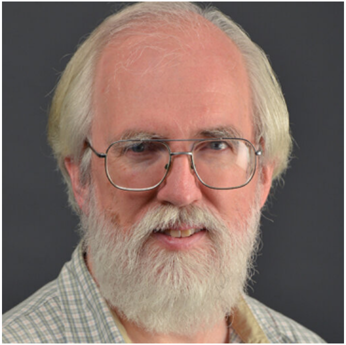
uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

Companion forms for scalar and matrix polynomials: a personal review

Fernando De Terán

Thanks to...



D. S. Mackey

And also to: **F. M. Dopico**, **M. I. Bueno**, **C. Hernando**, **J. Pérez**, **V. Noferini**, **F. Tisseur**, and **P. Van Dooren**.

Strong linearizations

Let $\mathscr{R}$ be a (commutative) **ring**.

Definition: $A, B \in \mathscr{R}^{m \times n}$ are unimodularly equivalent ($\sim_{ue}$) if there are $U \in \mathscr{R}^{m \times m}$ and $V \in \mathscr{R}^{n \times n}$, with $\det U, \det V$ being **units** in $\mathscr{R}$, s. t.

$$UAV = B.$$

(U, V are unimodular).

Strong linearizations

Let $\mathscr{R}$ be a (commutative) **ring**.

Definition: $A, B \in \mathscr{R}^{m\times n}$ are unimodularly equivalent ($\sim_{ue}$) if there are $U \in \mathscr{R}^{m\times m}$ and $V \in \mathscr{R}^{n\times n}$, with $\det U, \det V$ being **units** in $\mathscr{R}$, s. t.

$$UAV = B.$$

(U, V are unimodular).

Definition: $L(\lambda) = \lambda X + Y \in \mathbb{F}[\lambda]^{p\times q}$ is a linearization of $P(\lambda) = \sum_{i=0}^{d} \lambda^i A_i \in \mathbb{F}[\lambda]^{m\times n}$ if

$$L(\lambda) \sim_{ue} \operatorname{diag}(P(\lambda), I_{(d-1)n}).$$

The linearization is strong if, moreover,

$$\operatorname{rev} L(\lambda) \sim_{ue} \operatorname{diag}(\operatorname{rev} P(\lambda), I_{(d-1)n}).$$

$\operatorname{rev}(A_0 + \lambda A_1 + \cdots + \lambda^d A_d) := A_d + \lambda A_{d-1} + \cdots + \lambda^d A_0$ (**reversal**).

Strong linearizations

Let $\mathscr{R}$ be a (commutative) **ring**.

Definition: $A, B \in \mathscr{R}^{m\times n}$ are unimodularly equivalent ($\sim_{ue}$) if there are $U \in \mathscr{R}^{m\times m}$ and $V \in \mathscr{R}^{n\times n}$, with $\det U, \det V$ being **units** in $\mathscr{R}$, s. t.

$$UAV = B.$$

(U, V are unimodular).

Definition: $L(\lambda) = \lambda X + Y \in \mathbb{F}[\lambda]^{p\times q}$ is a linearization of $P(\lambda) = \sum_{i=0}^{d} \lambda^i A_i \in \mathbb{F}[\lambda]^{m\times n}$ if

$$L(\lambda) \sim_{ue} \operatorname{diag}(P(\lambda), I_{(d-1)n}).$$

The linearization is strong if, moreover,

$$\operatorname{rev} L(\lambda) \sim_{ue} \operatorname{diag}(\operatorname{rev} P(\lambda), I_{(d-1)n}).$$

$\operatorname{rev}(A_0 + \lambda A_1 + \cdots + \lambda^d A_d) := A_d + \lambda A_{d-1} + \cdots + \lambda^d A_0$ (**reversal**).

☞ Strong linearizations have the same (regular) spectral information as $P(\lambda)$.

What is a companion form?

Let $\mathbb{F}$ be a field.

A companion pencil for matrix polynomials $P(\lambda) = \sum_{i=0}^{d} \lambda^i A_i \in \mathbb{F}[\lambda]^{n\times n}$ is an $nd \times nd$ matrix pencil $\mathscr{C}_P(\lambda) = \lambda X + Y$ s. t. if X, Y are viewed as block $d \times d$ matrices with $n \times n$ blocks, then:

(a) each nonzero block of X and Y is either $\pm I$ or $\pm A_i$, for some $i = 0 : d$, and

(b) $\mathscr{C}_P$ is a **strong linearization** of $P(\lambda)$.

FDT, F. M. Dopico, D. S. Mackey.
Palindromic companion forms for matrix polynomials of odd degree.
JCAM, 236 (2011) 1464-1480.

What is a companion form?

Let $\mathbb{F}$ be a field.

