savchenko**. Funkts. Anal. Prilozh., 38 (2004), pp. 85–88 (in Russian). Transl. in Funct. Anal. Appl., 38 (2004), pp. 69–71.

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Regular vs singular

$P(\lambda) = A + \lambda B$, with $A, B \in \mathbb{C}^{m\times n}$, is:

- **Regular**: If $m = n$ and $\det(A + \lambda B) \neq 0$.
- **Singular** otherwise.

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- ∞ is an eigenvalue of $A + \lambda B \Leftrightarrow 0$ is an eigenvalue of $\operatorname{rev}(A + \lambda B) := B + \lambda A$.

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☞ Low-rank pencils are **singular**.

The Weierstrass Canonical Form (WCF)

$$J_k(a-\lambda) := \begin{bmatrix} a-\lambda & 1 & & \\ & \ddots & \ddots & \\ & & a-\lambda & 1 \\ & & & a-\lambda \end{bmatrix}_{k\times k}, \quad \operatorname{rev} J_k(\lambda) = \begin{bmatrix} 1 & \lambda & & \\ & \ddots & \ddots & \\ & & 1 & \lambda \\ & & & 1 \end{bmatrix}_{k\times k}$$

Theorem (WCF)

If $A+\lambda B$ is **regular**, there are $S,T\in\mathbb{C}^{n\times n}$ **invertible** such that

$$\begin{aligned} S(A+\lambda B)T \;=\; & \operatorname{diag}\big(J_{n_{1,1}}(\lambda_1-\lambda),\ldots,J_{n_{1,g_1}}(\lambda_1-\lambda),\ldots, \\ & J_{n_{\kappa,1}}(\lambda_\kappa-\lambda),\ldots,J_{n_{\kappa,g_\kappa}}(\lambda_\kappa-\lambda), \\ & \operatorname{rev} J_{n_{\kappa+1,1}}(\lambda),\ldots,\operatorname{rev} J_{n_{\kappa+1,g_{\kappa+1}}}(\lambda)\big). \end{aligned}$$

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The Weierstrass Canonical Form (cont.)

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$\lambda_1,\ldots,\lambda_\kappa$: Eigenvalues (**finite**) of $A+\lambda B$.

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Wlog: $n_{i,1}\geq\cdots\geq n_{i,g_i}$.

The Kronecker Canonical Form (KCF)

$$L_\alpha := \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \end{bmatrix}_{\alpha\times(\alpha+1)}$$

Theorem (KCF)

Given $A+\lambda B$, there are $S\in\mathbb{C}^{m\times m}$ and $T\in\mathbb{C}^{n\times n}$ **invertible** such that

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$\alpha_1,\ldots,\alpha_\eta$: Right minimal indices,

$\beta_1,\ldots,\beta_\xi$: Left minimal indices.

Low-rank perturbation of the WCF

Theorem

Let $A(\lambda)$ be a regular pencil with g_0 partial multiplicities of λ_0. Let $B(\lambda)$ another pencil such that $\operatorname{rank} B(\lambda_0) < g_0$.

Then, **generically**, $A(\lambda) + B(\lambda)$ has exactly $g_0 - \operatorname{rank} B(\lambda_0)$ λ_0-partial multiplicities:

- $g_0 - \operatorname{rank} B(\lambda_0) - \operatorname{rank} B_1$ are the smaller λ_0-partial multiplicities of $A(\lambda)$,
- $\operatorname{rank} B_1$ equal to 1.

FDT, F. M. Dopico, J. Moro. SIAM J. Matrix Anal. Appl., 30-2 (2008) 538-547.

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For matrix polynomials:

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Generic perturbations?

Q: How to generate “generic" perturbations?

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☞ **First idea** (previous slide):

$$\rho_0 := \text{rank}(B_0 + \lambda_0 B_1), \quad \rho_1 := \text{rank}(B_1), \quad \rho := \rho_0 + \rho_1 .$$

$\rho_0 < g_0$: guarantees that λ_0 is an e-val of $A_0 + B_0 + \lambda(A_1 + B_1)$.
($\rho_0 < g_0$ is the **low-rank condition**).

$B_0 + \lambda_0 B_1$: an arbitrary matrix with rank $\rho_0 < g_0$.
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Perturbation: $B_0 + \lambda B_1 = B_0 + \lambda_0 B_1 + (\lambda - \lambda_0) B_1 \rightsquigarrow$ has, generically, rank ρ (if $\rho < n$).

Generic perturbations? (II)

☞ **Second idea** (rank-1):

$$B_0 + \lambda B_1 \quad = \quad -\alpha u v^\top + \lambda \beta u v^\top \quad (u, v \in \mathbb{C}^n,\ \alpha, \beta \in \mathbb{C})$$

L. Batzke. Linear Algebra Appl., 458 (2014) 638–670.

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$u, v \in \mathbb{C}^n$ arbitrary.

