Generic eigenstructures of matrix pencils

Fernando De Terán

(Based on joint work with Froilán M. Dopico and Andrii Dmytryshyn)

uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

Outline

1. Basic setting
   - General (unstructured) pencils
   - Structured pencils
2. Regular pencils
3. Singular general (unstructured) pencils
4. Structured pencils (low-rank)
   - Palindromic pencils
   - $\top-$alternating pencils
   - Skew-symmetric pencils
   - Symmetric pencils
   - Hermitian pencils

Outline

1. Basic setting
   - General (unstructured) pencils
   - Structured pencils
2. Regular pencils
3. Singular general (unstructured) pencils
4. Structured pencils (low-rank)
   - Palindromic pencils
   - $\top$−alternating pencils
   - Skew-symmetric pencils
   - Symmetric pencils
   - Hermitian pencils

Pencils and eigenstructure

Matrix pencil: $A+\lambda B$, with $A,B\in\mathbb{C}^{m\times n}$ (or matrix pairs (A,B)).

$A+\lambda B$ is regular if $m=n$ and $\det(A+\lambda B)\not\equiv 0$.

Otherwise, it's singular ($m\neq n$ or $\det(A+\lambda B)\equiv 0$).

Pencils and eigenstructure

Matrix pencil: $A+\lambda B$, with $A,B\in\mathbb{C}^{m\times n}$ (or matrix pairs (A,B)).

$A+\lambda B$ is regular if $m=n$ and $\det(A+\lambda B)\not\equiv 0$.

Otherwise, it's singular ($m\neq n$ or $\det(A+\lambda B)\equiv 0$).

Definition

$A+\lambda B$ and $A'+\lambda B'$ are strictly equivalent if:

$$A'=PAQ,\ B'=PBQ,\qquad \text{for some } P,Q \text{ invertible.}$$

(Namely, $A'+\lambda B'=P(A+\lambda B)Q$).

Pencils and eigenstructure

Matrix pencil: $A+\lambda B$, with $A,B\in\mathbb{C}^{m\times n}$ (or matrix pairs (A,B)).

$A+\lambda B$ is regular if $m=n$ and $\det(A+\lambda B)\not\equiv 0$.

Otherwise, it's singular ($m\neq n$ or $\det(A+\lambda B)\equiv 0$).

Definition

$A+\lambda B$ and $A'+\lambda B'$ are strictly equivalent if:

$$A'=PAQ,\ B'=PBQ, \qquad \text{for some } P,Q \text{ invertible.}$$

(Namely, $A'+\lambda B'=P(A+\lambda B)Q$).

Eigenstructure of $A+\lambda B$: Set of invariants under strict equivalence.

The Kronecker canonical form (KCF)

Theorem (Kronecker Canonical Form, KCF)

Every pencil is strictly equivalent to a direct sum, uniquely determined (up to permutation), of blocks:

- **Blocks associated with finite evals** (μ): $J_k(\mu) := \begin{bmatrix} \lambda-\mu & 1 & & \\ & \ddots & \ddots & \\ & & \lambda-\mu & 1 \\ & & & \lambda-\mu \end{bmatrix}_{k\times k}$ $(k \geq 1)$.
- **Blocks associated with the ∞ eval**: $J_k(\infty) := \begin{bmatrix} 1 & \lambda & & \\ & \ddots & \ddots & \\ & & 1 & \lambda \\ & & & 1 \end{bmatrix}_{k\times k}$ $(k \geq 1)$.
- **Right singular blocks**: $R_k(\lambda) =: \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \end{bmatrix}_{k\times(k+1)}$ $(k \geq 0)$.
- **Left singular blocks**: $R_k(\lambda)^\top$ $(k \geq 0)$.

The Kronecker canonical form (KCF)

Theorem (Kronecker Canonical Form, KCF)

Every pencil is strictly equivalent to a direct sum, uniquely determined (up to permutation), of blocks:

- **Blocks associated with finite evals** (μ): $J_k(\mu) := \begin{bmatrix} \lambda-\mu & 1 & & \\ & \ddots & \ddots & \\ & & \lambda-\mu & 1 \\ & & & \lambda-\mu \end{bmatrix}_{k\times k}$ $(k \geq 1)$.
- **Blocks associated with the ∞ eval**: $J_k(\infty) := \begin{bmatrix} 1 & \lambda & & \\ & \ddots & \ddots & \\ & & 1 & \lambda \\ & & & 1 \end{bmatrix}_{k\times k}$ $(k \geq 1)$.
- **Right singular blocks**: $R_k(\lambda) =: \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \end{bmatrix}_{k\times(k+1)}$ $(k \geq 0)$.
- **Left singular blocks**: $R_k(\lambda)^\top$ $(k \geq 0)$.

☞ **Eigenstructure**: numbers and sizes of all blocks in the KCF.

The Kronecker canonical form (KCF)

Theorem (Kronecker Canonical Form, KCF)

Every pencil is strictly equivalent to a direct sum, uniquely determined (up to permutation), of blocks:

- **Blocks associated with finite evals** (μ): $J_k(\mu) := \begin{bmatrix} \lambda-\mu & 1 & & \\ & \ddots & \ddots & \\ & & \lambda-\mu & 1 \\ & & & \lambda-\mu \end{bmatrix}_{k\times k}$ $(k \geq 1)$.
- **Blocks associated with the ∞ eval**: $J_k(\infty) := \begin{bmatrix} 1 & \lambda & & \\ & \ddots & \ddots & \\ & & 1 & \lambda \\ & & & 1 \end{bmatrix}_{k\times k}$ $(k \geq 1)$.
- **Right singular blocks**: $R_k(\lambda) =: \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \end{bmatrix}_{k\times(k+1)}$ $(k \geq 0)$.
- **Left singular blocks**: $R_k(\lambda)^\top$ $(k \geq 0)$.

☞ **Eigenstructure**: numbers and sizes of all blocks in the KCF.

☞ $A+\lambda B$ is singular $\Leftrightarrow$ KCF($A+\lambda B$) contains singular blocks.

Orbits

Definition (orbit)

$\mathscr{O}(A+\lambda B) := \{P(A+\lambda B)Q : \; P, Q \text{ invertible}\}$.

(i.e.: it's the set of pencils which are strictly equivalent to $A+\lambda B$).

Orbits

Definition (orbit)

$\mathscr{O}(A+\lambda B) := \{P(A+\lambda B)Q : \; P, Q \text{ invertible}\}$.

(i.e.: it's the set of pencils which are strictly equivalent to $A+\lambda B$).

☞ Every orbit is (uniquely) determined by the KCF.

Orbits

Definition (orbit)

$\mathscr{O}(A+\lambda B) := \{P(A+\lambda B)Q : \ P, Q \text{ invertible}\}.$

(i.e.: it's the set of pencils which are strictly equivalent to $A+\lambda B$).

☞ Every orbit is (uniquely) determined by the KCF.

$\overline{\mathscr{O}}(L)$: closure of $\mathscr{O}(L)$ (in the standard topology of $\mathbb{C}^{m\times n}\times\mathbb{C}^{m\times n}$).

Orbits

Definition (orbit)

$\mathscr{O}(A+\lambda B) := \{P(A+\lambda B)Q : \ P, Q \text{ invertible}\}$.