A companion pencil for matrix polynomials $P(\lambda) = \sum_{i=0}^{d} \lambda^i A_i \in \mathbb{F}[\lambda]^{n\times n}$ is an $nd \times nd$ matrix pencil $\mathscr{C}_P(\lambda) = \lambda X + Y$ s. t. if X, Y are viewed as block $d \times d$ matrices with $n \times n$ blocks, then:

(a) each nonzero block of X and Y is either $\pm I$ or $\pm A_i$, for some $i = 0 : d$, and

(b) $\mathscr{C}_P$ is a **strong linearization** of $P(\lambda)$.

FDT, F. M. Dopico, D. S. Mackey.
Palindromic companion forms for matrix polynomials of odd degree.
JCAM, 236 (2011) 1464-1480.

☞ Simplicity of the constructions

What is a companion form?

Let $\mathbb{F}$ be a field.

A companion pencil for matrix polynomials $P(\lambda) = \sum_{i=0}^{d} \lambda^i A_i \in \mathbb{F}[\lambda]^{n\times n}$ is an $nd \times nd$ matrix pencil $\mathscr{C}_P(\lambda) = \lambda X + Y$ s. t. if X, Y are viewed as block $d \times d$ matrices with $n \times n$ blocks, then:

(a) each nonzero block of X and Y is either $\pm I$ or $\pm A_i$, for some $i = 0 : d$, and

(b) $\mathscr{C}_P$ is a **strong linearization** of $P(\lambda)$.

FDT, F. M. Dopico, D. S. Mackey.
Palindromic companion forms for matrix polynomials of odd degree.
JCAM, 236 (2011) 1464-1480.

☞ Simplicity of the constructions ...and all families of companion pencils knew at that time satisfied it!

What is a companion form?

Let $\mathbb{F}$ be a field.

A companion pencil for matrix polynomials $P(\lambda) = \sum_{i=0}^{d} \lambda^i A_i \in \mathbb{F}[\lambda]^{n\times n}$ is an $nd \times nd$ matrix pencil $\mathscr{C}_P(\lambda) = \lambda X + Y$ s. t. if X, Y are viewed as block $d \times d$ matrices with $n \times n$ blocks, then:

(a) each nonzero block of X and Y is either $\pm I$ or $\pm A_i$, for some $i = 0 : d$, and

(b) $\mathscr{C}_P$ is a **strong linearization** of $P(\lambda)$.

FDT, F. M. Dopico, D. S. Mackey.
Palindromic companion forms for matrix polynomials of odd degree.
JCAM, 236 (2011) 1464-1480.

Quite restrictive!

What is a companion form?

Let $\mathbb{F}$ be a field.

A companion pencil for matrix polynomials $P \in \mathscr{P}(d, m \times n, \mathbb{F})$ is a uniform template for building a pencil $\mathscr{C}_P \in \mathscr{P}(1, p \times q, \mathbb{F})$ from the entries in the coefficient matrices of P, in such a way that

- $\mathscr{C}_P$ is a strong linearization of P, for all $P \in \mathscr{P}(d, m \times n, \mathbb{F})$ (regular or singular).
- The construction of the coefficient matrices of $\mathscr{C}_P$ from those of P should involve no matrix operations other than scalar multiplication.

$\mathscr{P}(d, m \times n, \mathbb{F})$: the set of matrix polynomials over $\mathbb{F}$ of degree d with size $m \times n$.

FDT, F. M. Dopico, D. S. Mackey.
Spectral equivalence of matrix polynomials and the Index Sum Theorem.
LAA, 459 (2014) 264–333.

What is a companion form?

Let $\mathbb{F}$ be a field.

A companion pencil for matrix polynomials $P \in \mathscr{P}(d, m \times n, \mathbb{F})$ is a uniform template for building a pencil $\mathscr{C}_P \in \mathscr{P}(1, p \times q, \mathbb{F})$ from the entries in the coefficient matrices of P, in such a way that

- $\mathscr{C}_P$ is a strong linearization of P, for all $P \in \mathscr{P}(d, m \times n, \mathbb{F})$ (regular or singular).
- The construction of the coefficient matrices of $\mathscr{C}_P$ from those of P should involve no matrix operations other than scalar multiplication.

$\mathscr{P}(d, m \times n, \mathbb{F})$: the set of matrix polynomials over $\mathbb{F}$ of degree d with size $m \times n$.

☞ The size $p \times q$ depends on d, m, and n.

FDT, F. M. Dopico, D. S. Mackey.
Spectral equivalence of matrix polynomials and the Index Sum Theorem.
LAA, 459 (2014) 264–333.

