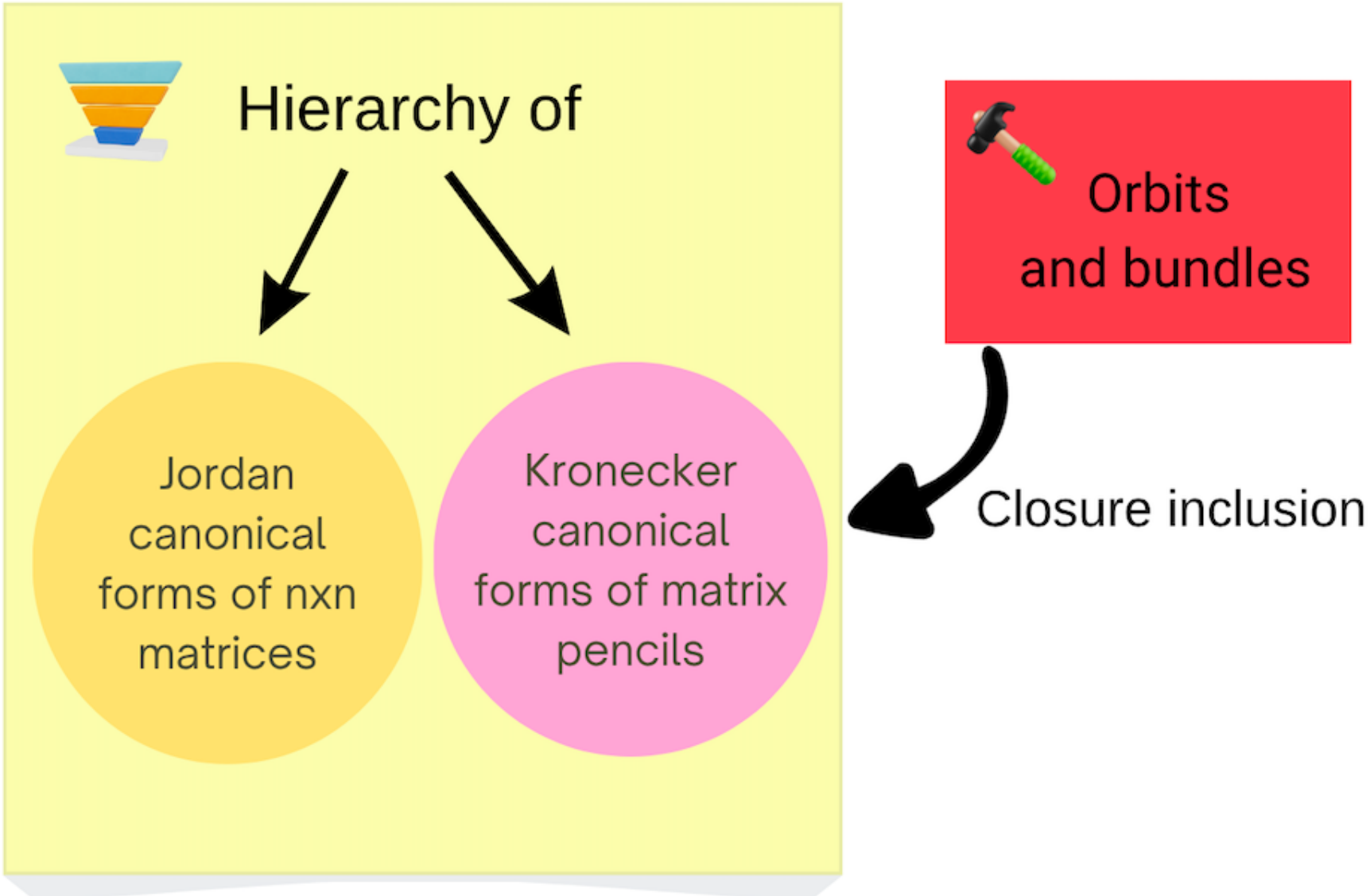
uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

New results on the inclusion of orbit and bundle closures of matrices and matrix pencils

Fernando De Terán

NA seminar (Oxford), June 4, 2026

What is this talk about?



Outline

Stratification of matrix orbits and bundle closures

Orbits and closures of matrix pencils

Orbit and closures of structured matrix pencils

Real matrices

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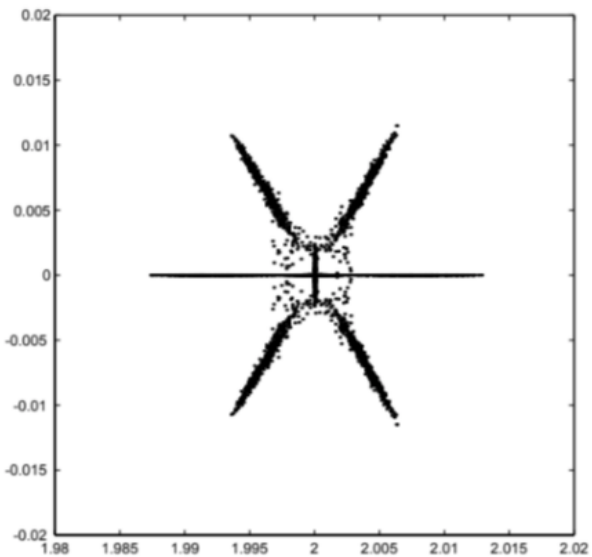
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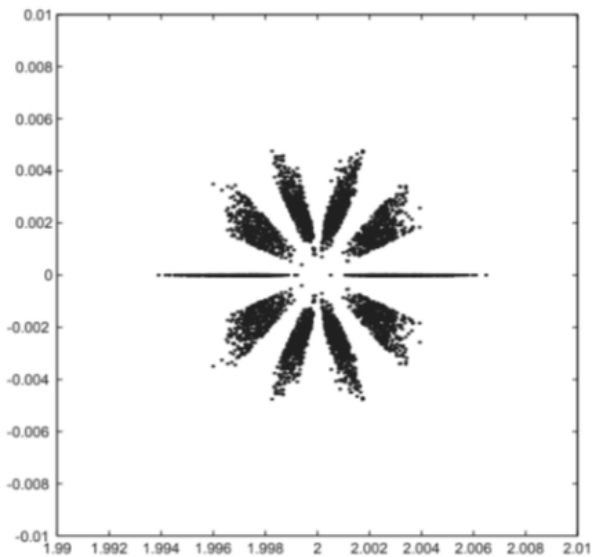
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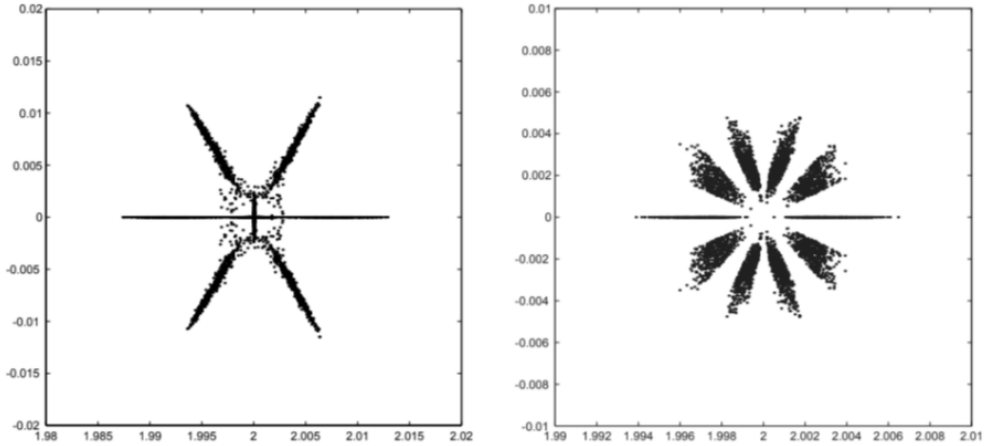
Example [Edelman-Elmroth-Kågström, SIMAX, 1999]: Perturbing $J_3(0) \oplus J_2(0)$:

- Evals of 1000 (real) normally distributed random perturbations of size $\sqrt{\varepsilon}$ (left).
- Filtered evals of 50.000 (real) normally distributed random perturbations of size $\sqrt{\varepsilon}$ (right).