(i.e.: it's the set of pencils which are strictly equivalent to $A+\lambda B$).

☞ Every orbit is (uniquely) determined by the KCF.

$\overline{\mathscr{O}}(L)$: closure of $\mathscr{O}(L)$ (in the standard topology of $\mathbb{C}^{m\times n}\times\mathbb{C}^{m\times n}$).

☞ $\overline{\mathscr{O}}(L)$ contains $\mathscr{O}(L)$ together with other pencils which are the **limit of sequences in** $\mathscr{O}(L)$.

Likeliness and genericity

We say that

$$\mathrm{KCF}(L_1) \text{ is more likely than } \mathrm{KCF}(L_2)$$
$$\Updownarrow$$
$$L_2 \in \overline{\mathscr{O}}(L_1)$$

Likeliness and genericity

We say that

$$\mathrm{KCF}(L_1) \text{ is more likely than } \mathrm{KCF}(L_2)$$
$$\Updownarrow$$
$$L_2 \in \overline{\mathscr{O}}(L_1)$$
$$\Updownarrow$$
$$\mathscr{O}(L_2) \subseteq \overline{\mathscr{O}}(L_1)$$

Likeliness and genericity

We say that

$$\text{KCF}(L_1) \text{ is more likely than } \text{KCF}(L_2)$$
$$\Updownarrow$$
$$L_2 \in \overline{\mathscr{O}}(L_1)$$
$$\Updownarrow$$
$$\mathscr{O}(L_2) \subseteq \overline{\mathscr{O}}(L_1)$$
$$\Updownarrow$$
$$\overline{\mathscr{O}}(L_2) \subseteq \overline{\mathscr{O}}(L_1)$$

Likeliness and genericity

We say that

$$\mathrm{KCF}(L_1) \text{ is more likely than } \mathrm{KCF}(L_2)$$
$$\Updownarrow$$
$$L_2 \in \overline{\mathscr{O}}(L_1)$$
$$\Updownarrow$$
$$\mathscr{O}(L_2) \subseteq \overline{\mathscr{O}}(L_1)$$
$$\Updownarrow$$
$$\overline{\mathscr{O}}(L_2) \subseteq \overline{\mathscr{O}}(L_1)$$

☞ Establishes an **order** relation between orbit closures.

Likeliness and genericity

We say that

$$\mathrm{KCF}(L_1) \text{ is more likely than } \mathrm{KCF}(L_2)$$
$$\Updownarrow$$
$$L_2 \in \overline{\mathscr{O}}(L_1)$$
$$\Updownarrow$$
$$\mathscr{O}(L_2) \subseteq \overline{\mathscr{O}}(L_1)$$
$$\Updownarrow$$
$$\overline{\mathscr{O}}(L_2) \subseteq \overline{\mathscr{O}}(L_1)$$

☞ Establishes an **order** relation between orbit closures.

Generic KCFs/eigenstructures: the “most likely" ones, namely:

K is **generic** if there is no $\widetilde{K} \neq K$ with $K \in \overline{\mathscr{O}}(\widetilde{K})$.

The structures

We consider the following structured pencils $A + \lambda B$ (with $\star = *, \top$):

- $\star$-palindromic: $B = A^\star$.
- $\star$-anti-palindromic: $B = -A^\star$.
- $\star$-alternating:
  - $\star$-even: $A^\star = A$, $B^\star = -B$.
  - $\star$-odd: $A^\star = -A$, $B^\star = B$.
- Symmetric: $A^\top = A$, $B^\top = B$.
- Skew-symmetric: $A^\top = -A$, $B^\top = -B$.
- Hermitian: $A^* = A$, $B^* = B$.
- Skew-Hermitian: $A^* = -A$, $B^* = -B$.

Strict equivalence vs congruence

☟ Strict equivalence does not preserve the previous structures!!

Strict equivalence vs congruence

☞ Strict equivalence does not preserve the previous structures!!

Definition

$A+\lambda B$ and $A'+\lambda B'$ are $\star$-congruent if

$$A' = P^\star AP,\ B' = P^\star BP, \qquad \text{for some } P \text{ invertible.}$$

(Namely, $A'+\lambda B' = P^\star(A+\lambda B)P$).

Strict equivalence vs congruence

☟ Strict equivalence does not preserve the previous structures!!

Definition

$A+\lambda B$ and $A'+\lambda B'$ are $\star$-congruent if

$$A' = P^\star AP, \; B' = P^\star BP, \qquad \text{for some } P \text{ invertible.}$$

(Namely, $A'+\lambda B' = P^\star(A+\lambda B)P$).

☝ Congruence preserves the previous structures.

Strict equivalence vs congruence

☟ Strict equivalence does not preserve the previous structures!!

Definition

$A+\lambda B$ and $A'+\lambda B'$ are $\star$-congruent if

$$A' = P^\star AP, \; B' = P^\star BP, \qquad \text{for some } P \text{ invertible.}$$

(Namely, $A'+\lambda B' = P^\star(A+\lambda B)P$).

☝ Congruence preserves the previous structures... and the eigenstructure!

☞ For structured pencils, we use congruence, instead of strict equivalence.

Congruence orbits and structured KCF

Definition ($\star$-congruence orbit)

$\mathscr{O}_c(A+\lambda B) := \{P^\star(A+\lambda B)P : \ P \text{ invertible}\}.$

(i.e.: it's the set of pencils which are $\star$-congruent to $A+\lambda B$).

Congruence orbits and structured KCF

Definition ($\star$-congruence orbit)

$\mathscr{O}_c(A+\lambda B) := \{P^\star(A+\lambda B)P : \ P \text{ invertible}\}.$

(i.e.: it's the set of pencils which are $\star$-congruent to $A+\lambda B$).

☟ KCF$(A+\lambda B)$ is not necessarily in $\mathscr{O}_c(A+\lambda B)$ (since it's not necessarily structured!).

Congruence orbits and structured KCF

Definition ($\star$-congruence orbit)

$\mathscr{O}_c(A+\lambda B) := \{P^\star(A+\lambda B)P : \ P \text{ invertible}\}.$

(i.e.: it's the set of pencils which are $\star$-congruent to $A+\lambda B$).

KCF($A+\lambda B$) is not necessarily in $\mathscr{O}_c(A+\lambda B)$ (since it's not necessarily structured!).

However, there is a structured KCF (SKCF($A+\lambda B$)) in $\mathscr{O}_c(A+\lambda B)$), encoding the eigenstructure of $A+\lambda B$.