Examples of companion pencils for $P(\lambda) = \sum_{i=0}^{d} \lambda^i A_i$

- 1st and 2nd Frobenius pencils:

$$F_1(\lambda) = \begin{bmatrix} A_{d-1} + \lambda A_d & A_{d-2} & \cdots & A_0 \\ -I & \lambda I & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & -I & \lambda I \end{bmatrix}, \qquad F_2(\lambda) = F_1(\lambda)^{\mathscr{B}}.$$

Examples of companion pencils for $P(\lambda) = \sum_{i=0}^{d} \lambda^i A_i$

- 1st and 2nd Frobenius pencils:

$$F_1(\lambda) = \begin{bmatrix} A_{d-1} + \lambda A_d & A_{d-2} & \cdots & A_0 \\ -I & \lambda I & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & -I & \lambda I \end{bmatrix}, \qquad F_2(\lambda) = F_1(\lambda)^{\mathscr{B}}.$$

- Fiedler pencils: Same entries as $F_1(\lambda)$ but more flexibility in the positions. Example ($d = 6$):

$$\begin{bmatrix} A_5 + \lambda A_6 & -I & 0 & 0 & 0 & 0 \\ A_4 & \lambda I & A_3 & -I & 0 & 0 \\ -I & 0 & \lambda I & 0 & 0 & 0 \\ 0 & 0 & A_2 & \lambda I & A_1 & -I \\ 0 & 0 & -I & 0 & \lambda I & 0 \\ 0 & 0 & 0 & 0 & A_0 & \lambda I \end{bmatrix}.$$

Examples of companion pencils for $P(\lambda)=\sum_{i=0}^{d}\lambda^i A_i$

- 1st and 2nd Frobenius pencils:

$$F_1(\lambda)=\begin{bmatrix} A_{d-1}+\lambda A_d & A_{d-2} & \cdots & A_0 \\ -I & \lambda I & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & -I & \lambda I \end{bmatrix}, \qquad F_2(\lambda)=F_1(\lambda)^{\mathscr{B}}.$$

- Fiedler pencils: Same entries as $F_1(\lambda)$ but more flexibility in the positions. Example ($d=6$):

$$\begin{bmatrix} A_5+\lambda A_6 & -I & 0 & 0 & 0 & 0 \\ A_4 & \lambda I & A_3 & -I & 0 & 0 \\ -I & 0 & \lambda I & 0 & 0 & 0 \\ 0 & 0 & A_2 & \lambda I & A_1 & -I \\ 0 & 0 & -I & 0 & \lambda I & 0 \\ 0 & 0 & 0 & 0 & A_0 & \lambda I \end{bmatrix}.$$

- Generalized Fiedler pencils: Same $I, \lambda I$ blocks as Fiedlers, but λI not necessarily on the main diagonal, and some blocks A_i, A_{i+1} may appear as $A_i+\lambda A_{i+1}$. Example ($d=5$):

$$\begin{bmatrix} A_4+\lambda A_5 & -I & 0 & 0 & 0 \\ -I & 0 & \lambda I & 0 & 0 \\ 0 & \lambda I & A_2+\lambda A_3 & -I & 0 \\ 0 & 0 & -I & 0 & \lambda I \\ 0 & 0 & 0 & \lambda I & A_0+\lambda A_1 \end{bmatrix}.$$

Examples of companion pencils for $P(\lambda) = \sum_{i=0}^{d} \lambda^i A_i$

- 1st and 2nd Frobenius pencils:

$$F_1(\lambda) = \begin{bmatrix} A_{d-1}+\lambda A_d & A_{d-2} & \cdots & A_0 \\ -I & \lambda I & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & -I & \lambda I \end{bmatrix}, \qquad F_2(\lambda) = F_1(\lambda)^{\mathscr{B}}.$$

- Fiedler pencils: Same entries as $F_1(\lambda)$ but more flexibility in the positions. Example ($d = 6$):

$$\begin{bmatrix} A_5+\lambda A_6 & -I & 0 & 0 & 0 & 0 \\ A_4 & \lambda I & A_3 & -I & 0 & 0 \\ -I & 0 & \lambda I & 0 & 0 & 0 \\ 0 & 0 & A_2 & \lambda I & A_1 & -I \\ 0 & 0 & -I & 0 & \lambda I & 0 \\ 0 & 0 & 0 & 0 & A_0 & \lambda I \end{bmatrix}.$$

- Generalized Fiedler pencils: Same $I, \lambda I$ blocks as Fiedlers, but λI not necessarily on the main diagonal, and some blocks A_i, A_{i+1} may appear as $A_i + \lambda A_{i+1}$. Example ($d = 5$):

$$\begin{bmatrix} A_4+\lambda A_5 & -I & 0 & 0 & 0 \\ -I & 0 & \lambda I & 0 & 0 \\ 0 & \lambda I & A_2+\lambda A_3 & -I & 0 \\ 0 & 0 & -I & 0 & \lambda I \\ 0 & 0 & 0 & \lambda I & A_0+\lambda A_1 \end{bmatrix}.$$

☞ Allow us to create structured linearizations ((skew-)symmetric, alternating, (anti-)palindromic).