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$$\begin{aligned} \rho &= \operatorname{rank}(B_0 + \lambda B_1) = \operatorname{rank}(B_0 + \lambda_0 B_1 + (\lambda - \lambda_0) B_1) \\ &= \rho_0 + \rho_1 > \rho_0 = \operatorname{rank}(A + \lambda_0 B) \end{aligned}$$

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- $(-\alpha + \lambda\beta) u v^\top$ has α/β as e-val.

☞ Singular pencils do not have e-vals (generically).

See, for instance,

J. W. Demmel, A. Edelman. Linear Algebra Appl. 230 (1995) 61–87.

Generic perturbations!

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Generic perturbations!

☞ **Third idea**: Use a decomposition of low-rank pencils as a sum of rank-1 pencils.

- This decomposition is **new**.
- It is obtained from the KCF.
- It allows to obtain a parametrization of the $m \times n$ pencils with rank $\leq r$, for $r \leq \min\{m, n\}$.

Rank-1 decomposition

We express an $m \times n$ pencil with rank $\leq r$ as

$$v_1(\lambda)w_1(\lambda)^\top + \cdots + v_s(\lambda)w_s(\lambda)^\top + v_{s+1}(\lambda)w_{s+1}(\lambda)^\top + \cdots + v_r(\lambda)w_r(\lambda)^\top$$

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$\mathbb{P}_r = \mathfrak{C}_0 \cup \mathfrak{C}_1 \cup \cdots \cup \mathfrak{C}_r$ ($m \times n$ pencils with rank $\leq r$)

Parametrization of the set of pencils with rank $\leq r$

$$\mathfrak{C}_s := \{v_1 w_1(\lambda)^\top + \cdots + v_s w_s(\lambda)^\top + v_{s+1}(\lambda) w_{s+1}{}^\top + \cdots + v_r(\lambda) w_r{}^\top\}.$$

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$\alpha \in \mathbb{C}^{rn}$:	degree-0 coefficients of v_i's.
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Low-rank perturbation of the WCF (revisited)

Theorem (DT, Dopico'16)

$L(\lambda)$ a **regular** $n \times n$ pencil, with

- $\lambda_1, \ldots, \lambda_\kappa$: e-vals.
- $n_{i,1} \geq \cdots \geq n_{i,g_i} > 0$: partial multiplicities ($i = 1, \ldots, \kappa$).

Let $0 \leq s \leq r$ and $\Phi_s : \mathbb{C}^{3rn} \to \mathfrak{C}_s$ as before.

There is a generic set $\mathscr{G}_s \subseteq \mathbb{C}^{3rn}$ s.t., **for all** $E(\lambda) \in \Phi_s(\mathscr{G}_s)$, the pencil $L + E$ is **regular** and

$n_{i,r+1} \geq \cdots \geq n_{i,g_i}$ are the **partial multiplicities of $L + E$ at λ_i**.

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☞ All e-vals of $L + E$ different to those in L are **simple** **(DT-Mehl-Mehrmann)**.

Some remarks

$$\mathbb{P}_r = \mathfrak{C}_0 \cup \mathfrak{C}_1 \cup \cdots \cup \mathfrak{C}_r,$$

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- Comparison with the result in [DT, Dopico, 2008]:

The perturbations in that work add $\rho_1 = \operatorname{rank} B_1$ e-vals λ_0 (generically simple).

Outline

1 Motivation and related work

2 The unstructured case for pencils

3 The case of structured pencils
- The $\top$-palindromic case
- Other symmetry structures

Symmetry structures we consider

A pencil $A+\lambda B$ ($A,B\in\mathbb{C}^{n\times n}$) is:

- *Hermitian* if $A=A^*, B=B^*$;
- *symmetric* if $A=A^\top, B=B^\top$;
- *anti-Hermitian* if $A^*=-A, B^*=-B$;
- *anti-symmetric* if $A^\top=-A, B^\top=-B$;
- $\star$-*even* if $A^\star=A, B^\star=-B$;
- $\star$-*odd* if $A^\star=-A, B^\star=B$;
- $\star$-*palindromic* if $A^\star=B$;
- $\star$-*anti-palindromic* if $A^\star=-B$.

($\star=\top,*$).

$\star$-*alternating*: $\star$-even $\cup$ $\star$-odd.

Why these symmetries?