Likeliness and genericity (structured)

If L_1 and L_2 have the same structure, we say that

$$\text{SKCF}(L_1) \text{ is more likely than } \text{SKCF}(L_2)$$

$$\Updownarrow$$

$$L_2 \in \overline{\mathscr{O}}_c(L_1)$$

Likeliness and genericity (structured)

If L_1 and L_2 have the same structure, we say that

$$\mathrm{SKCF}(L_1) \text{ is more likely than } \mathrm{SKCF}(L_2)$$

$$\Updownarrow$$

$$L_2 \in \overline{\mathscr{O}}_c(L_1)$$

$$\Updownarrow$$

$$\mathscr{O}_c(L_2) \subseteq \overline{\mathscr{O}_c}(L_1)$$

Likeliness and genericity (structured)

If L_1 and L_2 have the same structure, we say that

$$\mathrm{SKCF}(L_1) \text{ is more likely than } \mathrm{SKCF}(L_2)$$

$$\Updownarrow$$

$$L_2 \in \overline{\mathscr{O}}_c(L_1)$$

$$\Updownarrow$$

$$\mathscr{O}_c(L_2) \subseteq \overline{\mathscr{O}_c}(L_1)$$

$$\Updownarrow$$

$$\overline{\mathscr{O}_c}(L_2) \subseteq \overline{\mathscr{O}_c}(L_1)$$

Likeliness and genericity (structured)

If L_1 and L_2 have the same structure, we say that

$$\mathrm{SKCF}(L_1) \text{ is more likely than } \mathrm{SKCF}(L_2)$$
$$\Updownarrow$$
$$L_2 \in \overline{\mathscr{O}}_c(L_1)$$
$$\Updownarrow$$
$$\mathscr{O}_c(L_2) \subseteq \overline{\mathscr{O}_c}(L_1)$$
$$\Updownarrow$$
$$\overline{\mathscr{O}_c}(L_2) \subseteq \overline{\mathscr{O}_c}(L_1)$$

☞ Establishes an **order** relation between $\star$-congruence orbit closures.

Likeliness and genericity (structured)

If L_1 and L_2 have the same structure, we say that

$$\text{SKCF}(L_1) \text{ is more likely than } \text{SKCF}(L_2)$$
$$\Updownarrow$$
$$L_2 \in \overline{\mathscr{O}_c}(L_1)$$
$$\Updownarrow$$
$$\mathscr{O}_c(L_2) \subseteq \overline{\mathscr{O}_c}(L_1)$$
$$\Updownarrow$$
$$\overline{\mathscr{O}_c}(L_2) \subseteq \overline{\mathscr{O}_c}(L_1)$$

☞ Establishes an **order** relation between $\star$-congruence orbit closures.

Generic SKCFs/eigenstructures: the “most likely" ones, namely:

K is **generic** (structured) if there is no $\widetilde{K} \neq K$ (structured) with $K \in \overline{\mathscr{O}_c}(\widetilde{K})$.

Our main goal

To describe the generic eigenstructure(s) of:

- General (unstructured) $m \times n$ pencils of rank at most $r \leq \min\{m,n\}$.
- Structured $n \times n$ pencils with rank at most $r \leq \min\{m,n\}$.

Rank of a pencil: size of the largest non-identically zero minor (namely, the rank over $\mathbb{C}(\lambda)$).

Our main goal

To describe the generic eigenstructure(s) of:

- General (unstructured) $m \times n$ pencils of rank at most $r \leq \min\{m,n\}$.
- Structured $n \times n$ pencils with rank at most $r \leq \min\{m,n\}$.

Rank of a pencil: size of the largest non-identically zero minor (namely, the rank over $\mathbb{C}(\lambda)$).

☞ Includes the case $r = \min\{m,n\}$ (limit case).

Unstructured vs structured

Q: Are the generic eigenstructures of structured pencils just the ones of structured pencils which are compatible with the structure?

Unstructured vs structured

Q: Are the generic eigenstructures of structured pencils just the ones of structured pencils which are compatible with the structure?

A: NOT necessarily!

Unstructured vs structured

Q: Are the generic eigenstructures of structured pencils just the ones of structured pencils which are compatible with the structure?

A: NOT necessarily!

If L_1, L_2 have the same structure:

$$L_1 \in \overline{\mathscr{O}}(L_2) \not\Rightarrow L_1 \in \overline{\mathscr{O}_c}(L_2)$$

Unstructured vs structured

Q: Are the generic eigenstructures of structured pencils just the ones of structured pencils which are compatible with the structure?

A: NOT necessarily!

If L_1, L_2 have the same structure:

$$L_1 \in \overline{\mathcal{O}}(L_2) \not\Rightarrow L_1 \in \overline{\mathcal{O}_c}(L_2)$$

Example

$$L_1(\lambda) = \begin{bmatrix} \lambda - \lambda_1 & 0 & 0 \\ 0 & \lambda - \lambda_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad L_2(\lambda) = \begin{bmatrix} 0 & 0 & \lambda \\ 0 & 0 & 1 \\ \lambda & 1 & 0 \end{bmatrix}$$

are both symmetric and

$$L_1 \in \overline{\mathcal{O}}(L_2), \qquad \text{but} \qquad L_1 \notin \overline{\mathcal{O}_c}(L_2).$$

Outline

Generic eigenstructure of regular pencils

Theorem

The generic KCF of $n \times n$ pencils is:

$$\begin{bmatrix} \lambda - \lambda_1 & & & \\ & \lambda - \lambda_2 & & \\ & & \ddots & \\ & & & \lambda - \lambda_n \end{bmatrix},$$

with $\lambda_i \neq \lambda_j$ for $i \neq j$.

Generic eigenstructure of regular pencils

Theorem

The generic KCF of $n \times n$ pencils is:

$$\begin{bmatrix} \lambda - \lambda_1 & & & \\ & \lambda - \lambda_2 & & \\ & & \ddots & \\ & & & \lambda - \lambda_n \end{bmatrix},$$

with $\lambda_i \neq \lambda_j$ for $i \neq j$.

☞ There are infinitely many **generic** orbits, depending on the eigenvalues.

Outline

1. Basic setting
   - General (unstructured) pencils
   - Structured pencils
2. Regular pencils
3. Singular general (unstructured) pencils
4. Structured pencils (low-rank)
   - Palindromic pencils
   - $\top$−alternating pencils
   - Skew-symmetric pencils
   - Symmetric pencils
   - Hermitian pencils

Singular pencils

Theorem [Waterhouse'84, Demmel-Edelman'95]

The generic $n \times n$ singular pencils have the following KCFs:

$$\mathrm{diag}(R_j, R_{n-j-1}^\top),$$

for $j = 0, \ldots, n-1$.

J. Demmel, A. Edelman. The dimension of matrices (matrix pencils) with given Jordan (Kronecker) canonical forms. LAA 230 (1995) 61–87.

W. C. Waterhouse. The codimension of singular matrix pairs. LAA 57 (1984) 227—245.

Singular pencils

Theorem [Waterhouse'84, Demmel-Edelman'95]

The generic $n \times n$ singular pencils have the following KCFs:

$$\mathrm{diag}(R_j, R_{n-j-1}^\top),$$

for $j = 0, \ldots, n-1$.

J. Demmel, A. Edelman. The dimension of matrices (matrix pencils) with given Jordan (Kronecker) canonical forms. LAA 230 (1995) 61–87.

W. C. Waterhouse. The codimension of singular matrix pairs. LAA 57 (1984) 227—245.

☞ Generic rectangular pencils do not have eigenvalues!

Singular pencils

Theorem [Waterhouse'84, Demmel-Edelman'95]

The generic $n \times n$ singular pencils have the following KCFs:

$$\operatorname{diag}(R_j, R_{n-j-1}^\top),$$

for $j = 0, \ldots, n-1$.