Examples of companion pencils for $P(\lambda) = \sum_{i=0}^{d} \lambda^i A_i$

- 1st and 2nd Frobenius pencils:

$$F_1(\lambda) = \begin{bmatrix} A_{d-1} + \lambda A_d & A_{d-2} & \cdots & A_0 \\ -I & \lambda I & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & -I & \lambda I \end{bmatrix}, \qquad F_2(\lambda) = F_1(\lambda)^{\mathscr{B}}.$$

- Fiedler pencils: Same entries as $F_1(\lambda)$ but more flexibility in the positions. Example ($d = 6$):

$$\begin{bmatrix} A_5 + \lambda A_6 & -I & 0 & 0 & 0 & 0 \\ A_4 & \lambda I & A_3 & -I & 0 & 0 \\ -I & 0 & \lambda I & 0 & 0 & 0 \\ 0 & 0 & A_2 & \lambda I & A_1 & -I \\ 0 & 0 & -I & 0 & \lambda I & 0 \\ 0 & 0 & 0 & 0 & A_0 & \lambda I \end{bmatrix}.$$

- Generalized Fiedler pencils: Same $I, \lambda I$ blocks as Fiedlers, but λI not necessarily on the main diagonal, and some blocks A_i, A_{i+1} may appear as $A_i + \lambda A_{i+1}$. Example ($d = 5$):

$$\begin{bmatrix} A_4 + \lambda A_5 & -I & 0 & 0 & 0 \\ -I & 0 & \lambda I & 0 & 0 \\ 0 & \lambda I & A_2 + \lambda A_3 & -I & 0 \\ 0 & 0 & -I & 0 & \lambda I \\ 0 & 0 & 0 & \lambda I & A_0 + \lambda A_1 \end{bmatrix}.$$

☞ Allow to create structured linearizations ((skew-)symmetric, alternating, (anti-)palindromic).

Examples of companion pencils (II)

- Generalized Fiedler pencils with repetition: Some A_i's can appear more than once. The number of I's and λI's can be different to those in Fiedler's (and with different signs). Example ($d = 4$):

$$\begin{bmatrix} -I & \lambda A_4 & 0 & 0 \\ \lambda A_4 & A_2 + \lambda A_3 & A_1 & I \\ 0 & A_1 & A_0 - \lambda A_1 & -\lambda I \\ 0 & I & -\lambda I & 0 \end{bmatrix}$$

Examples of companion pencils (II)

- Generalized Fiedler pencils with repetition: Some A_i's can appear more than once. The number of I's and λI's can be different to those in Fiedler's (and with different signs). Example ($d = 4$):

$$\begin{bmatrix} -I & \lambda A_4 & 0 & 0 \\ \lambda A_4 & A_2 + \lambda A_3 & A_1 & I \\ 0 & A_1 & A_0 - \lambda A_1 & -\lambda I \\ 0 & I & -\lambda I & 0 \end{bmatrix}$$

(symmetric linearization).

Examples of companion pencils (II)

- Generalized Fiedler pencils with repetition: Some A_i's can appear more than once. The number of I's and λI's can be different to those in Fiedler's (and with different signs). Example ($d=4$):

$$\begin{bmatrix} -I & \lambda A_4 & 0 & 0 \\ \lambda A_4 & A_2+\lambda A_3 & A_1 & I \\ 0 & A_1 & A_0-\lambda A_1 & -\lambda I \\ 0 & I & -\lambda I & 0 \end{bmatrix}$$

(symmetric linearization).

- Block-Kronecker pencils: Choose p, q with $p+q=d-1$:

$$\left[\begin{array}{cccc|cccc} X_{11}+\lambda Y_{11} & X_{12}+\lambda Y_{12} & \cdots & X_{1,p+1}+\lambda Y_{1,p+1} & -I & & & \\ X_{21}+\lambda Y_{21} & \iddots & \iddots & \vdots & \lambda I & \ddots & & \\ \vdots & \iddots & \iddots & X_{q,p+1}+\lambda Y_{q,p+1} & & \ddots & -I & \\ X_{q+1,1}+\lambda Y_{q+1,1} & \cdots & X_{q+1,p}+\lambda Y_{q+1,p} & X_{q+1,p+1}+\lambda Y_{q+1,p+1} & & & \lambda I & \\ \hline -I & \lambda I & & & & & & \\ & \ddots & \ddots & & & & & \\ & & -I & \lambda I & & & & \end{array}\right]_{dn\times dn}$$

Examples of companion pencils (II)

- Generalized Fiedler pencils with repetition: Some A_i's can appear more than once. The number of I's and λI's can be different to those in Fiedler's (and with different signs). Example ($d = 4$):

$$\begin{bmatrix} -I & \lambda A_4 & 0 & 0 \\ \lambda A_4 & A_2 + \lambda A_3 & A_1 & I \\ 0 & A_1 & A_0 - \lambda A_1 & -\lambda I \\ 0 & I & -\lambda I & 0 \end{bmatrix}$$

(symmetric linearization).