☞ They are the symmetries that frequently arise in applications:

- **I. Abou Hamad , B. Israels, P.A. Rikvold, S.V. Poroseva**. In: Computer Simulation Studies in Condensed-Matter Physics XXIII (CSP10), 4 (2010) 125–129.
- **R. Albert, I. Albert, G. L. Nakarado**. Phys. Rev. E, 69 (2004) 025103.
- **N.H. Du, V.H. Linh, V. Mehrmann**. En: Differential Algebraic Equation Forum, Surveys in Differential-Algebraic Equations I, Springer (2013) 63–96.
- **D. Mackey, N. Mackey, C. Mehl, V. Mehrmann**. SIAM J. Matrix Anal. Appl., 28 (2006) 1029–1051.
- **M. C. Petri**. Energy Sciences and Engineering Directorate, U.S. Department of Homeland Security (tech. report), 2008.
- ...

Differences with the general case (not structured)

- The set of perturbations is different (smaller): we only consider structured perturbations.
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Spectral symmetries

If $A+\lambda A^\top$ is $\top$-palindromic:

- λ_0 an e-val of $A \Rightarrow 1/\lambda_0$ e-val of A.
- The partial multiplicities of λ_0 and $1/\lambda_0$ **coincide**.
- Each odd partial multiplicity of 1 appears **an even number of times**.
- Each even partial multiplicity of -1 appears **an even number of times**.
- The right and left minimal indices **coincide**.

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WCF of $\top$-palindromic pencils

Theorem

Any pencil $A+\lambda A^\top$ is congruent to a direct sum (unique, up to permutation) of blocks:

- $\begin{bmatrix} 0 & L_\alpha(\lambda) \\ \operatorname{rev} L_\alpha(\lambda)^\top & 0 \end{bmatrix}_{(2\alpha+1)\times(2\alpha+1)}.$

  (Associated to a right and left minimal index both equal to α).

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- $\begin{bmatrix} 0 & J_\beta(1+\lambda) \\ J_\beta(1+\lambda)^\top & 0 \end{bmatrix}$ (β even).

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- $\begin{bmatrix} 0 & J_\delta(\lambda_0-\lambda) \\ \mathrm{rev}\, J_\delta(\lambda_0-\lambda)^\top & 0 \end{bmatrix}.$

  (Associated to a couple of partial multiplicities of $\lambda_0, 1/\lambda_0$).

WCF of $\top$-palindromic pencils (cont.)

Theorem (cont.)

- $$\left[\begin{array}{cccc|cccc} & & & & & & & \lambda-1 \\ & & & & & & \lambda-1 & 1 \\ & & & & & \iddots & \iddots & \\ & & & & \lambda-1 & 1 & & \\ \hline & & & 1-\lambda & \lambda+1 & & & \\ & & 1-\lambda & \lambda & & & & \\ & \iddots & \iddots & & & & & \\ 1-\lambda & \lambda & & & & & & \end{array}\right]_{(2\varepsilon)\times(2\varepsilon)} .$$

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 & & & & & \iddots & 1 \\
 & & & & \lambda+1 & \iddots & \\
 & & & \lambda+1 & 1 & & \\
\hline
 & & 1+\lambda & \lambda & & & \\
 & \iddots & \iddots & & & & \\
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Rank-1 decomposition of $\top$-palindromico pencils

Theorem

$E(\lambda)$, a $\top$-palindromic $n \times n$ pencil, with $\operatorname{rank} E = r \leq n$, can be written as

$$E(\lambda) = \begin{cases} \begin{array}{c} v_1 w_1^\top(\lambda) + \cdots + v_s w_s^\top(\lambda) + \\ (\operatorname{rev} w_1)(\lambda) v_1^\top + \cdots + (\operatorname{rev} w_s)(\lambda) v_s^\top, \end{array} & \text{if } r \text{ is even}, \\ \begin{array}{c} (1+\lambda) u u^\top + v_1 w_1^\top(\lambda) + \cdots + v_s w_s^\top(\lambda) + \\ (\operatorname{rev} w_1)(\lambda) v_1^\top + \cdots + (\operatorname{rev} w_s)(\lambda) v_s^\top \end{array}, & \text{if } r \text{ is odd}, \end{cases}$$

donde $s = \lfloor r/2 \rfloor$, $\deg u = \deg v_1 = \cdots = \deg v_s = 0$ y $\deg w_1, \ldots, \deg w_s \leq 1$.

FDT. SIAM J. Matrix Anal. Appl., 39 (2018) 1116–1134.

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FDT. SIAM J. Matrix Anal. Appl., 39 (2018) 1116–1134.

Generic change (not structured)?