J. Demmel, A. Edelman. The dimension of matrices (matrix pencils) with given Jordan (Kronecker) canonical forms. LAA 230 (1995) 61–87.

W. C. Waterhouse. The codimension of singular matrix pairs. LAA 57 (1984) 227—245.

☞ Generic rectangular pencils do not have eigenvalues!

☞ The number of generic orbits is n (**finite**).

Rectangular pencils

Theorem [Van Dooren'79] after Wilkinson and Wonham, [Demmel-Edelman'95]

The generic $m \times n$ pencils, with $n > m$, have the following KCF:

$$\mathrm{diag}(\overbrace{R_{\alpha+1},\ldots,R_{\alpha+1}}^{s},\overbrace{R_{\alpha},\ldots,R_{\alpha}}^{n-m-s}),$$

where $\alpha = \lfloor m/(n-m) \rfloor$ and $m \equiv s \bmod (n-m)$.

J. Demmel, A. Edelman. The dimension of matrices (matrix pencils) with given Jordan (Kronecker) canonical forms. LAA 230 (1995) 61–87.

P. Van Dooren. The generalized eigenstructure problem: Applications in linear systems theory. PhD thesis, Kath. Univ. Leuven, Leuven, Belgium, 1979.

Rectangular pencils

Theorem [Van Dooren'79] after Wilkinson and Wonham, [Demmel-Edelman'95]

The generic $m \times n$ pencils, with $n > m$, have the following KCF:

$$\operatorname{diag}(\overbrace{R_{\alpha+1}, \ldots, R_{\alpha+1}}^{s}, \overbrace{R_{\alpha}, \ldots, R_{\alpha}}^{n-m-s}),$$

where $\alpha = \lfloor m/(n-m) \rfloor$ and $m \equiv s \bmod (n-m)$.

J. Demmel, A. Edelman. The dimension of matrices (matrix pencils) with given Jordan (Kronecker) canonical forms. LAA 230 (1995) 61–87.

P. Van Dooren. The generalized eigenstructure problem: Applications in linear systems theory. PhD thesis, Kath. Univ. Leuven, Leuven, Belgium, 1979.

☞ There is **only one** generic orbit!

Rectangular pencils

Theorem [Van Dooren'79] after Wilkinson and Wonham, [Demmel-Edelman'95]

The generic $m \times n$ pencils, with $n > m$, have the following KCF:

$$\mathrm{diag}(\overbrace{R_{\alpha+1}, \ldots, R_{\alpha+1}}^{s}, \overbrace{R_{\alpha}, \ldots, R_{\alpha}}^{n-m-s}),$$

where $\alpha = \lfloor m/(n-m) \rfloor$ and $m \equiv s \bmod (n-m)$.

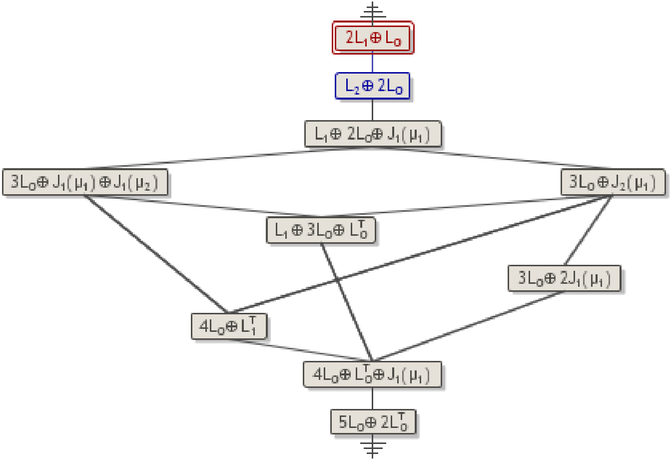
J. Demmel, A. Edelman. The dimension of matrices (matrix pencils) with given Jordan (Kronecker) canonical forms. LAA 230 (1995) 61–87.

P. Van Dooren. The generalized eigenstructure problem: Applications in linear systems theory. PhD thesis, Kath. Univ. Leuven, Leuven, Belgium, 1979.

☞ There is **only one** generic orbit!

☞ There is an analogous result for $m > n$.

The case $m = 2, n = 5$



Stratigraph:

https://www.umu.se/en/research/projects/stratigraph-and-mcs-toolbox/

Low-rank pencils

Rank of a pencil: size of the largest non-identically zero minor.

Theorem [DT-Dopico'08]

Let $1 \leq r \leq \min\{m,n\}-1$. Then, the generic KCFs of $m \times n$ pencils with rank at most r are

$$K_a(\lambda) = \mathrm{diag}(\overbrace{R_{\alpha+1},\dots,R_{\alpha+1}}^{s},\overbrace{R_{\alpha},\dots,R_{\alpha}}^{n-r-s},\overbrace{R^{\top}_{\beta+1},\dots,R^{\top}_{\beta+1}}^{t},\overbrace{R^{\top}_{\beta},\dots,R^{\top}_{\beta}}^{m-r-t}),$$

for $a = 0:r$, $\alpha = \lfloor a/(n-r)\rfloor, s \equiv a \bmod (n-r), \beta = \lfloor (r-a)/(m-r)\rfloor, t \equiv r-a \bmod (m-r)$.

FDT, FM. Dopico. A note on generic Kronecker orbits of matrix pencils with fixed rank. SIMAX 30 (2008) 491–496

Low-rank pencils

Rank of a pencil: size of the largest non-identically zero minor.

Theorem [DT-Dopico'08]

Let $1 \leq r \leq \min\{m,n\} - 1$. Then, the generic KCFs of $m \times n$ pencils with rank at most r are

$$K_a(\lambda) = \operatorname{diag}(\overbrace{R_{\alpha+1}, \ldots, R_{\alpha+1}}^{s}, \overbrace{R_{\alpha}, \ldots, R_{\alpha}}^{n-r-s}, \overbrace{R_{\beta+1}^{\top}, \ldots, R_{\beta+1}^{\top}}^{t}, \overbrace{R_{\beta}^{\top}, \ldots, R_{\beta}^{\top}}^{m-r-t}),$$

for $a = 0 : r$, $\alpha = \lfloor a/(n-r) \rfloor, s \equiv a \bmod (n-r), \beta = \lfloor (r-a)/(m-r) \rfloor, t \equiv r - a \bmod (m-r)$.

☞ There are $r+1$ different orbits!!

Low-rank pencils

Rank of a pencil: size of the largest non-identically zero minor.

Theorem [DT-Dopico'08]

Let $1 \leq r \leq \min\{m,n\}-1$. Then, the generic KCFs of $m \times n$ pencils with rank at most r are

$$K_a(\lambda) = \operatorname{diag}(\overbrace{R_{\alpha+1},\ldots,R_{\alpha+1}}^{s},\overbrace{R_{\alpha},\ldots,R_{\alpha}}^{n-r-s},\overbrace{R_{\beta+1}^{\top},\ldots,R_{\beta+1}^{\top}}^{t},\overbrace{R_{\beta}^{\top},\ldots,R_{\beta}^{\top}}^{m-r-t}),$$

for $a=0:r$, $\alpha = \lfloor a/(n-r) \rfloor, s \equiv a \bmod (n-r), \beta = \lfloor (r-a)/(m-r) \rfloor, t \equiv r-a \bmod (m-r)$.