- Block-Kronecker pencils: Choose p, q with $p + q = d - 1$:

$$\left[\begin{array}{cccc|ccc} X_{11} + \lambda Y_{11} & X_{12} + \lambda Y_{12} & \cdots & X_{1,p+1} + \lambda Y_{1,p+1} & -I & & \\ X_{21} + \lambda Y_{21} & \iddots & \iddots & \vdots & \lambda I & \ddots & \\ \vdots & \iddots & \iddots & X_{q,p+1} + \lambda Y_{q,p+1} & & \ddots & -I \\ X_{q+1,1} + \lambda Y_{q+1,1} & \cdots & X_{q+1,p} + \lambda Y_{q+1,p} & X_{q+1,p+1} + \lambda Y_{q+1,p+1} & & & \lambda I \\ \hline -I & \lambda I & & & & & \\ & \ddots & \ddots & & & & \\ & & -I & \lambda I & & & \end{array}\right]_{dn \times dn}$$

$X_{q+1,p+1} = A_0$

Examples of companion pencils (II)

- Generalized Fiedler pencils with repetition: Some A_i's can appear more than once. The number of I's and λI's can be different to those in Fiedler's (and with different signs). Example ($d=4$):

$$\begin{bmatrix} -I & \lambda A_4 & 0 & 0 \\ \lambda A_4 & A_2+\lambda A_3 & A_1 & I \\ 0 & A_1 & A_0-\lambda A_1 & -\lambda I \\ 0 & I & -\lambda I & 0 \end{bmatrix}$$

(symmetric linearization).

- Block-Kronecker pencils: Choose p, q with $p+q=d-1$:

$$\left[\begin{array}{cccc|cccc} X_{11}+\lambda Y_{11} & X_{12}+\lambda Y_{12} & \cdots & X_{1,p+1}+\lambda Y_{1,p+1} & -I & & & \\ X_{21}+\lambda Y_{21} & \iddots & \iddots & \vdots & \lambda I & \ddots & & \\ \vdots & \iddots & \iddots & X_{q,p+1}+\lambda Y_{q,p+1} & & \ddots & -I & \\ X_{q+1,1}+\lambda Y_{q+1,1} & \cdots & X_{q+1,p}+\lambda Y_{q+1,p} & X_{q+1,p+1}+\lambda Y_{q+1,p+1} & & & \lambda I & \\ \hline -I & \lambda I & & & & & & \\ & \ddots & \ddots & & & & & \\ & & -I & \lambda I & & & & \end{array}\right]_{dn\times dn}$$

$$X_{q+1,p}+X_{q,p+1}+Y_{q+1,p+1}=A_1$$

Examples of companion pencils (II)

- Generalized Fiedler pencils with repetition: Some A_i's can appear more than once. The number of I's and λI's can be different to those in Fiedler's (and with different signs). Example ($d=4$):

$$\begin{bmatrix} -I & \lambda A_4 & 0 & 0 \\ \lambda A_4 & A_2+\lambda A_3 & A_1 & I \\ 0 & A_1 & A_0-\lambda A_1 & -\lambda I \\ 0 & I & -\lambda I & 0 \end{bmatrix}$$

(symmetric linearization).

- Block-Kronecker pencils: Choose p,q with $p+q=d-1$:

$$\left[\begin{array}{cccc|ccc} X_{11}+\lambda Y_{11} & X_{12}+\lambda Y_{12} & \cdots & X_{1,p+1}+\lambda Y_{1,p+1} & -I & & \\ X_{21}+\lambda Y_{21} & \iddots & \iddots & \vdots & \lambda I & \ddots & \\ \vdots & \iddots & \iddots & X_{q,p+1}+\lambda Y_{q,p+1} & & \ddots & -I \\ X_{q+1,1}+\lambda Y_{q+1,1} & \cdots & X_{q+1,p}+\lambda Y_{q+1,p} & X_{q+1,p+1}+\lambda Y_{q+1,p+1} & & & \lambda I \\ \hline -I & \lambda I & & & & & \\ & \ddots & \ddots & & & & \\ & & -I & \lambda I & & & \end{array}\right]_{dn\times dn}$$

$$X_{11}+Y_{12}+Y_{21}=A_{d-1}$$

Examples of companion pencils (II)

- Generalized Fiedler pencils with repetition: Some A_i's can appear more than once. The number of I's and λI's can be different to those in Fiedler's (and with different signs). Example ($d=4$):

$$\begin{bmatrix} -I & \lambda A_4 & 0 & 0 \\ \lambda A_4 & A_2+\lambda A_3 & A_1 & I \\ 0 & A_1 & A_0-\lambda A_1 & -\lambda I \\ 0 & I & -\lambda I & 0 \end{bmatrix}$$ (symmetric linearization).