Explanation of the pictures



Theorem [Lidskii'65, Vishyk-Lyusternik'60]

If JCF(A) has a Jordan block $J_k(\lambda)$ then, for most perturbations B, there are k eigenvalues of $A+\varepsilon B$ of the form

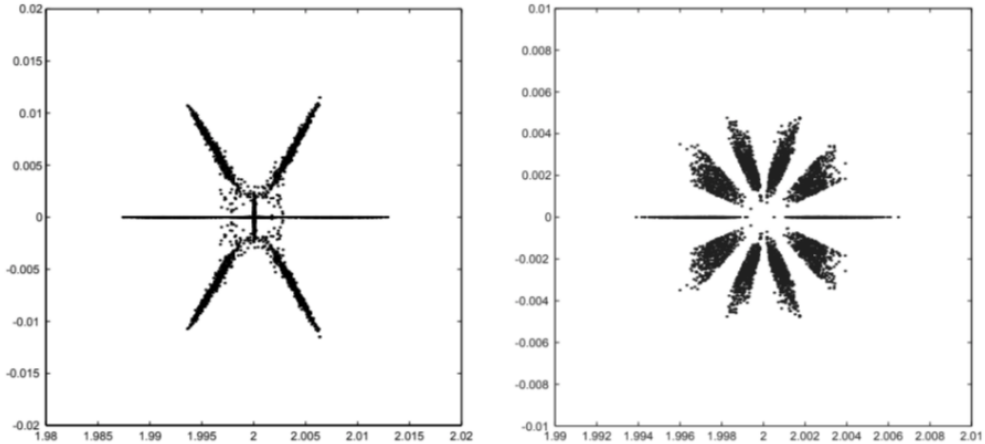
$$\lambda_j(\varepsilon) = \lambda + \xi_j \varepsilon^{1/k} + o(\varepsilon^{1/k}), \quad j = 1, \ldots, k,$$

where ξ_j, for $j = 1, \ldots, k$, are the kth roots of some $\xi \in \mathbb{C}$.

V. B. Lidskii. Perturbation theory of non-conjugate operators.
U.S.S.R. Comput. Math. and Math. Phys. 1 (1965), pp. 73-85.

M. I. Vishik and L. A. Lyusternik. The solution of problems on perturbation in the case of matrices and selfconjugate and non-selfconjugate differential equations.
Uspeshki mat. Nauk. 15 (1960), pp. 3-80.

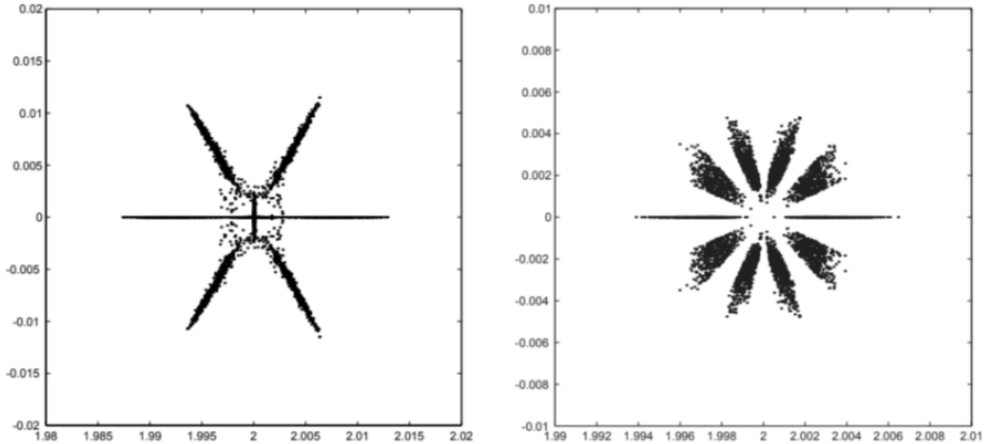
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☞ There are arbitrarily small perturbations of $J_3(0) \oplus J_2(0)$ whose JCF is $J_5(0)$:

$$J_\varepsilon := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim J_5(0)$$

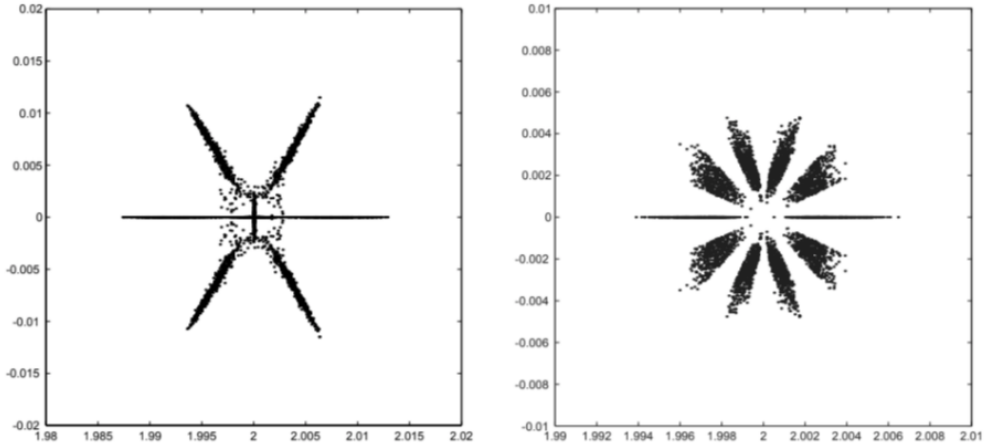
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☞ $J_3(0) \oplus J_2(0) \in \overline{\mathscr{O}}(J_5(0))$, with

$\mathscr{O}(A) := \{\text{Matrices with the same JCF as } A\}$ (similarity orbit)

Similarity orbits: closure inclusion

$\mathscr{O}(A) = \{P^{-1}AP : \ P \text{ invertible}\}$ (similarity orbit).

$B \in \overline{\mathscr{O}}(A)$: JCF($A$) **dominates** JCF($B$).

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☞ The “dominance" of JCFs corresponds to **orbit closure** inclusions (which is a partial **order** relation).

☞ If $B \in \overline{\mathscr{O}}(A)$, then A, B have the same eigenvalues.

Characterization of orbit closure inclusion

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$$\begin{array}{lcll} J_{1,\lambda} & := & (m_{11}(\lambda), m_{12}(\lambda), \ldots), & \text{with } m_{11}(\lambda) \geq m_{12}(\lambda) \geq \cdots \\ J_{2,\lambda} & := & (m_{21}(\lambda), m_{22}(\lambda), \ldots), & \text{with } m_{21}(\lambda) \geq m_{22}(\lambda) \geq \cdots \end{array}$$

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Theorem [Den Boer-Thijsse'80],[Markus-Parilis'83]

$\overline{\mathscr{O}}(J_1) \subseteq \overline{\mathscr{O}}(J_2) \Leftrightarrow J_{1,\lambda} \prec J_{2,\lambda}$, for all $\lambda \in \mathbb{C}$.