☞ There are $r+1$ different orbits!! (just 1 when $r=\min\{m,n\}$ and $m \neq n$).

Low-rank pencils

Rank of a pencil: size of the largest non-identically zero minor.

Theorem [DT-Dopico'08]

Let $1 \leq r \leq \min\{m,n\} - 1$. Then, the generic KCFs of $m \times n$ pencils with rank at most r are

$$K_a(\lambda) = \operatorname{diag}(\overbrace{R_{\alpha+1},\ldots,R_{\alpha+1}}^{s},\overbrace{R_{\alpha},\ldots,R_{\alpha}}^{n-r-s},\overbrace{R_{\beta+1}^{\top},\ldots,R_{\beta+1}^{\top}}^{t},\overbrace{R_{\beta}^{\top},\ldots,R_{\beta}^{\top}}^{m-r-t}),$$

for $a = 0:r$, $\alpha = \lfloor a/(n-r) \rfloor, s \equiv a \bmod (n-r), \beta = \lfloor (r-a)/(m-r) \rfloor, t \equiv r - a \bmod (m-r)$.

☞ There are $r+1$ different orbits!! (just 1 when $r = \min\{m,n\}$ and $m \neq n$).
☞ When $r = m-1 = n-1$, then $s = t = 0$, $n-r-s = m-r-t = 1$ and $\alpha = a, \beta = n-a-1$, so the generic KCFs are $\operatorname{diag}(R_a, R_{n-a-1}^{\top})$ [Waterhouse'84].

Low-rank pencils (II)

$\mathscr{P}(m \times n)$: Set of $m \times n$ pencils.
$\mathscr{P}_r(m \times n)$: Set of $m \times n$ pencils with rank at most r.

Theorem [DT-Dopico'08]

$$\mathscr{P}_r(m \times n) = \bigcup_{0 \leq a \leq r} \overline{\mathscr{O}}(K_a).$$

FDT, FM. Dopico. A note on generic Kronecker orbits of matrix pencils with fixed rank. SIMAX 30 (2008) 491–496

(Also: $\overline{\mathscr{O}}(K_a) \not\subseteq \overline{\mathscr{O}}(K_{a'})$ when $a \neq a'$).

Low-rank pencils (II)

$\mathscr{P}(m \times n)$: Set of $m \times n$ pencils.
$\mathscr{P}_r(m \times n)$: Set of $m \times n$ pencils with rank at most r.

Theorem [DT-Dopico'08]

$$\mathscr{P}_r(m \times n) = \bigcup_{0 \le a \le r} \overline{\mathscr{O}}(K_a).$$

FDT, FM. Dopico. A note on generic Kronecker orbits of matrix pencils with fixed rank. SIMAX 30 (2008) 491–496

(Also: $\overline{\mathscr{O}}(K_a) \not\subseteq \overline{\mathscr{O}}(K_{a'})$ when $a \neq a'$).

Q: Are all sets $\overline{\mathscr{O}}(K_a)$, for $a = 0, \ldots, r$, of the same size?

Low-rank pencils (II)

$\mathscr{P}(m \times n)$: Set of $m \times n$ pencils.
$\mathscr{P}_r(m \times n)$: Set of $m \times n$ pencils with rank at most r.

Theorem [DT-Dopico'08]

$$\mathscr{P}_r(m \times n) = \bigcup_{0 \leq a \leq r} \overline{\mathscr{O}}(K_a).$$

FDT, FM. Dopico. A note on generic Kronecker orbits of matrix pencils with fixed rank. SIMAX 30 (2008) 491–496

(Also: $\overline{\mathscr{O}}(K_a) \not\subseteq \overline{\mathscr{O}}(K_{a'})$ when $a \neq a'$).

Q: Are all sets $\overline{\mathscr{O}}(K_a)$, for $a = 0, \ldots, r$, of the same dimension?

Low-rank pencils (III)

Theorem [DT-Dopico'08]

$$\dim \overline{\mathscr{O}}(K_a) = r(2m+n-r) - a(m-n).$$

Low-rank pencils (III)

Theorem [DT-Dopico'08]

$$\dim \overline{\mathscr{O}}(K_a) = r(2m+n-r) - a(m-n).$$

The largest orbit is

- $\mathscr{O}(K_0)$, where $K_0 = [\,\overbrace{0}^{n-r}\,| \bigoplus R_{\beta+1}^\top \oplus \bigoplus R_\beta^\top]$, if $m \geq n$.
- $\mathscr{O}(K_r)$, where $K_r = \begin{matrix} \\ m-r\{ \end{matrix} \begin{bmatrix} \bigoplus R_{\alpha+1} \oplus \bigoplus R_\alpha \\ 0 \end{bmatrix}$, if $m < n$.

Low-rank pencils (III)

Theorem [DT-Dopico'08]

$$\dim \overline{\mathscr{O}}(K_a) = r(2m+n-r) - a(m-n).$$

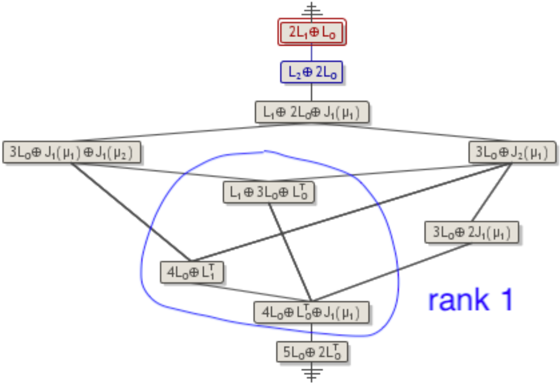
The largest orbit is

- $\mathscr{O}(K_0)$, where $K_0 = [\,\overbrace{\;0\;}^{n-r}\,| \bigoplus R_{\beta+1}^\top \oplus \bigoplus R_\beta^\top]$, if $m \geq n$.
- $\mathscr{O}(K_r)$, where $K_r = \begin{matrix} \\ m-r\{ \end{matrix} \begin{bmatrix} \bigoplus R_{\alpha+1} \oplus \bigoplus R_\alpha \\ 0 \end{bmatrix}$, if $m < n$.

Corollary [DT-Dopico'08]

$$\dim \mathscr{P}_r(m\times n) = \begin{cases} r(2m+n-r) & \text{if } m \geq n, \\ r(m+2n-r) & \text{if } m < n. \end{cases}$$

The case $m = 2, n = 5, r = 1$



Outline

1. Basic setting
   - General (unstructured) pencils
   - Structured pencils
2. Regular pencils
3. Singular general (unstructured) pencils
4. Structured pencils (low-rank)
   - Palindromic pencils
   - $\top-$alternating pencils
   - Skew-symmetric pencils
   - Symmetric pencils
   - Hermitian pencils

The full rank case

$\top-$palindromic: $A+\lambda A^\top$

Theorem [DT-Dopico'11]

The generic KCFs of $n\times n$ $\top-$palindromic pencils are

$$\mathrm{diag}(\lambda+\mu_1,\lambda+1/\mu_1,\ldots,\lambda+\mu_{n/2},\lambda+1/\mu_{n/2}), \qquad (n \text{ even}),$$
$$\mathrm{diag}(\lambda+\mu_1,\lambda+1/\mu_1,\ldots,\lambda+\mu_{(n-1)/2},\lambda+1/\mu_{(n-1)/2},\lambda+1), \qquad (n \text{ odd}),$$

with $0,\pm1\neq\mu_i\neq\mu_j$ for $i\neq j$.

