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Departamento de Matemáticas

On the consistency of $X^\top AX = B$ when B is either symmetric or skew

Fernando De Terán

Joint work with **A. Borobia** and **R. Canogar**

Nick Higham (1961-2024)

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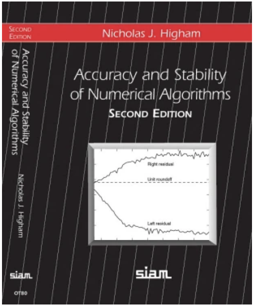

The problem

Goal

Provide necessary and sufficient conditions for the equation

$$X^\top AX = B$$

to be consistent, when B is symmetric.

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$(\cdot)^\top$: **transpose**.

☞ A is not necessarily symmetric. When A is symmetric the result is **well-known**:

$$X^\top AX = B \text{ is consistent} \Leftrightarrow \operatorname{rank} B \leq \operatorname{rank} A$$

(even when $m \neq n$).

A remark

$$\boxed{X^\top A X = B,\ A \in \mathbb{C}^{n\times n}, B \in \mathbb{C}^{m\times m}}$$

☞ If X is **invertible**, then A must be symmetric. ✓

Then...

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GOAL

Fixed $A \in \mathbb{C}^{n\times n}$, which is the largest m such that $X^\top AX = B$ is consistent, with $B \in \mathbb{C}^{m\times m}$ symmetric/skew?

$X^\top AX = B$ and bilinear forms

The problem is equivalent to

Given a bilinear form $\mathbb{A} : \mathbb{C}^n \to \mathbb{C}^n$, find the largest dimension of a subspace $V \subseteq \mathbb{C}^n$, such that $\mathbb{A}_{|V} : V \to V$ is symmetric and non-degenerate.

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(If A is a matrix of $\mathbb{A}$ in some basis, and the columns of X are a basis of V, then $X^\top AX$ is a matrix for $\mathbb{A}_{|V}$.)

(So $\dim V = m$).

Some references on this problem

- A, B with entries over finite fields (or fields with characteristic 2):

J. H. M. Wedderburn.
The automorphic transformation of a bilinear form.
Ann. of Math. 2, 23 (1921) 122–134.

L. Carlitz.
Representations by skew forms in a finite field.
Arch. Math., V (1954) 19–31.

J. H. Hodges.
A skew matrix equation over a finite field.
Math. Nachr., 17 (1966) 49–55.

P. G. Buckhiester.
Rank r solutions to the matrix equation $XAX^t = C$, A alternate, over GF(2^y).
Trans. Amer. Math. Soc., 189 (1974) 201–209.

- Recent references (connected to applications):

P. Benner, D. Palitta.
On the solution of the non-symmetric T-Riccati equation.
Electron. Trans. Numer. Anal., 54 (2021) 66–88.

P. Benner, B. Iannazzo, B. Meini, D. Palitta.
Palindromic linearization and numerical solution of nonsymmetric algebraic T-Riccati equations.
BIT 62 (2022) 1649-1672.

Some examples

$X^\top AX = B$ with ...

- $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is consistent ($X = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$)

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(B is symmetric in all cases, but A **is not**).

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$X^\top J_n(0)X = I_m \oplus 0_{s\times s}$ is consistent $\Leftrightarrow n \geq 2m-1,\ n > 1$.

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  ($m = 2, n = 3$)

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  ($m = 3, n = 4$)

$X^\top J_n(0) X = I_m \oplus 0_{s \times s}$ is consistent $\Leftrightarrow n \geq 2m - 1,\ n > 1$.

The Canonical form for congruence (CFC)

$$J_k(\lambda) := \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix}, \quad \Gamma_k := \begin{bmatrix} 0 & & & & & (-1)^{k+1} \\ & & & & \iddots & (-1)^k \\ & & & -1 & \iddots & \\ & & 1 & 1 & & \\ & -1 & -1 & & & \\ 1 & 1 & & & & 0 \end{bmatrix}, \quad H_{2k}(\lambda) := \begin{bmatrix} 0 & I_k \\ J_k(\lambda) & 0 \end{bmatrix}.$$

Theorem (CFC) [Horn & Sergeichuk, 2006]

Each square complex matrix is congruent to a direct sum, uniquely determined up to permutation of addends, of matrices of the form:

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$$(\Gamma_1 = [1], \qquad H_2(-1) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.)$$

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Notation: C_M= CFC of M.

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☞ We can restrict ourselves to A and B given in CFC.

Some basic remarks

- B symmetric $\Leftrightarrow C_B = I_m \oplus 0_{s\times s}$
- $X^\top(A \oplus 0_{\ell\times\ell})X = B \oplus 0_{s\times s}$ is consistent $\Leftrightarrow X^\top AX = B$ is consistent.