$\prec$: Majorization: $\sum_{j=1}^{k} m_{1j}(\lambda) \leq \sum_{j=1}^{k} m_{2j}(\lambda)$ and $\sum_{j=1}^{\infty} m_{1j}(\lambda) = \sum_{j=1}^{\infty} m_{2j}(\lambda)$, ,

for all $k = 1, 2, \ldots$

Example

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- $\boxed{\overline{\mathscr{O}}(J_3(0) \oplus J_2(0)) \subseteq \overline{\mathscr{O}}(J_5(0))}$ Since: $(3,2) \prec (5,0)$

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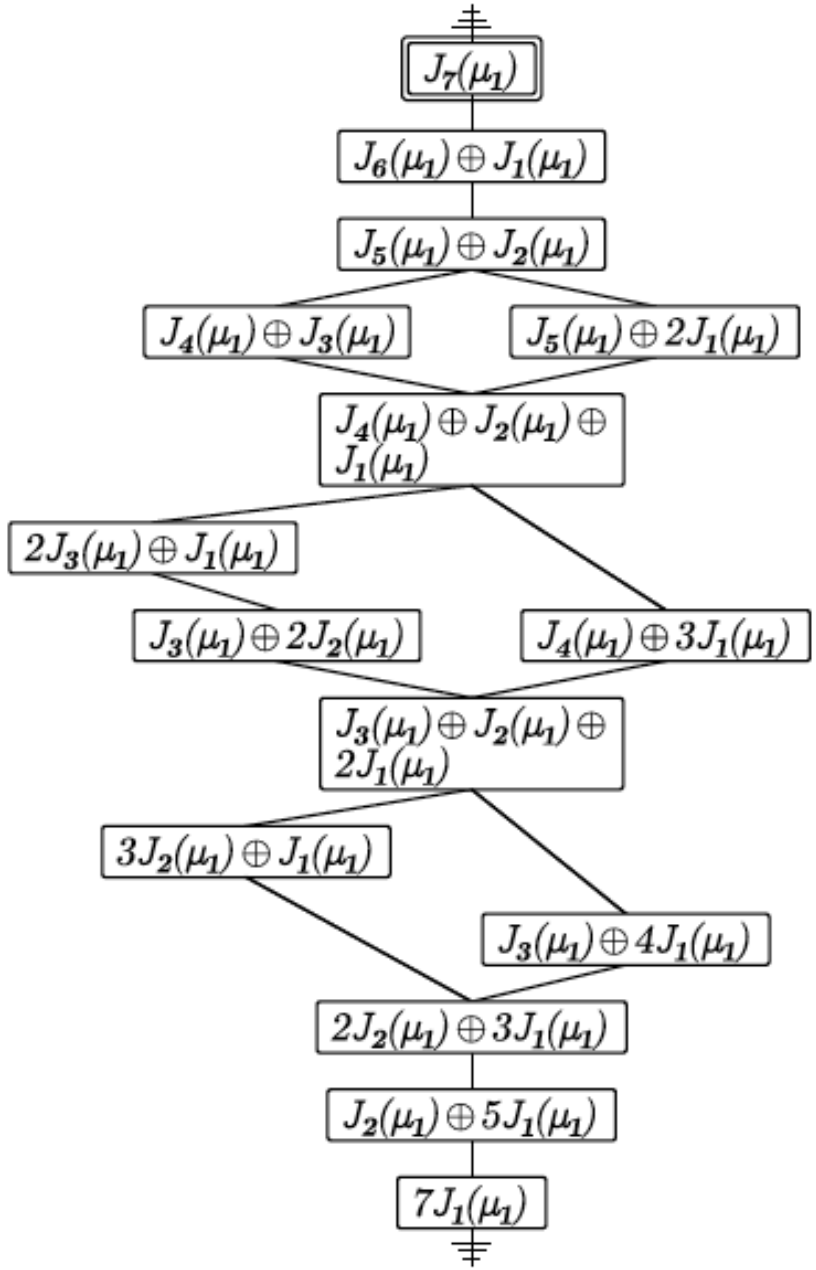
- Set:

$$\left.\begin{array}{l} J_1 = J_3(0) \oplus J_3(0) \oplus J_2(0) \oplus J_2(1) \oplus J_2(1) \\ J_2 = J_4(0) \oplus J_2(0) \oplus J_2(0) \oplus J_3(1) \oplus J_1(1) \end{array}\right\} \quad \boxed{\overline{\mathscr{O}}(J_1) \subseteq \overline{\mathscr{O}}(J_2)}$$

Since: $(3,3,2) \prec (4,2,2)$ and $(2,2) \prec (3,1)$

Stratification

Full Hasse diagram of the inclusion relation between orbit closures of 7×7 matrices.
(`Stratigraph 4.0`)



Covering

An **edge** between J_1 and J_2 in the Hasse diagram means:

- $\overline{\mathscr{O}}(J_1) \subseteq \overline{\mathscr{O}}(J_2)$ and
- there is **no** J_3 s. t. $\overline{\mathscr{O}}(J_1) \subsetneq \overline{\mathscr{O}}(J_3) \subsetneq \overline{\mathscr{O}}(J_2)$

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☞A characterization for the **covering** is provided in



A. Edelman, E. Elmroth, B. Kågström.
SIAM J. Matrix Anal. Appl. 20-3 (1999) 667–699.

in terms of “minimal" changes in the JCF.

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The bundle of $A \in \mathbb{C}^{n\times n}$, $\mathscr{B}(A)$, is the set of matrices with the same JCF as A, up to the specific values of the eigenvalues.

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If $\mathrm{JCF}(A) = J_3(0) \oplus J_2(0) \oplus J_4(1) \oplus J_1(1) \oplus J_5(-1)$, then

$$\mathscr{B}(A) = P^{-1}(J_3(\mu_1) \oplus J_2(\mu_1) \oplus J_4(\mu_2) \oplus J_1(\mu_2) \oplus J_5(\mu_3))P, \quad \mu_1 \neq \mu_2 \neq \mu_3 \neq \mu_1.$$

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☞ A bundle is an infinite union of orbits.

Inclusion relationship between bundle closures (I)

Coalescence of e-vals:

$$J_{k_1}(\mu_1) \oplus J_{k_2}(\mu_2) \quad = \quad \left[\begin{array}{ccccc|ccccc} \mu_1 & 1 & & & & & & & & \\ & \mu_1 & \ddots & & & & & & & \\ & & \ddots & 1 & & & & & & \\ & & & \mu_1 & & & & & & \\ \hline & & & & & \mu_2 & 1 & & & \\ & & & & & & \mu_2 & \ddots & & \\ & & & & & & & \ddots & 1 & \\ & & & & & & & & \mu_2 \end{array}\right]$$

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$$\boxed{J_{k_1+k_2}(\mu)\in\overline{\mathscr{B}}\left(J_{k_1}(\mu_1)\oplus J_{k_2}(\mu_2)\right)}$$

Inclusion relationship between bundle closures (II)

Coalescence of eigenvalues: **Add up** the corresponding entries in the Segre characteristic.

Example:

$J_5(\mu_1) \oplus J_3(\mu_1) \oplus J_2(\mu_1) \oplus J_4(\mu_2) \oplus J_1(\mu_2) \rightsquigarrow J_9(\mu) \oplus J_4(\mu) \oplus J_2(\mu)$,

since $(5,3,2)+(4,1,0)=(9,4,2)$.

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Theorem [Edelman-Elmroth-Kågström'99]

If A_2 has at least as many different e-vals as A_1, then $\overline{\mathscr{B}}(A_1) \subseteq \overline{\mathscr{B}}(A_2)$ iff there is some $\widetilde{A}_2$ obtained from A_2 by coalescing e-vals, and $\mathrm{Segre}(A_1,\lambda) \prec \mathrm{Segre}(\widetilde{A}_2,\lambda)$, for all $\lambda \in \mathbb{C}$.