FDT, FM. Dopico. The solution of the equation $XA+AX^T=0$ and its application to the theory of orbits. LAA 434 (2011) 44–67.

The full rank case

$\top-$palindromic: $A+\lambda A^\top$

Theorem [DT-Dopico'11]

The generic KCFs of $n\times n$ $\top-$palindromic pencils are

$$\begin{array}{lr} \operatorname{diag}(\lambda+\mu_1,\lambda+1/\mu_1,\ldots,\lambda+\mu_{n/2},\lambda+1/\mu_{n/2}), & (n\text{ even}),\\ \operatorname{diag}(\lambda+\mu_1,\lambda+1/\mu_1,\ldots,\lambda+\mu_{(n-1)/2},\lambda+1/\mu_{(n-1)/2},\lambda+1), & (n\text{ odd}), \end{array}$$

with $0,\pm1\neq\mu_i\neq\mu_j$ for $i\neq j$.

FDT, FM. Dopico. The solution of the equation $XA+AX^T=0$ and its application to the theory of orbits. LAA 434 (2011) 44–67.

The full rank case

$\top-$palindromic: $A+\lambda A^\top$

Theorem [DT-Dopico'11]

The generic KCFs of $n\times n$ $\top-$palindromic pencils are

$$\begin{array}{lr} \mathrm{diag}(\lambda+\mu_1,\lambda+1/\mu_1,\ldots,\lambda+\mu_{n/2},\lambda+1/\mu_{n/2}), & (n \text{ even}),\\ \mathrm{diag}(\lambda+\mu_1,\lambda+1/\mu_1,\ldots,\lambda+\mu_{(n-1)/2},\lambda+1/\mu_{(n-1)/2},\lambda+1), & (n \text{ odd}),\end{array}$$

with $0,\pm1\neq\mu_i\neq\mu_j$ for $i\neq j$.

FDT, FM. Dopico. The solution of the equation $XA+AX^T=0$ and its application to the theory of orbits. LAA 434 (2011) 44–67.

The full rank case

$\top-$palindromic: $A+\lambda A^\top$

Theorem [DT-Dopico'11]

The generic KCFs of $n\times n$ $\top-$palindromic pencils are

$$\begin{array}{lr} \mathrm{diag}(\lambda+\mu_1,\lambda+1/\mu_1,\ldots,\lambda+\mu_{n/2},\lambda+1/\mu_{n/2}), & (n \text{ even}),\\ \mathrm{diag}(\lambda+\mu_1,\lambda+1/\mu_1,\ldots,\lambda+\mu_{(n-1)/2},\lambda+1/\mu_{(n-1)/2},\lambda+1), & (n \text{ odd}), \end{array}$$

with $0,\pm1\neq\mu_i\neq\mu_j$ for $i\neq j$.

FDT, FM. Dopico. The solution of the equation $XA+AX^T=0$ and its application to the theory of orbits. LAA 434 (2011) 44–67.

The full rank case

$\top-$palindromic: $A+\lambda A^\top$

Theorem [DT-Dopico'11]

The generic KCFs of $n\times n$ $\top-$palindromic pencils are

$$\begin{array}{lr} \mathrm{diag}(\lambda+\mu_1,\lambda+1/\mu_1,\ldots,\lambda+\mu_{n/2},\lambda+1/\mu_{n/2}), & (n \text{ even}), \\ \mathrm{diag}(\lambda+\mu_1,\lambda+1/\mu_1,\ldots,\lambda+\mu_{(n-1)/2},\lambda+1/\mu_{(n-1)/2},\lambda+1), & (n \text{ odd}), \end{array}$$

with $0,\pm1\neq\mu_i\neq\mu_j$ for $i\neq j$.

FDT, FM. Dopico. The solution of the equation $XA+AX^T=0$ and its application to the theory of orbits. LAA 434 (2011) 44–67.

☞ There are **infinitely many** generic orbits (depend on the eigenvalues).

Generic eigenstructures for low-rank pencils

$\mathrm{rev}\,(A+\lambda B) := B+\lambda A.$

$$M_d^\sharp(\lambda) := \begin{bmatrix} 0 & R_d(\lambda)^\top \\ \mathrm{rev}\,R_d(\lambda) & 0 \end{bmatrix}_{(2d+1)\times(2d+1)}.$$

Theorem [DT'18]

The generic SKCF of $\top-$palindromic $n\times n$ pencils with rank at most $0\le r<n$ is

$$K^e_{\mathrm{pal}}(\lambda) := \mathrm{diag}(\overbrace{M^\sharp_{\alpha+1}(\lambda),\ldots,M^\sharp_{\alpha+1}(\lambda)}^{s},\overbrace{M^\sharp_{\alpha}(\lambda),\ldots,M^\sharp_{\alpha}(\lambda)}^{n-r-s}), \quad (n \text{ even}),$$
$$K^o_{\mathrm{pal}}(\lambda) := \mathrm{diag}(1+\lambda, K^e_{\mathrm{pal}}(\lambda)), \quad (n \text{ odd}),$$

where $r/2=(n-r)\alpha+s$ (Euclidean division of $r/2$ by $n-r$).

FDT. A geometric description of the set of palindromic and alternating matrix pencils with bounded rank. SIMAX 39 (2018) 1116–1134.

Generic eigenstructures for low-rank pencils

$\operatorname{rev}(A+\lambda B) := B+\lambda A.$

$$M_d^\sharp(\lambda) := \begin{bmatrix} 0 & R_d(\lambda)^\top \\ \operatorname{rev} R_d(\lambda) & 0 \end{bmatrix}_{(2d+1)\times(2d+1)}.$$

Theorem [DT'18]

The generic SKCF of $\top-$palindromic $n\times n$ pencils with rank at most $0\le r<n$ is

$$\begin{aligned} K_{\rm pal}^e(\lambda) &:= \operatorname{diag}(\overbrace{M_{\alpha+1}^\sharp(\lambda),\dots,M_{\alpha+1}^\sharp(\lambda)}^{s},\overbrace{M_\alpha^\sharp(\lambda),\dots,M_\alpha^\sharp(\lambda)}^{n-r-s}), \quad (n \text{ even}),\\ K_{\rm pal}^o(\lambda) &:= \operatorname{diag}(1+\lambda, K_{\rm pal}^e(\lambda)), \quad (n \text{ odd}),\end{aligned}$$

where $r/2=(n-r)\alpha+s$ (Euclidean division of $r/2$ by $n-r$).

FDT. A geometric description of the set of palindromic and alternating matrix pencils with bounded rank. SIMAX 39 (2018) 1116–1134.

☞ There is only one generic orbit.

Dimension

$\mathrm{Pal}_r(n)$: Set of $\top-$palindromic $n \times n$ pencils $(A + \lambda A^\top)$ with rank at most r.