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- $X^\top (A \oplus 0_{\ell\times\ell})X = B \oplus 0_{s\times s}$ is consistent $\Leftrightarrow X^\top AX = B$ is consistent.

(We can get rid of possible null diagonal blocks in the CFC of A and B, namely blocks $J_1(0)$. In particular, B may be assumed to be invertible).

A necessary condition

Theorem

$A \in \mathbb{C}^{m\times m}$. If $X^\top AX = B$ is consistent, with B symmetric, then:

$$\operatorname{rank} B \leq \min\{m - d_A - \frac{\operatorname{rank}(A - A^\top)}{2}, \operatorname{rank}(A + A^\top)\},$$

with $d_A = \dim(\operatorname{Nul} A \cap \operatorname{Nul} A^\top)$.

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Proof: If $X^\top AX = B$, for some X, then

- $X^\top(A + A^\top)X = 2B$

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- $X^\top(A+A^\top)X = 2B \Rightarrow \operatorname{rank} B = \operatorname{rank}(X^\top(A+A^\top)X) \leq \operatorname{rank}(A+A^\top)$.

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- $X^\top(A-A^\top)X = 0$. Using

$$\operatorname{rank}(MN) \ge \operatorname{rank} M + \operatorname{rank} N - m \qquad (M \in \mathbb{C}^{p\times m}, N \in \mathbb{C}^{m\times q}),$$

we get:

$$0 = \operatorname{rank}(X^\top(A-A^\top)X) \ge 2\operatorname{rank} X + \operatorname{rank}(A-A^\top) - 2m \ge 2\operatorname{rank} B + \operatorname{rank}(A-A^\top) - 2m$$
$$\Rightarrow \operatorname{rank} B \le m - \frac{\operatorname{rank}(A-A^\top)}{2}.$$

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$A \in \mathbb{C}^{m\times m}$. If $X^\top AX = B$ is consistent, with B symmetric, then:

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Proof: If $X^\top AX = B$, for some X, then

- $X^\top(A + A^\top)X = 2B \Rightarrow \operatorname{rank} B = \operatorname{rank}(X^\top(A + A^\top)X) \leq \operatorname{rank}(A + A^\top)$.
- $X^\top(A - A^\top)X = 0$. Using

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we get:

$$0 = \operatorname{rank}(X^\top(A - A^\top)X) \geq 2\operatorname{rank} X + \operatorname{rank}(A - A^\top) - 2m \geq 2\operatorname{rank} B + \operatorname{rank}(A - A^\top) - 2m$$
$$\Rightarrow \operatorname{rank} B \leq m - \tfrac{\operatorname{rank}(A - A^\top)}{2}.$$

CFC(A)= $\begin{bmatrix} \widehat{A} & 0 \\ 0 & 0_{d_A} \end{bmatrix}$, and we apply the previous inequality to $X^\top \widehat{A} X = B$ (which is consistent).□

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☞ Comes from $X^\top(A+A^\top)X = 2B$ and $X^\top(A-A^\top)X = 0$.

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Q: Is it sufficient???

It is sufficient (for most $A \in \mathbb{C}^{m\times m}$)!

Theorem

$A \in \mathbb{C}^{m\times m}$ whose CFC does not have $H_4(1)$ blocks, B symmetric. Then

$$X^\top AX = B \text{ is consistent}$$
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$$(H_4(1) = \begin{bmatrix} 0 & I_2 \\ J_2(1) & 0 \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]).$$

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☞ What happens when CFC(A) contains blocks $H_4(1)$?

$$A = H_4(1) \rightsquigarrow X^\top AX = I_3 \text{ is not consistent, but}$$
$$\min\{4 - d_A - \tfrac{\operatorname{rank}(A-A^\top)}{2}, \operatorname{rank}(A+A^\top)\} = \min\{3,4\} = 3.$$

Comments on the proof

- The CFC is our main tool.
- We prove the condition is sufficient for a direct sum of blocks of Type I, Type II, and Type III independently.
- When we put all pieces (canonical blocks) together, the necessary conditions shows up to be sufficient!

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The generic case

The “generic" CFC in $\mathbb{C}^{n\times n}$ is:

$$\mathrm{CFC}_g(n) := \begin{cases} H_2(\mu_1)\oplus\cdots\oplus H_2(\mu_k), & \text{if } n=2k, \\ H_2(\mu_1)\oplus\cdots\oplus H_2(\mu_k)\oplus\Gamma_1, & \text{if } n=2k+1 \end{cases}$$

($\mu_1,\ldots,\mu_k$ different to each other and to $\mu_1^{-1},\ldots,\mu_k^{-1}$, ±1).

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Theorem

If $C_A=\mathrm{CFC}_g(n)$, then

$$X^\top AX = B \qquad (B \text{ symmetric})$$

is consistent if and only if $\operatorname{rank} B \leq n/2$.