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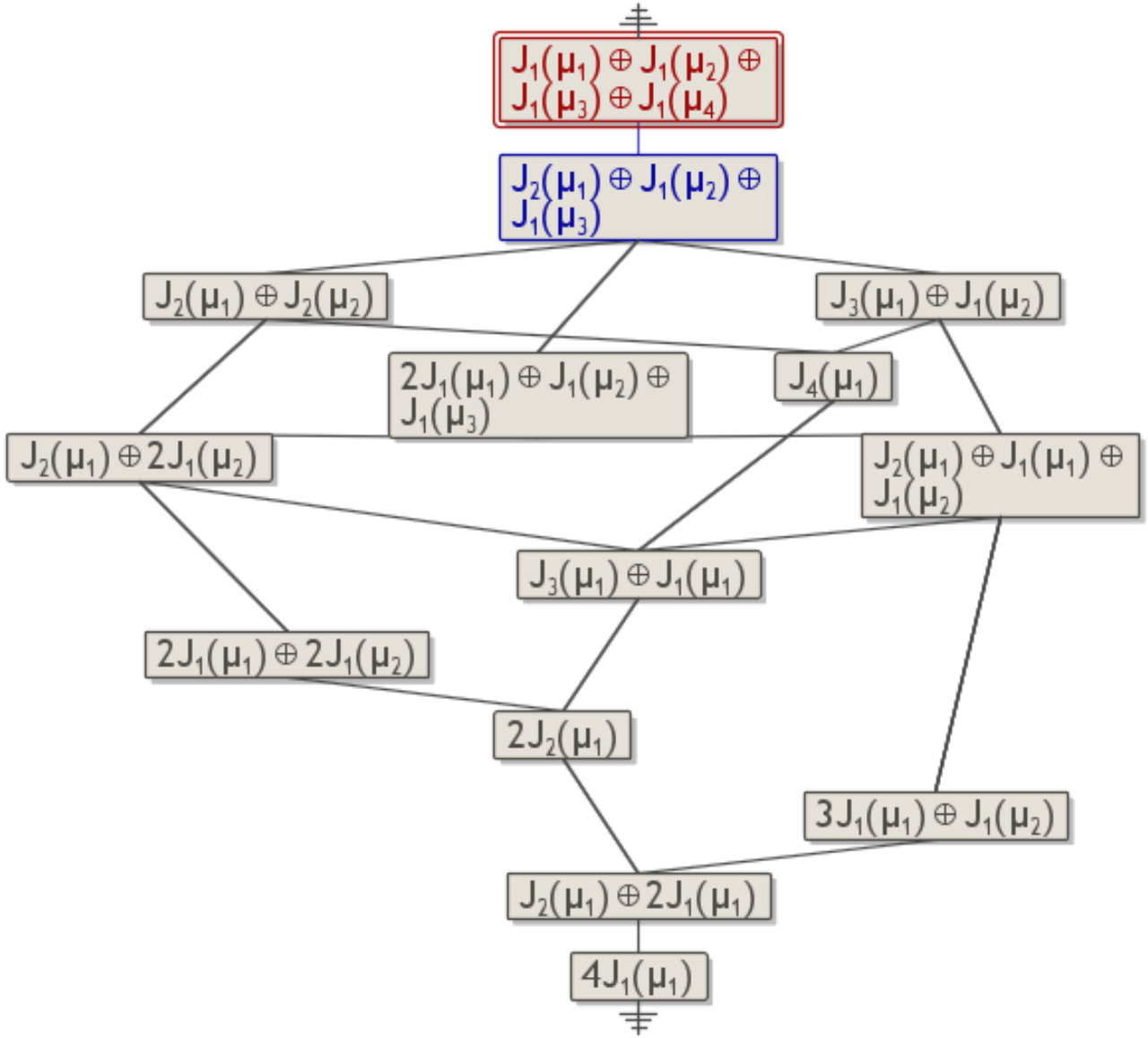
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Stratification of bundle closures

Full Hasse diagram of the inclusion relation between bundle closures of 4×4 matrices.
(`Stratigraph 4.0`)



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Orbit and closures of structured matrix pencils

Real matrices

Matrices vs Matrix Pencils

Standard eigenvalue problem: $A\mathbf{v} = \lambda\mathbf{v}$ ($A \in \mathbb{C}^{n\times n}, \mathbf{v} \in \mathbb{C}^n, \lambda \in \mathbb{C}$).

Generalized eigenvalue problem (GEP): $A\mathbf{v} = \lambda B\mathbf{v}$
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If $P \in \mathbb{C}^{m\times m}, Q \in \mathbb{C}^{n\times n}$ are invertible:

$$B\mathbf{x}' = A\mathbf{x} \Leftrightarrow (PBQ)(Q^{-1}\mathbf{x})' = (PAQ)(Q^{-1}\mathbf{x}) \overset{\mathbf{y}=Q^{-1}\mathbf{x}}{\Longleftrightarrow} (PBQ)\mathbf{y}' = (PAQ)\mathbf{y}.$$

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Matrices vs Matrix Pencils

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☞ **Spectral information:** Set of invariants under strict equivalence.

Canonical forms

	Relation	Canonical form
SEP	Similarity: $P^{-1}AP$	Jordan canonical form
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KCF: Four types of blocks:

- **Blocks associated with finite evals** (μ):

  $J_k(\mu) := \begin{bmatrix} z-\mu & 1 & & \\ & \ddots & \ddots & \\ & & z-\mu & 1 \\ & & & z-\mu \end{bmatrix}_{k\times k}$ $(k \geq 1)$.

- **Blocks associated with the** ∞ **eval**:

  $J_k(\infty) := \begin{bmatrix} 1 & z & & \\ & \ddots & \ddots & \\ & & 1 & z \\ & & & 1 \end{bmatrix}_{k\times k}$ $(k \geq 1)$.

- **Right singular blocks**: $R_k(z) =: \begin{bmatrix} z & 1 & & \\ & z & 1 & \\ & & \ddots & \ddots \\ & & & z \quad 1 \end{bmatrix}_{k\times(k+1)}$ $(k \geq 0)$.

- **Left singular blocks**: $R_k(z)^\top$ $(k \geq 0)$.

Orbit closures

L, L_1, L_2 are $m \times n$ pencils.

$\mathscr{O}(L) = \{PLQ : \ P, Q \text{ invertible}\}$ (strict equivalence orbit)

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Proof: If $P_n L_2 Q_n \to L_1$, then $(PP_n)L_2(Q_n Q) \to PL_1 Q$. □

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Q: Characterization of the inclusion relationship between orbit closures?

Orbit closures: domination rules

Weyr characteristic of $N = (n_1, n_2, \ldots, n_k, 0, 0, \ldots)$, is $W(N) := (w_1(N), w_2(N), \ldots)$ with $w_i(N) = \#\{n_j : \ n_j \geq i\})$.

$r(L)$: Weyr characteristic of the sizes of right singular blocks in KCF(L).

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$W(\mu, L)$: Weyr characteristic of the sizes of Jordan blocks of μ in KCF(L).

Now: $(m_1, m_2, \ldots) \prec_w (n_1, n_2, \ldots) \Leftrightarrow \sum_{i=1}^{k} m_i \leq \sum_{i=1}^{k} n_i$, for all $k \geq 1$ (Weak majorization).