- Block-Kronecker pencils: Choose p, q with $p+q=d-1$:

$$\left[\begin{array}{cccc|ccc} X_{11}+\lambda Y_{11} & X_{12}+\lambda Y_{12} & \cdots & X_{1,p+1}+\lambda Y_{1,p+1} & -I & & \\ X_{21}+\lambda Y_{21} & \iddots & \iddots & \vdots & \lambda I & \ddots & \\ \vdots & \iddots & \iddots & X_{q,p+1}+\lambda Y_{q,p+1} & & \ddots & -I \\ X_{q+1,1}+\lambda Y_{q+1,1} & \cdots & X_{q+1,p}+\lambda Y_{q+1,p} & X_{q+1,p+1}+\lambda Y_{q+1,p+1} & & & \lambda I \\ \hline -I & \lambda I & & & & & \\ & \ddots & \ddots & & & & \\ & & -I & \lambda I & & & \end{array}\right]_{dn\times dn}$$

$Y_{11}=A_d$

Examples of companion pencils (II)

- Generalized Fiedler pencils with repetition: Some A_i's can appear more than once. The number of I's and λI's can be different to those in Fiedler's (and with different signs). Example ($d=4$):

$$\begin{bmatrix} -I & \lambda A_4 & 0 & 0 \\ \lambda A_4 & A_2+\lambda A_3 & A_1 & I \\ 0 & A_1 & A_0-\lambda A_1 & -\lambda I \\ 0 & I & -\lambda I & 0 \end{bmatrix}$$

(symmetric linearization).

- Block-Kronecker pencils: Choose p,q with $p+q=d-1$:

$$\left[\begin{array}{cccc|ccc} X_{11}+\lambda Y_{11} & X_{12}+\lambda Y_{12} & \cdots & X_{1,p+1}+\lambda Y_{1,p+1} & -I & & \\ X_{21}+\lambda Y_{21} & \iddots & \iddots & \vdots & \lambda I & \ddots & \\ \vdots & \iddots & \iddots & X_{q,p+1}+\lambda Y_{q,p+1} & & \ddots & -I \\ X_{q+1,1}+\lambda Y_{q+1,1} & \cdots & X_{q+1,p}+\lambda Y_{q+1,p} & X_{q+1,p+1}+\lambda Y_{q+1,p+1} & & & \lambda I \\ \hline -I & \lambda I & & & & & \\ & \ddots & \ddots & & & & \\ & & -I & \lambda I & & & \end{array}\right]_{dn\times dn}$$

Very flexible family.

Examples of companion pencils (II)

- Generalized Fiedler pencils with repetition: Some A_i's can appear more than once. The number of I's and λI's can be different to those in Fiedler's (and with different signs). Example ($d=4$):

$$\begin{bmatrix} -I & \lambda A_4 & 0 & 0 \\ \lambda A_4 & A_2+\lambda A_3 & A_1 & I \\ 0 & A_1 & A_0-\lambda A_1 & -\lambda I \\ 0 & I & -\lambda I & 0 \end{bmatrix}$$

(symmetric linearization).

- Block-Kronecker pencils: Choose p,q with $p+q=d-1$:

$$\left[\begin{array}{cccc|ccc} X_{11}+\lambda Y_{11} & X_{12}+\lambda Y_{12} & \cdots & X_{1,p+1}+\lambda Y_{1,p+1} & -I & & \\ X_{21}+\lambda Y_{21} & \iddots & \iddots & \vdots & \lambda I & \ddots & \\ \vdots & \iddots & \iddots & X_{q,p+1}+\lambda Y_{q,p+1} & & \ddots & -I \\ X_{q+1,1}+\lambda Y_{q+1,1} & \cdots & X_{q+1,p}+\lambda Y_{q+1,p} & X_{q+1,p+1}+\lambda Y_{q+1,p+1} & & & \lambda I \\ \hline -I & \lambda I & & & & & \\ & \ddots & \ddots & & & & \\ & & -I & \lambda I & & & \end{array}\right]_{dn\times dn}$$

Can be more general $\rightsquigarrow$ Block-minimal-bases pencils.

Examples of companion pencils (II)

- Generalized Fiedler pencils with repetition: Some A_i's can appear more than once. The number of I's and λI's can be different to those in Fiedler's (and with different signs). Example ($d = 4$):

$$\begin{bmatrix} -I & \lambda A_4 & 0 & 0 \\ \lambda A_4 & A_2 + \lambda A_3 & A_1 & I \\ 0 & A_1 & A_0 - \lambda A_1 & -\lambda I \\ 0 & I & -\lambda I & 0 \end{bmatrix} \qquad \text{(symmetric linearization).}$$