Theorem [DT'18]

$\mathrm{Pal}_r(n)$ $(r < n)$ is an irreducible algebraic set of dimension

$$\dim \mathrm{Pal}_r(n) = \begin{cases} \frac{r}{2} \cdot (3n - r), & \text{if } r \text{ is even}, \\ \frac{r-1}{2} \cdot (3n - r) + n, & \text{if } r \text{ is odd}. \end{cases}$$

Moreover

$$\mathrm{Pal}_r(n) = \begin{cases} \overline{\mathscr{O}}_c(K^e_{\mathrm{pal}}), & \text{if } n \text{ is even}, \\ \overline{\mathscr{O}}_c(K^o_{\mathrm{pal}}), & \text{if } n \text{ is odd}. \end{cases}$$

FDT. A geometric description of the set of palindromic and alternating matrix pencils with bounded rank. SIMAX 39 (2018) 1116–1134.

Dimension

$\mathrm{Pal}_r(n)$: Set of $\top-$palindromic $n \times n$ pencils $(A + \lambda A^\top)$ with rank at most r.

Theorem [DT'18]

$\mathrm{Pal}_r(n)$ $(r < n)$ is an irreducible algebraic set of dimension

$$\dim \mathrm{Pal}_r(n) = \begin{cases} \frac{r}{2} \cdot (3n - r), & \text{if } r \text{ is even,} \\ \frac{r-1}{2} \cdot (3n - r) + n, & \text{if } r \text{ is odd.} \end{cases}$$

Moreover

$$\mathrm{Pal}_r(n) = \begin{cases} \overline{\mathscr{O}}_c(K^e_{\mathrm{pal}}), & \text{if } n \text{ is even,} \\ \overline{\mathscr{O}}_c(K^o_{\mathrm{pal}}), & \text{if } n \text{ is odd.} \end{cases}$$

FDT. A geometric description of the set of palindromic and alternating matrix pencils with bounded rank. SIMAX 39 (2018) 1116–1134.

☞ Similar results for $\top-$anti-palindromic $(A - \lambda A^\top)$.

The set $\mathrm{Even}_r(n)$

$\mathrm{Even}_r(n)$: Set of $\top$−even $n\times n$ pencils ($A+\lambda B^\top$, $A^\top = A, B^\top = -B$) with rank at most r

$$M_d^\flat(\lambda) := \begin{bmatrix} 0 & R_d(\lambda)^\top \\ R_d(-\lambda) & 0 \end{bmatrix}_{(2d+1)\times(2d+1)}.$$

$$\begin{array}{lcll} K^e_{\mathrm{even}}(\lambda): & = & \mathrm{diag}(\overbrace{M^\flat_{\alpha+1}(\lambda),\dots,M^\flat_{\alpha+1}(\lambda)}^{s},\overbrace{M^\flat_{\alpha}(\lambda),\dots,M^\flat_{\alpha}(\lambda)}^{n-r-s}), & \frac{r}{2} = (n-r)\alpha+s. \\ K^o_{\mathrm{even}}(\lambda): & = & \mathrm{diag}(1,K^e_{\mathrm{even}}(\lambda)), & \frac{r-1}{2} = (n-r)\alpha+s. \end{array}$$

Theorem [DT'18]

$\mathrm{Even}_r(n)$ ($r<n$) is an irreducible algebraic set with dimension

$$\dim \mathrm{Even}_r = \begin{cases} \frac{r}{2}\cdot(3n-r), & \text{if } r \text{ is even,} \\ \frac{r-1}{2}\cdot(3n-r)+n, & \text{if } r \text{ is odd.} \end{cases}$$

Moreover,

$$\mathrm{Even}_r(n) = \begin{cases} \overline{\mathscr{O}}_c(K^e_{\mathrm{even}}) & \text{if } n \text{ is even,} \\ \overline{\mathscr{O}}_c(K^o_{\mathrm{even}}), & \text{if } n \text{ is odd.} \end{cases}$$

The set $\mathrm{Odd}_r(n)$

$\mathrm{Odd}_r(n)$: Set of $\top-$odd $n\times n$ pencils ($A+\lambda B, A^\top=-A, B^\top=B$) with rank at most r.

☞ Similar result to the one for $\mathrm{Even}_r(n)$.

☞ Are immediate consequences of the results for $\top-$palindromic and $\top-$anti-palindromic using **Cayley transforms**:

$$\mathscr{C}_{-1}(A_0+\lambda A_1)=A_0-A_1+\lambda(A_0+A_1),\qquad \mathscr{C}_{+1}(A_0+\lambda A_1)=A_0+A_1+\lambda(A_1-A_0),$$

since

- $\mathscr{C}_{+1}:\mathrm{Pal}_r(n)\to\mathrm{Even}_r(n)$ is an isomorphism of algebraic sets ($\mathscr{C}_{+1}^{-1}=\frac{1}{2}\mathscr{C}_{-1}$).
- $\mathscr{C}_{+1}(\mathscr{O}_c(K^e_{\mathrm{even}}))=\mathscr{O}_c(K^e_{\mathrm{pal}})$ and $\mathscr{C}_{+1}(\mathscr{O}_c(K^o_{\mathrm{even}}))=\mathscr{O}_c(K^o_{\mathrm{pal}})$.
- $A_0+\lambda A_1$ is $\top-$odd $\Leftrightarrow$ $A_1+\lambda A_0$ is $\top-$even.

The set $\mathrm{Skew}_r(n)$

$\mathrm{Skew}_r(n)$: Set of skew-symmetric $n \times n$ pencils ($A + \lambda B, A^\top = -A, B^\top = -B$) with rank at most r.

☞ The rank of a skew-symmetric pencil is even.

$$M_d^\dagger(\lambda) := \begin{bmatrix} 0 & -R_d(-\lambda)^\top \\ R_d(-\lambda) & 0 \end{bmatrix}_{(2d+1)\times(2d+1)}.$$

Theorem [Dopico-Dmytryshyn'18]

For $2w < n$:

$$\mathrm{Skew}_{2w}(n) = \overline{\mathscr{O}}_c(\mathrm{diag}(\overbrace{M_{\alpha+1}^\dagger, \ldots, M_{\alpha+1}^\dagger}^{s}, \overbrace{M_\alpha^\dagger, \ldots, M_\alpha^\dagger}^{n-2w-s})),$$

where $\alpha = \lfloor w/(n-2w) \rfloor$ and $s \equiv w \bmod (n-2w)$.

A. Dmytryshyn, FM. Dopico. Generic skew-symmetric matrix polynomials with fixed rank and fixed odd grade. LAA 536 (2018) 1–18.

The set $\mathrm{Skew}_r(n)$

$\mathrm{Skew}_r(n)$: Set of skew-symmetric $n \times n$ pencils ($A + \lambda B, A^\top = -A, B^\top = -B$) with rank at most r.