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Theorem [Pokrzywa'86]

$\overline{\mathscr{O}}(L_1) \subseteq \overline{\mathscr{O}}(L_2)$ iff:

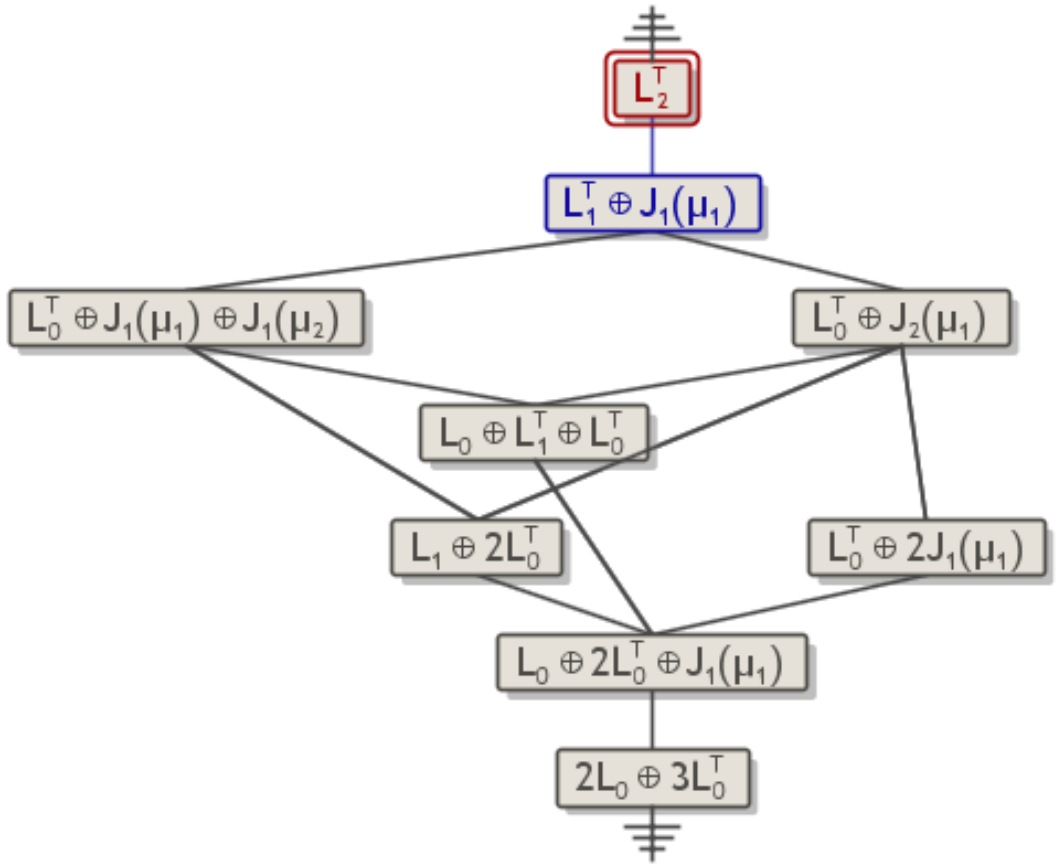
(i) $r(L_1) \prec_{\mathrm{w}} r(L_2) + (h, h, \ldots)$,

(ii) $\ell(L_1) \prec_{\mathrm{w}} \ell(L_2) + (h, h, \ldots)$,

(iii) $W(\mu, L_2) \prec_{\mathrm{w}} W(\mu, L_1) + (h, h, \ldots)$, $\forall \mu \in \overline{\mathbb{C}}$,

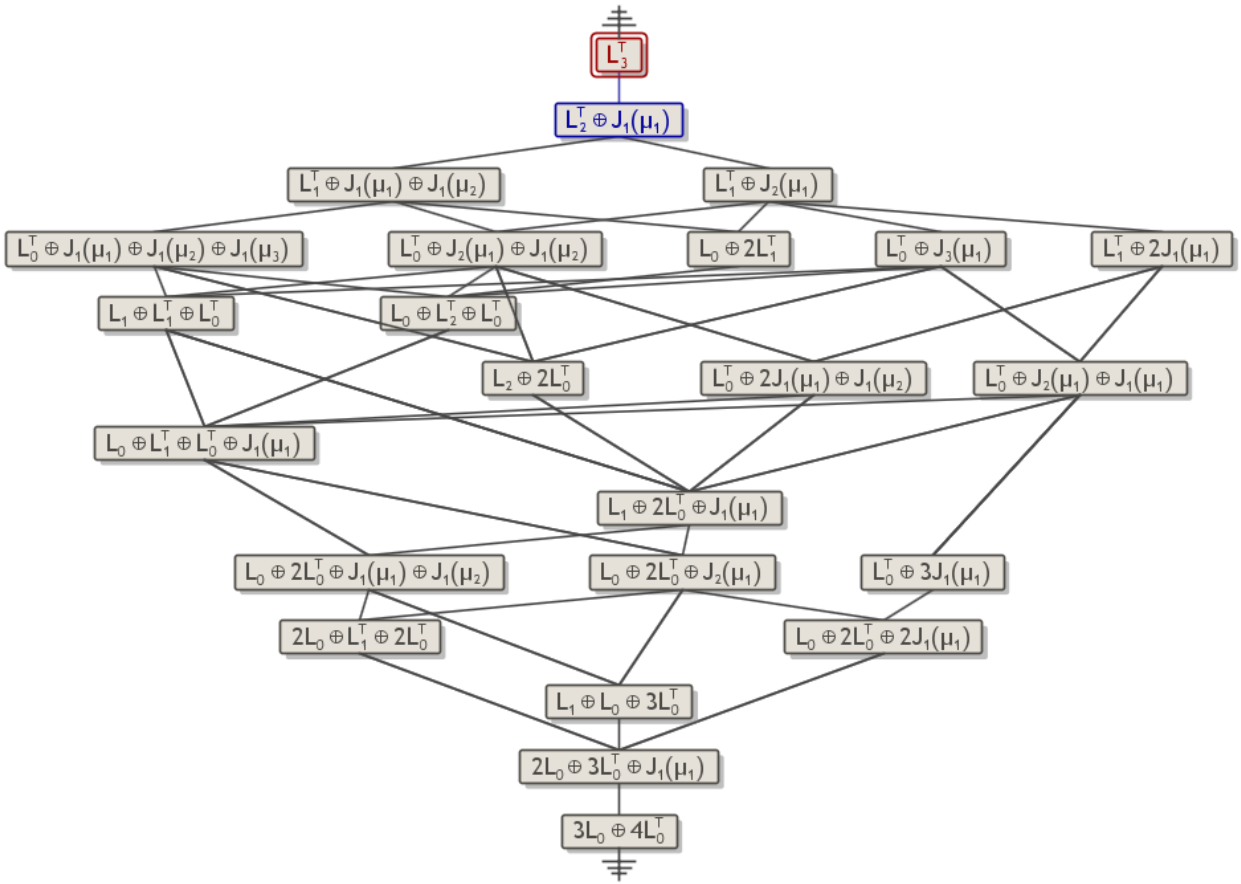
where $h := \operatorname{rank} L_2 - \operatorname{rank} L_1$.

Domination rules: Visualization



Stratification of closure orbits of 3×2 pencils

Domination rules: Visualization



Stratification of closure orbits of 4×3 pencils

Made with `Stratigraph`:

`https://www.umu.se/en/research/projects/stratigraph-and-mcs-toolbox/`

Orbit closures: summary

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- A characterization for the inclusion is known.
- This allows us to known whether a given KCF can be obtained after an arbitrarily small perturbation of another one.
- Also a characterization of the covering is known.
- This allows us to obtain the full stratification.

Bundle: definition

Definition (bundle)

The bundle of L, $\mathscr{B}(L)$, is the set of matrix pencils with the same KCF as L, up to the specific values of the evals.

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Example: If

$$L = R_1(z) \oplus J_1(\mu) = \left[\begin{array}{cc|c} z & 1 & 0 \\ 0 & 0 & z-\mu \end{array}\right],$$

then

$$\begin{aligned} \mathscr{B}(L) \quad = \quad & \left\{ P \left[\begin{array}{cc|c} z & 1 & 0 \\ 0 & 0 & z-\alpha \end{array}\right] Q : \ P, Q \text{ invertible}, \ \alpha \in \mathbb{C} \right\} \\ & \cup \left\{ P \left[\begin{array}{cc|c} z & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] Q : \ \ P, Q \text{ invertible} \right\}. \end{aligned}$$

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☞ A bundle is a union of orbits (infinite, provided that there are eigenvalues).