- Block-Kronecker pencils: Choose p, q with $p + q = d - 1$:

$$\left[\begin{array}{cccc|cccc} X_{11}+\lambda Y_{11} & X_{12}+\lambda Y_{12} & \cdots & X_{1,p+1}+\lambda Y_{1,p+1} & -I & & & \\ X_{21}+\lambda Y_{21} & \iddots & \iddots & \vdots & \lambda I & \ddots & & \\ \vdots & \iddots & \iddots & X_{q,p+1}+\lambda Y_{q,p+1} & & & \ddots & -I \\ X_{q+1,1}+\lambda Y_{q+1,1} & \cdots & X_{q+1,p}+\lambda Y_{q+1,p} & X_{q+1,p+1}+\lambda Y_{q+1,p+1} & & & & \lambda I \\ \hline -I & \lambda I & & & & & & \\ & \ddots & \ddots & & & & & \\ & & -I & \lambda I & & & & \end{array}\right]_{dn\times dn}$$

☞ Can be more general ⇝ Block-minimal-bases pencils.

☞ Include, up to block permutation, all Fiedler-like pencils.

A summary of known companion pencils

Frobenius companion pencils

(long time ago)

Fiedler pencils

[Antoniou-Vologiannidis'04], [DT-Dopico-Mackey'10] (after [Fiedler'02])

Generalized Fiedler pencils

[Antoniou-Vologiannidis'04], [Bueno-DT-Dopico'11]

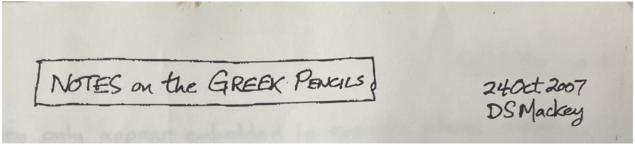
Generalized Fiedler pencils with repetition

[Vologiannidis-Antoniou'11], [Bueno-DT'14]

Block minimal bases pencils [Dopico-Lawrence-Pérez-VanDooren'18]

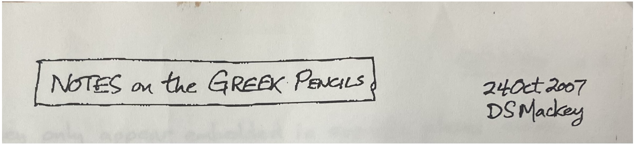
Some history on the name of Fiedler pencils

"Greek pencils":



Some history on the name of Fiedler pencils

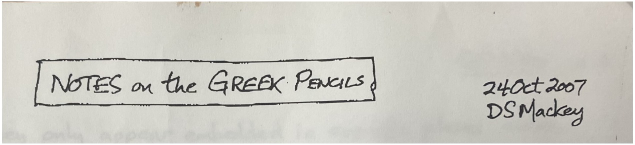
"Greek pencils":



E. N. Antoniou, S. Vologiannidis.
A new family of companion forms of polynomial matrices.
ELA, 11 (2004) 78–87.

Some history on the name of Fiedler pencils

“Greek pencils":



E. N. Antoniou, S. Vologiannidis.
A new family of companion forms of polynomial matrices.
ELA, 11 (2004) 78–87.

Shortly after this...

> *“We should probably decide on a better name. I suggested in Berlin that maybe it would be appropriate to call them the "Fiedler companion pencils", since it was really Fiedler who first(?) suggested factoring the companion matrix and then rearranging the factors."*
>
> (e-mail from August 30, 2008)

M.Fiedler.
A note on companion matrices.
LAA, 372 (2003) 325–331.

Why companion “form"?

It is the word used in

E. N. Antoniou, S. Vologiannidis.
A new family of companion forms of polynomial matrices.
ELA, 11 (2004) 78–87.

Why companion "form"?

It is the word used in

E. N. Antoniou, S. Vologiannidis.
A new family of companion forms of polynomial matrices.
ELA, 11 (2004) 78–87.

It allows us to include also companion ℓ-ifications:

FDT, F. M. Dopico, D. S. Mackey.
Spectral equivalence of matrix polynomials and the Index Sum theorem.
LAA, 459 (2014) 264–333.

Which are used in:

[DT-Dopico-Van Dooren, LAA 495 (2016) 344–372], [Bini-Robol, LAA 502 (2016) 275–298], [Van Dooren-Dopico, LAA 542 (2018) 246–281], [Melman, LAMA 67 (2019) 598–612], [Chan-Corless-González-Vega-Sendra-Sendra, LAA 563 (2019) 373–399], [Dopico-Pérez-Van Dooren, LAA 562 (2019) 163–204], [Song-Maier-Luskin, J. Comput. Phys. 423 (2020) 109871], [Zhan-Dyachencko, JCAM 383 (2021) 113113], [DT-Hernando-Pérez, ELA 37 (2021) 35–71], [Drmač-Šain-Glibić, TOMS 48 (2022), Art. 4],...

Why companion “form"?