☞ The rank of a skew-symmetric pencil is even.

$$M_d^\dagger(\lambda) := \begin{bmatrix} 0 & -R_d(-\lambda)^\top \\ R_d(-\lambda) & 0 \end{bmatrix}_{(2d+1)\times(2d+1)}.$$

Theorem [Dopico-Dmytryshyn'18]

For $2w < n$:

$$\mathrm{Skew}_{2w}(n) = \overline{\mathscr{O}}_c(\mathrm{diag}(\overbrace{M_{\alpha+1}^\dagger, \dots, M_{\alpha+1}^\dagger}^{s}, \overbrace{M_\alpha^\dagger, \dots, M_\alpha^\dagger}^{n-2w-s})),$$

where $\alpha = \lfloor w/(n-2w) \rfloor$ and $s \equiv w \mod (n-2w)$.

☞ Again, only one generic orbit (no eigenvalues).

The set $\mathrm{Skew}_r(n)$

$\mathrm{Skew}_r(n)$: Set of skew-symmetric $n \times n$ pencils ($A + \lambda B, A^\top = -A, B^\top = -B$) with rank at most r.

☞ The rank of a skew-symmetric pencil is even.

$$M_d^\dagger(\lambda) := \begin{bmatrix} 0 & -R_d(-\lambda)^\top \\ R_d(-\lambda) & 0 \end{bmatrix}_{(2d+1)\times(2d+1)}.$$

Theorem [Dopico-Dmytryshyn'18]

For $2w < n$:

$$\mathrm{Skew}_{2w}(n) = \overline{\mathscr{O}}_c(\mathrm{diag}(\overbrace{M_{\alpha+1}^\dagger, \ldots, M_{\alpha+1}^\dagger}^{s}, \overbrace{M_\alpha^\dagger, \ldots, M_\alpha^\dagger}^{n-2w-s})),$$

where $\alpha = \lfloor w/(n-2w) \rfloor$ and $s \equiv w \bmod (n-2w)$.

☞ Again, only one generic orbit (no eigenvalues).

☞ If n is odd, $\mathrm{Skew}_{n-1}(n) = \mathrm{Skew}(n)$.

The generic low rank symmetric eigenstructures

$$M_d(\lambda) := \begin{bmatrix} 0 & R_d(\lambda)^\top \\ R_d(\lambda) & 0 \end{bmatrix}_{(2d+1)\times(2d+1)}.$$

Theorem [DT-Dmytryshyn-Dopico'20]

The generic SKCF of $n \times n$ symmetric pencils with rank at most r $(1 \leq r \leq n-1)$ are:

$$K_a^s(\lambda) := \operatorname{diag}(\underbrace{M_{\alpha+1},\ldots,M_{\alpha+1}}_{s},\underbrace{M_\alpha,\ldots,M_\alpha}_{n-r-s},\lambda-\mu_1,\ldots,\lambda-\mu_{r-2a}),$$

for $a = 0,1,\ldots,\lfloor \frac{r}{2} \rfloor$, where $a = (n-r)\alpha + s$, and $\mu_1,\ldots,\mu_{r-2a}$ are arbitrary complex numbers (different from each other).

FDT, A. Dmytryshyn, F.M. Dopico. Generic symmetric matrix pencils with bounded rank. J. Spectr. Theor. 10 (2020) 905-926

The generic low rank symmetric eigenstructures

$$M_d(\lambda) := \begin{bmatrix} 0 & R_d(\lambda)^\top \\ R_d(\lambda) & 0 \end{bmatrix}_{(2d+1)\times(2d+1)}.$$

Theorem [DT-Dmytryshyn-Dopico'20]

The generic SKCF of $n \times n$ symmetric pencils with rank at most r $(1 \leq r \leq n-1)$ are:

$$K_a^s(\lambda) := \operatorname{diag}(\underbrace{M_{\alpha+1}, \ldots, M_{\alpha+1}}_{s}, \underbrace{M_{\alpha}, \ldots, M_{\alpha}}_{n-r-s}, \lambda - \mu_1, \ldots, \lambda - \mu_{r-2a}),$$

for $a = 0, 1, \ldots, \lfloor \frac{r}{2} \rfloor$, where $a = (n-r)\alpha + s$, and $\mu_1, \ldots, \mu_{r-2a}$ are arbitrary complex numbers (different from each other).

FDT, A. Dmytryshyn, F.M. Dopico. Generic symmetric matrix pencils with bounded rank. J. Spectr. Theor. 10 (2020) 905-926

☞ The generic low rank symmetric pencils have eigenvalues!!!

The generic eigenstructures of Hermitian pencils

Theorem [DT-Dmytryshyn-Dopico'22]

The generic SKCFs of Hermitian $n \times n$ pencils are:

$$\operatorname{diag}\left(\begin{bmatrix} 0 & \lambda - \overline{\mu_1} \\ \lambda - \mu_1 & \end{bmatrix}, \ldots, \begin{bmatrix} 0 & \lambda - \overline{\mu_d} \\ \lambda - \mu_d & \end{bmatrix}, \lambda - a_1, \ldots, \lambda - a_c, -\lambda + a_{c+1}, \ldots, -\lambda + a_{n-2d}\right),$$

where $a_1, \ldots, a_{n-2d} \in \mathbb{R}$, $\mu_1, \ldots, \mu_d \in \mathbb{C} \setminus \mathbb{R}$ have positive imaginary part, $a_i \neq a_j$, and $\mu_i \neq \mu_j$, for $i \neq j$.

FDT, A. Dmytryshyn, FM. Dopico. Generic eigenstructures of Hermitian pencils. Submitted

Generic eigenstructures of low-rank Hermnitian pencils

Theorem [DT-Dmytrsyhyn-Dopico'22]

The generic SKCFs of Hermitian $n \times n$ pencils with rank at most r ($0 \leq r \leq n-1$) are:

$$\operatorname{diag}(\overbrace{M_{\alpha+1},\ldots,M_{\alpha+1}}^{s},\overbrace{M_{\alpha},\ldots,M_{\alpha}}^{n-r-s},\lambda-a_1,\ldots,\lambda-a_c,-\lambda+a_{c+1},\ldots,-\lambda+a_{r-2d}),$$

where $a_1,\ldots,a_{r-2d}\in\mathbb{R}, a_i\neq a_j$ for $i\neq j$, and $c=0,1,\ldots,r-2d$.

FDT, A. Dmytryshyn, FM. Dopico. Generic eigenstructures of Hermitian pencils. Submitted

Generic eigenstructures of low-rank Hermnitian pencils

Theorem [DT-Dmytrsyhyn-Dopico'22]

The generic SKCFs of Hermitian $n \times n$ pencils with rank at most r $(0 \leq r \leq n-1)$ are:

$$\mathrm{diag}(\overbrace{M_{\alpha+1},\ldots,M_{\alpha+1}}^{s},\overbrace{M_{\alpha},\ldots,M_{\alpha}}^{n-r-s},\lambda-a_1,\ldots,\lambda-a_c,-\lambda+a_{c+1},\ldots,-\lambda+a_{r-2d}),$$

where $a_1,\ldots,a_{r-2d}\in\mathbb{R}, a_i\neq a_j$ for $i\neq j$, and $c=0,1,\ldots,r-2d$.

FDT, A. Dmytryshyn, FM. Dopico. Generic eigenstructures of Hermitian pencils. Submitted

☞ Generically, all eigenvalues (if any) are real!!!


THANK YOU!!!!