Bundle closures: inclusion relation

Problem: Characterize the inclusion $L_1 \in \overline{\mathscr{B}}(L_2)$.

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Q: Same domination rules as for the orbits?

A: NO. Different eigenvalues may coalesce.

☞ Coalescence of evals: Take the union of their Weyr characteristics. (i.e.: add up the sizes of the Jordan blocks).

Rules for the inclusion of bundle closures

Theorem (domination rules for bundle closures)

$\overline{\mathscr{B}}(\widetilde{L}) \subseteq \overline{\mathscr{B}}(L)$ if and only if $\mathrm{KCF}(\widetilde{L})$ is obtained from $\mathrm{KCF}(L)$ after coalescing eigenvalues and applying the dominance rules for closure orbit inclusion.



FD, F. M. Dopico.
On bundles of matrix pencils under strict equivalence.
Linear Algebra Appl. 658 (2023) 1-31.

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☞A characterization for the covering was provided in

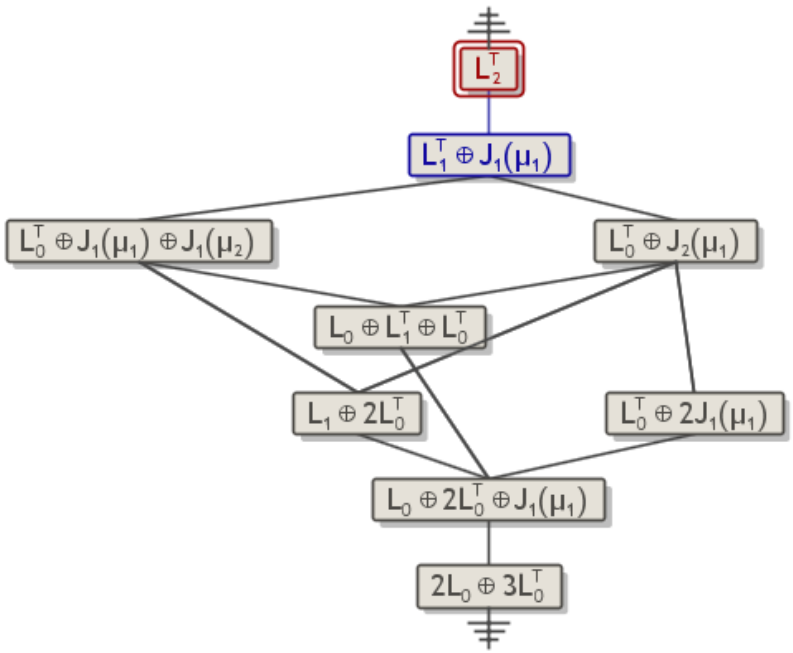


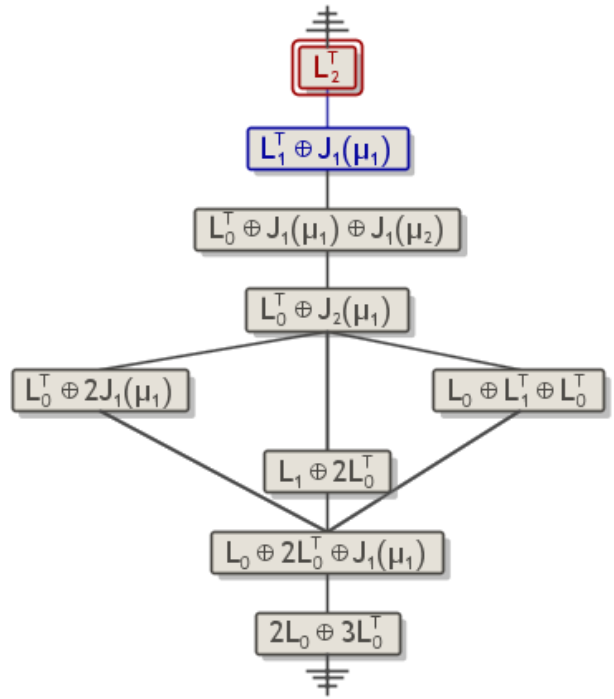
A. Edelman, E. Elmroth, B. Kågström.
SIAM J. Matrix Anal. Appl. 20-3 (1999) 667–699.

in terms of “minimal" changes in the KCF.

Stratigraph again

Let's compare bundles and orbits:

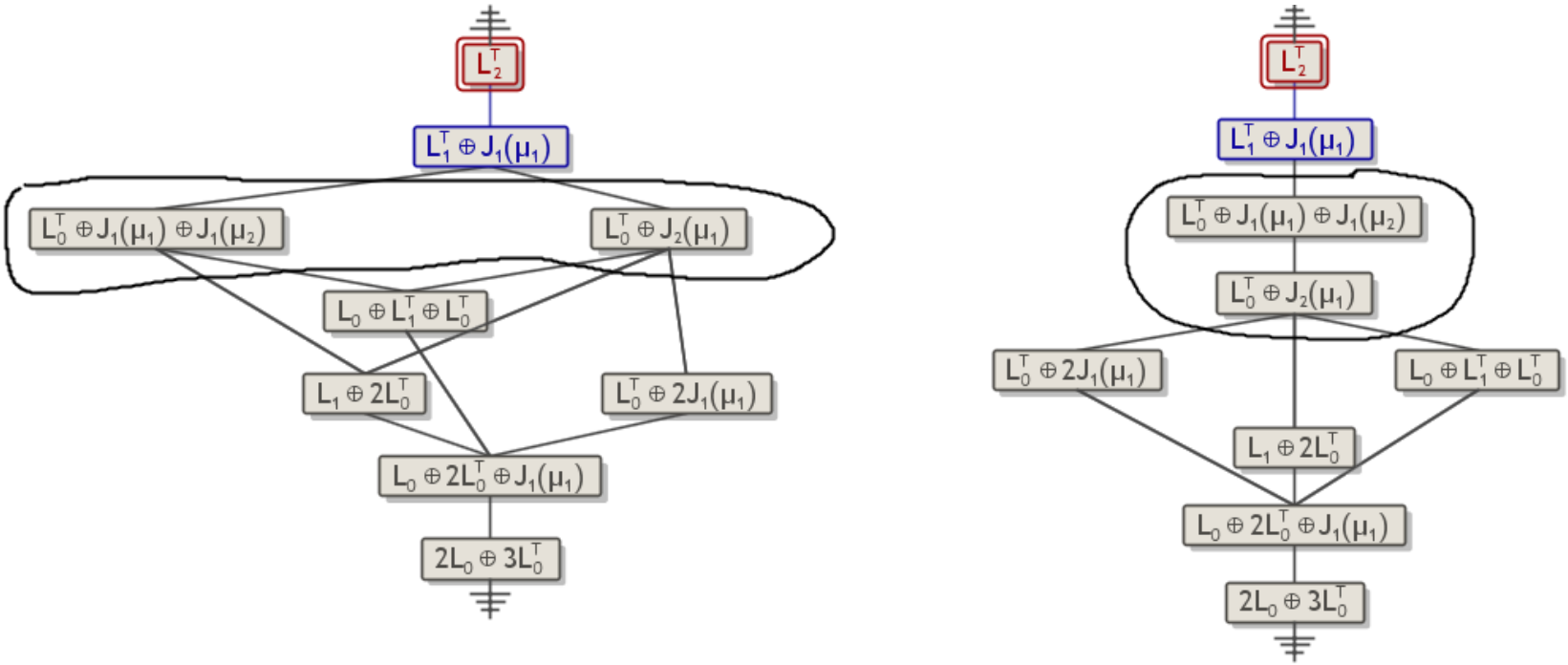




Orbits (left) and bundles (right) of 3×2 pencils

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Some further results

- $\widetilde{L} \in \overline{\mathscr{B}}(L) \Rightarrow \mathscr{B}(\widetilde{L}) \subseteq \overline{\mathscr{B}}(L)$.