Assume that

$$n_{i,1} \geq n_{i,2} \geq \cdots \geq n_{i,r} = n_{i,r+1} = \ldots = n_{i,r+d} > n_{i,r+d+1} > \cdots$$

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It cannot happen!
(there would be an odd number of partial multiplicities $n_{i,r+1}$ in $L+E$ at $\lambda_0 = 1$)

The generic change

Partial multiplicities of λ_i in $L(\lambda) = A + \lambda A^\top$:

$$n_{i,1} \geq \cdots n_{i,r} \geq \cdots \geq n_{i,g_i}.$$

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If $E(\lambda)$ is $\top$-palindromic and $\operatorname{rank} E = r$, **generically**, the partial multiplicities of λ_i in $L + E$ are:

e-val λ_i	case	multiplicities
$\lambda_i = 1$	$n_{i,r+1}$ odd + (P)	$(n_{i,r+1}+1, n_{i,r+2}, \ldots, n_{i,g_i})$
	otherwise	$(n_{i,r+1}, n_{i,r+2}, \ldots, n_{i,g_i})$
$\lambda_i = -1$	r even, $n_{i,r+1}$ even + (P)	$(n_{i,r+1}+1, n_{i,r+2}, \ldots, n_{i,g_i})$
	r even, otherwise	$(n_{i,r+1}, n_{i,r+2}, \ldots, n_{i,g_i})$
	r odd, $n_{i,r+1}$ even + (P)	$(n_{i,r+1}+1, n_{i,r+2}, \ldots, n_{i,g_i}, 1)$
	r odd, otherwise	$(n_{i,r+1}, n_{i,r+2}, \ldots, n_{i,g_i}, 1)$
$\lambda_i \in \mathbb{C} \setminus \{\pm 1\}$	all	$(n_{i,r+1}, n_{i,r+2}, \ldots, n_{i,g_i})$

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Parametrization of $\top$-palindromic pencils with rank $\leq r$

$$E(\lambda)=\begin{cases} \begin{array}{c} v_1w_1^\top+\cdots+v_sw_s^\top+ \\ (\mathrm{rev}\, w_1)v_1^\top+\cdots+(\mathrm{rev}\, w_s)v_s^\top, \end{array} & \text{if } r \text{ is even} \\ \begin{array}{c} (1+\lambda)uu^\top+v_1w_1^\top+\cdots+v_sw_s^\top+ \\ (\mathrm{rev}\, w_1)v_1^\top+\cdots+(\mathrm{rev}\, w_s)v_s^\top \end{array}, & \text{if } r \text{ is odd,} \end{cases}$$

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Let:

$$\Phi : \quad \mathbb{C}^{\lfloor \frac{3r}{2} \rfloor n} \to \mathrm{Pal}_r^\top, \qquad \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} \mapsto E(\lambda),$$

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- $\gamma \in \mathbb{C}^{rn}$: Degree-0 coefficients of w_i's.
- $\delta \in \mathbb{C}^{sn}$: Degree-1 coefficients of w_i's.

The main result

Theorem

Let

- $L(\lambda)$: $\top$-palindromic regular $n \times n$ pencils with e-vals $\lambda_1, \ldots, \lambda_\kappa$ and partial multiplicities: $n_{i,1} \geq n_{i,2} \geq \cdots \geq n_{i,g_i} > 0$ $(k = 1, \ldots, \kappa)$,
- $r \in \mathbb{Z}^+$, and
- $\Phi : \mathbb{C}^{\lfloor \frac{3r}{2} \rfloor n} \to \mathrm{Pal}_r^\top$ the previous map.

Then, there is a generic set $\mathscr{G} \subseteq \mathbb{C}^{\lfloor \frac{3r}{2} \rfloor n}$ such that, for all $E(\lambda) \in \Phi(\mathscr{G})$, the pencil $L + E$:

- is regular and
- their partial multiplicities are those in the **previous table.**

All e-vals of $L + E$ other than those of L are simple.

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(In particular, if $r \geq g_i$ then λ_i is not an e-val of $L + E$.)

What happens for the remaining structures?

- $\top$-anti-palindromic and $\top$-alternating:
  - Analogous results.
  - $\top$-alternating: It can be obtained from $\top$-pal. and $\top$-anti-pal. through **Cayley transforms**.
- (anti-)Hermitian, (anti-)Symmetric, $*$-(anti-)palindromic and $*$-alternating:
  - Same generic behavior as in the non structured case.
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Related to

- **FDT, F. M. Dopico, J. M. Landsberg.** An explicit description of the irreducible components of the set of matrix pencils with bounded normal rank. Linear Algebra Appl. 520 (2017) 80-103.

Idea of the proof

- If, for $x \in \mathbb{C}^{\lfloor \frac{3r}{2} \rfloor n}$ **arbitrarily small** the change in the table holds, then this change is generic.
- Find some $x \in \mathbb{C}^{\lfloor \frac{3r}{2} \rfloor n}$ (with $\|x\| < \varepsilon$) for which this change holds.
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