B skew: Characterization for consistency

Theorem

$A \in \mathbb{C}^{m\times m}$ and B skew-symmetric matrix. If CFC(A) does not have Γ_2 blocks, then:

$$X^{\mathsf{T}}AX = B \text{ is consistent}$$
$$\Updownarrow$$
$$\operatorname{rank} B \leq \min\{m - d_A - \frac{\operatorname{rank}(A+A^{\mathsf{T}})}{2}, \operatorname{rank}(A - A^{\mathsf{T}})\},$$

where $d_A = \dim(\operatorname{Nul} A \cap \operatorname{Nul} A^{\mathsf{T}})$.

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($\Gamma_2 = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$).

☞ When CFC(A) contains blocks Γ_2, it is not necessarily true:

$$X^{\top}\Gamma_2^{\oplus 4}X = H_2(-1)^{\oplus 3}$$

is not consistent, but $\operatorname{rank} H_2(-1)^{\oplus 3} = 6 = \min\{6,8\}$.

The $*$-congruence

M^*: Conjugate transpose of M.

A, B are $*$-congruent if $P^*AP = B$, for some invertible P.

☞ $X^*AX = B$ is consistent $\Leftrightarrow$ $X^*C_AX = C_B$ is consistent, for any C_A and C_B $*$-congruent with A and B (respectively).

The $*$-CFC

$$\text{Type-0}: \quad J_k(0) = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & 0 \end{bmatrix};$$

$$\text{Type-I}: \quad \mu\widetilde{\Gamma}_k = \widetilde{\Gamma}_k(\mu) = \mu \begin{bmatrix} 1 & 1 & & & & 0 \\ -1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & -1 & 0 & 1 & \\ & & & 1 & 0 & \ddots \\ 0 & & & & \ddots & \ddots \end{bmatrix} \quad \text{with } \mu \in \mathbb{F},\ |\mu| = 1; \text{ and}$$

$$\text{Type-II}: \quad \widetilde{H}_{2k}(\mu) = \begin{bmatrix} 0 & 1 & & & & & \\ \mu & 0 & 1 & & & \mathbf{0} & \\ & 0 & 0 & 1 & & & \\ & & \mu & 0 & 1 & & \\ & & & 0 & \ddots & \ddots & \\ & \mathbf{0} & & & \ddots & 0 & 1 \\ & & & & & \mu & 0 \end{bmatrix}_{2k\times 2k} \quad \mu \in \mathbb{F},\ 0 < |\mu| < 1.$$

Theorem [Horn-Sergeichuk, 2006]

Every square complex matrix A is $*$-congruent to a direct sum of blocks of Types 0, I, and II (uniquely determined up to permutation).

Ongoing work

We are trying to get a characterization for

$$X^*AX = B \qquad \text{to be consistent,}$$

with B being either Hermitian or skew-Hermitian.

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☞ The inertia comes into play!!!

Open questions

Characterize the consistency of:

- $X^\top AX = B$, B symmetric when C_A contains blocks $H_4(1)$.
- $X^\top AX = B$, B skew-symmetric when C_A contains blocks Γ_2.
- $X^* AX = B$, with B Hermitian or skew-Hermitian.
- $X^\top AX = B$ with B symmetric but A, B, X having **real entries**.
- (Hard) $X^\top AX = B$, with B arbitrary.

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$X^\top AX = B$ with A, B real: inertia comes into play

ELSEVIER Linear Algebra and its Applications 275–276 (1998) 261–279

LINEAR ALGEBRA AND ITS APPLICATIONS

Modifying the inertia of matrices arising in optimization

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Submitted by V. Mehrmann

Corollary 3.3. *Let $A \in \mathbb{R}^{n\times n}$ be symmetric, and let $X \in \mathbb{R}^{n\times m} (n \geqslant m)$ be of full rank. Then*

$$\begin{aligned} \text{inertia}(A) - (n-m, n-m, n-m) &\leqslant \text{inertia}(X^\top AX) \\ &\leqslant \text{inertia}(A) + (0, 0, n-m). \end{aligned}$$

References

A. Borobia, R. Canogar, FDT.
On the consistency of the matrix equation $X^T AX = B$ when B is symmetric.
Mediterr. J. Math. (2021) 18:40 (electronic)

A. Borobia, R. Canogar, FDT.
The equation $X^T AX = B$ with B skew-symmetric: How much of a bilinear form is skew-symmetric?
Lin. Multilin. Algebra, in press.

A. Borobia, R. Canogar, FDT.
On the consistency of the matrix equation $X^T AX = B$ when B is symmetric: the case where CFC(A) includes skew-symmetric blocks.
RACSAM, 117 (2023) Article number: 61.

A. Borobia, R. Canogar, FDT.
On the consistency of the matrix equation $X^T AX = B$ when B is skew-symmetric: improving the previous characterization.
Lin. Multilin. Algebra, in press.