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Some further results

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- Bundle closures are “stratified manifolds" (namely, the union of the bundle itself with other bundles of smaller dimension).
- Bundles are open in their closure. The same is true for bundles of matrices (under similarity) and matrix polynomials (under strict equivalence).


FD, F. M. Dopico.
On bundles of matrix pencils under strict equivalence.
Linear Algebra Appl. 658 (2023) 1-31.


FD, F. M. Dopico, V. Koval, P. Pagacz.
On bundle closures of matrix pencils and matrix polynomials.
Ann. Sc. Norm. Super. Pisa, Cl.Sci. In press (2025).

Outline

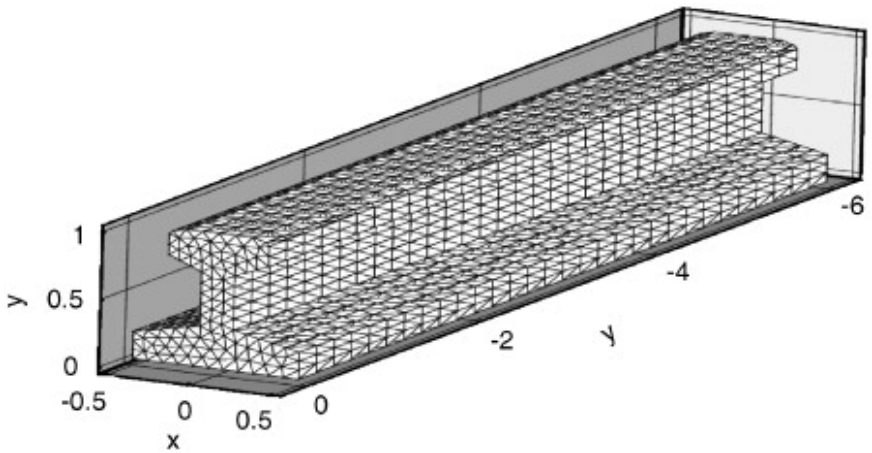
Stratification of matrix orbits and bundle closures

Orbits and closures of matrix pencils

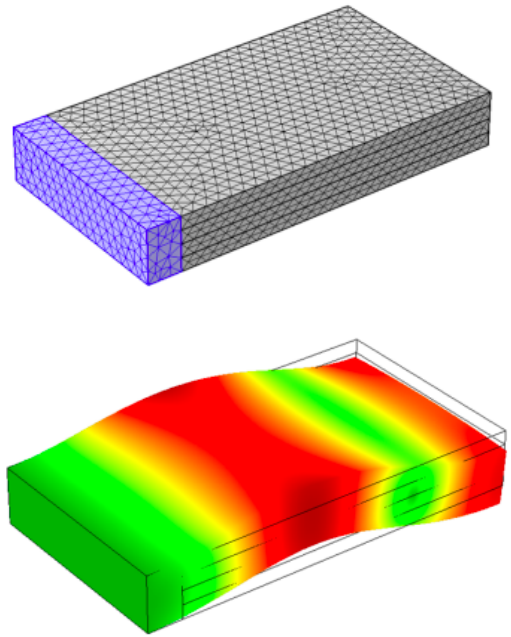
Orbit and closures of structured matrix pencils

Real matrices

Structured matrix pencils



Palindromic



Symmetric

K.-W. E. Chu, T.-M. Huang, W.-W. Lin, C.-T. Wu.

Vibration of fast trains, palindromic eigenvalue problems and structure-preserving doubling algorithms.

J. Comput. Appl. Math. 219 (2008) 237–252.

G. V. Muratova, T. S. Martynova, P. A. Oganesyan.

Numerical solution of partial symmetric generalized eigenvalue problems in piezo device modal analysis.

Commun. Appl. Math. Comp. 7 (2025) 1002–1015.

Which structures?

$$L(z) = A + zB \qquad A, B \in \mathbb{C}^{n\times n}$$

- Symmetric: $A^\top = A, B^\top = B$.
- Skew-Symmetric: $A^\top = -A, B^\top = -B$.
- Hermitian: $A^* = A, B^* = B$.
- Skew-Hermitian: $A^* = -A, B^* = -B$.
- $\star$-alternating: $A^\star = A, B^\star = -B$ or $A^\star = -A, B^\star = B$ $(\star = \top, *)$.
- $\star$-palindromic: $A^\star = B$ $(\star = \top, *)$.
- $\star$-anti-palindromic: $A^\star = -B$ $(\star = \top, *)$.

Strict equivalence

☞ Strict equivalence does not preserve the structure:

$$A + zB \text{ structured } \not\Rightarrow P(A + zB)Q \text{ structured.}$$

Strict equivalence vs congruence

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We need to consider **congruence** orbits and bundles:

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☞ **Yes**! (Structured $\star$-Congruence canonical forms are available).

Strict equivalence vs congruence (II)

Key: It may happen that $L_1 \in \overline{\mathscr{O}}(L_2)$ but $L_1 \notin \overline{\mathscr{O}}_c(L_2)$.

Example: Two symmetric pencils:

$$\begin{bmatrix} \lambda-\lambda_1 & 0 & 0 \\ 0 & \lambda-\lambda_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \overline{\mathscr{O}}\left(\left[\begin{array}{cc|c} 0 & 0 & \lambda \\ 0 & 0 & 1 \\ \hline \lambda & 1 & 0 \end{array}\right] \right)$$

but

$$\begin{bmatrix} \lambda-\lambda_1 & 0 & 0 \\ 0 & \lambda-\lambda_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \notin \overline{\mathscr{O}}_c\left(\left[\begin{array}{cc|c} 0 & 0 & \lambda \\ 0 & 0 & 1 \\ \hline \lambda & 1 & 0 \end{array}\right] \right).$$

Known so far (I)

- For skew-symmetric pencils:
  - Characterization of the covering and inclusion of **orbit** closures in

    A. Dmytryshyn, B. Kågström.
    Orbit closure hierarchies of skew-symmetric matrix pencils.
    SIAM J. Matrix Anal. Appl. 35 (2014) 1429–1443.
  - Same for **bundle** closures in

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    Bundles of skew-symmetric matrix pencils and their minimal degenerations.
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    Linear Algebra. Appl. 744 (2026) 53–81.
- For symmetric pencils:
  - **Dimension** of orbits and bundles in:

    A. Dmytryshyn, B. Kågström, V. V. Sergeichuk.
    Symmetric matrix pencils: codimension counts and the solution of a pair of matrix equations.
    Electron. J. Linear Algebra 27 (2014) 1–18
  - The "**generic**" KCFs of pencils with **bounded rank** in:

    FD, A. Dmytryshyn, F. M. Dopico.
    Generic symmetric matrix pencils with bounded rank.
    J. Spectr. Theor. 10 (2020) 905-926.

Known so far (II)

- For Hermitian pencils:
  - **Generic** KCFs obtained in

    FD, A. Dmytryshyn, F. M. Dopico.
    Generic eigenstructure of Hermitian pencils.
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    Electron. J. Linear Algebra. 42 (2026) 84-104.

- For $\top$-palindromic and $\top$-alternating:
  - **Generic** KCFs of pencils with **bounded rank** in:

    FD
    A geometric description of the sets of palindromic and alternating matrix pencils with bounded rank.
    SIAM J. Matrix Anal. Appl. 39 (2018) 1116-1134.

Open problems

Characterize the

- inclusion, and
- covering

for orbits and bundle closures of structured pencils:

- Symmetric,
- Hermitian,
- $\star$-palindromic,
- $\star$-alternating.

Outline

Stratification of matrix orbits and bundle closures

Orbits and closures of matrix pencils

Orbit and closures of structured matrix pencils

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Matrices again

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Matrices again

For **complex** $n \times n$ matrices, the **generic** JCF is

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For **complex** $n \times n$ matrices, the **generic** JCF is

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In other words:

$$\mathbb{C}^{n \times n} = \overline{\mathscr{B}} \left(\bigoplus_{i=1}^{n} J_1(\mu_i) \right).$$

Matrices again

For **complex** $n \times n$ matrices, the **generic** JCF is

$$J_1(\mu_1) \oplus \cdots \oplus J_1(\mu_n) = \begin{bmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{bmatrix}, \text{ with } \mu_1, \ldots, \mu_n \text{ all different.}$$

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- $2j$ complex conjugate evals and
- $n-2j$ real eigenvalues,

for some j.

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Q: What happens for real matrices?

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- $n-2j$ real eigenvalues,

for some j.

Q: All possible j appear?

Probability of the number of real e-vals

For $n \times n$ **Gaussian** real matrices, the density function and the expected number of real e-vals were provided in:



A. Edelman.

The probability that a random real Gaussian matrix has k real eigenvalues, related distributions, and the circular law.
J. Multivar. Anal. 60 (1997) 203–232



A. Edelman, E. Kostlan, M. Shub.

How Many Eigenvalues of a Random Matrix are Real?
J. Amer. Math. Soc. 7 (1994) 247–267.

Probability of real eigenvalues [Edelman'97]

n	k	$p_{n,k}$	
1	1	1	1
2	2	$\frac{1}{2}\sqrt{2}$	0.70711
	0	$1-\frac{1}{2}\sqrt{2}$	0.29289
3	3	$\frac{1}{4}\sqrt{2}$	0.35355
	1	$1-\frac{1}{4}\sqrt{2}$	0.64645
4	4	$\frac{1}{8}$	0.125
	2	$-\frac{1}{4}+\frac{11}{16}\sqrt{2}$	0.72227
	0	$\frac{9}{8}-\frac{11}{16}\sqrt{2}$	0.15273
5	5	$\frac{1}{32}$	0.03125
	3	$-\frac{1}{16}+\frac{13}{32}\sqrt{2}$	0.51202
	1	$\frac{33}{32}-\frac{13}{32}\sqrt{2}$	0.45673
6	6	$\frac{1}{256}$	0.00552
	4	$\frac{271}{1024}-\frac{3}{256}\sqrt{2}$	0.24808
	2	$-\frac{271}{512}+\frac{107}{128}\sqrt{2}$	0.65290
	0	$\frac{1295}{1024}-\frac{53}{64}\sqrt{2}$	0.09350
7	7	$\frac{1}{2048}\sqrt{2}$	0.00069
	5	$\frac{355}{4096}-\frac{3}{2048}\sqrt{2}$	0.08460
	3	$-\frac{355}{2048}+\frac{1087}{2048}\sqrt{2}$	0.57727
	1	$\frac{4451}{4096}-\frac{1085}{2048}\sqrt{2}$	0.33744
8	8	$\frac{1}{16384}$	0.00006
	6	$-\frac{1}{4096}+\frac{3851}{262144}\sqrt{2}$	0.02053
	4	$\frac{53519}{131072}-\frac{11553}{262144}\sqrt{2}$	0.34599
	2	$-\frac{53487}{65536}+\frac{257185}{262144}\sqrt{2}$	0.57131
	0	$\frac{184551}{131072}-\frac{249483}{262144}\sqrt{2}$	0.06210
9	9	$\frac{1}{262144}$	0.00000
	7	$-\frac{1}{65536}+\frac{5297}{2097152}\sqrt{2}$	0.00256
	5	$\frac{82347}{524288}-\frac{15891}{20\text{-}97152}\sqrt{2}$	0.14635
	3	$-\frac{82339}{262144}+\frac{1345555}{2097152}\sqrt{2}$	0.59328
	1	$\frac{606625}{524288}-\frac{1334961}{2097152}\sqrt{2}$	0.25681

$p_{n,k}$: probability of a random Gaussian $n \times n$ matrix to have exactly k real e-vals

Our result

Theorem [DT-Dopico'2026]

$$\mathbb{R}^{n\times n} = \bigcup_{t=0}^{\lfloor n/2\rfloor} \overline{\mathscr{B}}\left(\bigoplus_{i=1}^{t}\begin{bmatrix} a_i & b_i \\ -b_i & a_i\end{bmatrix} \oplus \bigoplus_{i=1}^{n-2t}[c_i]\right),$$

with $a_i, b_i, c_i \in \mathbb{R}$, $b_i > 0$, and $(a_i, b_i) \neq (a_{i'}, b_{i'})$, $c_i \neq c_{i'}$ for $i \neq i'$.

☞ Every $n \times n$ real matrix is the limit of $\{A_k\}_{k\in\mathbb{N}}$, with **all** A_k having **exactly** $n-2j$ complex e-vals, for some $0 \leq j \leq \lfloor n/2\rfloor$.

Numerical experiments

n	k	F	$F/10^7$
8	8	594	$5.94 \cdot 10^{-5}$
	6	205759	$2.06 \cdot 10^{-2}$
	4	3456910	$3.46 \cdot 10^{-1}$
	2	5713924	$5.71 \cdot 10^{-1}$
	0	622813	$6.23 \cdot 10^{-2}$
9	9	46	$4.6 \cdot 10^{-6}$
	7	35384	$3.53 \cdot 10^{-3}$
	5	1462469	$1.46 \cdot 10^{-1}$
	3	5931622	$5.93 \cdot 10^{-1}$
	1	2570479	$2.57 \cdot 10^{-1}$

n	k	F	$F/10^7$
10	10	2	$2 \cdot 10^{-7}$
	8	4325	$4.22 \cdot 10^{-4}$
	6	444855	$4.45 \cdot 10^{-2}$
	4	4172775	$4.17 \cdot 10^{-1}$
	2	4944333	$4.94 \cdot 10^{-1}$
	0	433710	$4.34 \cdot 10^{-2}$
15	11	12	$1.2 \cdot 10^{-6}$
	9	5444	$5.44 \cdot 10^{-4}$
	7	335896	$3.36 \cdot 10^{-2}$
	5	3142390	$3.14 \cdot 10^{-1}$
	3	5248744	$5.25 \cdot 10^{-1}$
	1	1267514	$1.27 \cdot 10^{-1}$

Number (k) of real e-vals of 10^7 `rand(n)` matrices with size $n \times n$

Numerical experiments

k	F	$F/10^6$
15	11240	$1.12 \cdot 10^{-2}$
13	91675	$9.17 \cdot 10^{-2}$
11	269714	$2.70 \cdot 10^{-1}$
9	352263	$3.52 \cdot 10^{-1}$
7	212891	$2.13 \cdot 10^{-1}$
5	56619	$5.66 \cdot 10^{-2}$
3	5477	$5.48 \cdot 10^{-3}$
1	121	$1.21 \cdot 10^{-4}$

Number (k) of real eigenvalues of 10^6 matrices `randn(15)+diag(2,4,6,...,2*15)`.



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Numerical experiments

k	F	$F/10^6$
15	11240	$1.12 \cdot 10^{-2}$
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3	5477	$5.48 \cdot 10^{-3}$
1	121	$1.21 \cdot 10^{-4}$

Number (k) of real eigenvalues of 10^6 matrices `randn(15)+diag(2,4,6,...,2*15)`.



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Thank you!

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