It is the word used in

E. N. Antoniou, S. Vologiannidis.
A new family of companion forms of polynomial matrices.
ELA, 11 (2004) 78–87.

It allows us to include also companion ℓ-ifications:

FDT, F. M. Dopico, D. S. Mackey.
Spectral equivalence of matrix polynomials and the Index Sum theorem.
LAA, 459 (2014) 264–333.

Which are used in:

[DT-Dopico-Van Dooren, LAA 495 (2016) 344–372], [Bini-Robol, LAA 502 (2016) 275–298], [Van Dooren-Dopico, LAA 542 (2018) 246–281], [Melman, LAMA 67 (2019) 598–612], [Chan-Corless-González-Vega-Sendra-Sendra, LAA 563 (2019) 373–399], [Dopico-Pérez-Van Dooren, LAA 562 (2019) 163–204], [Song-Maier-Luskin, J. Comput. Phys. 423 (2020) 109871], [Zhan-Dyachencko, JCAM 383 (2021) 113113], [DT-Hernando-Pérez, ELA 37 (2021) 35–71], [Drmač-Šain-Glibić, TOMS 48 (2022), Art. 4],...

☞ The word “ℓ-ification" (including “quadratification") was made up by Steve.

Are they useful in practice?

☞ They are natural tools to be used in the Polynomial Eigenvalue Problem (PEP):

Finding $\lambda_0 \in \mathbb{C}, 0 \neq v \in \mathbb{C}^n$: $P(\lambda_0)v = 0$
λ_0: eigenvalue, v: eigenvector

$(\det P(\lambda) \not\equiv 0)$

Are they useful in practice?

☞ They are natural tools to be used in the Polynomial Eigenvalue Problem (PEP):

Finding $\lambda_0 \in \mathbb{C}, 0 \neq v \in \mathbb{C}^n$: $P(\lambda_0)v = 0$ λ_0: eigenvalue, v: eigenvector

$(\det P(\lambda) \not\equiv 0)$

The backward error and conditioning have been analyzed in:

- For Fiedlers:

  FDT.
  Backward error and conditioning of Fiedler companion linearizations.
  Math. Comp. 89 (2020) 1459-1300.

- For block-Kronecker (b'err):

  F. M. Dopico, P. Lawrence J. Pérez, P. Van Dooren.
  Block Kronecker linearizations of matrix polynomials and their backward errors.
  Numer. Math. 140 (2018) 373–426.

Are they useful in practice?

☞ They are natural tools to be used in the Polynomial Eigenvalue Problem (PEP):

$$\text{Finding } \lambda_0 \in \mathbb{C}, 0 \neq v \in \mathbb{C}^n\text{: } P(\lambda_0)v = 0$$

λ_0: eigenvalue, v: eigenvector

$(\det P(\lambda) \not\equiv 0)$

The backward error and conditioning have been analyzed in:

- For Fiedlers:
  - FDT.
    Backward error and conditioning of Fiedler companion linearizations.
    Math. Comp. 89 (2020) 1459-1300.
- For block-Kronecker (b'err):
  - F. M. Dopico, P. Lawrence J. Pérez, P. Van Dooren.
    Block Kronecker linearizations of matrix polynomials and their backward errors.
    Numer. Math. 140 (2018) 373–426.

☞ The main tools come from:

[Tisseur, LAA 309 (2000) 339–361], [Higham-Mackey-Tisseur, SIMAX 28 (2006) 1005–1028], [Higham-Li-Tisseur, SIMAX 29 (2007) 1218–1241], [Higham-Grammont-Tisseur, LAA 435 (2011) 623–640].

Are they useful? (ctd)

☞ The numerical features (b'err & conditioning) are similar to those of the Frobenius pencils.

Are they useful? (ctd)

☞ The numerical features (b'err & conditioning) are similar to those of the Frobenius pencils.

☞ None of them seems to present advantages (b'err & conditioning) compared to the Frobenius pencils.

Are they useful? (ctd)

☞ The numerical features (b'err & conditioning) are similar to those of the Frobenius pencils.

☞ None of them seems to present advantages (b'err & conditioning) compared to the Frobenius pencils.

> *"One thing that came up in Nick Higham's talk gave a strong indication that the conditioning and backward error properties of the Greek pencils (at least for regular P) will very likely be much like those of the standard Frobenius companion pencils. Hence they are probably completely OK for numerical computation, but also there may not be any particular advantage in using them either."*
> (e-mail from August 30, 2008)

Exploit the structure!

☟ The Frobenius companion forms do not preserve any of the standard symmetry structures of matrix polynomials arising in applications:

- $\star$-Symmetric: $P^\star(\lambda) = P(\lambda)$.
- $\star$-Skew-symmetric: $P^\star(\lambda) = -P(\lambda)$.
- $\star$-palindromic: $P^\star(\lambda) = \mathrm{rev}\, P(\lambda)$.
- $\star$-alternating (odd/even): $P^\star(\lambda) = \pm P(-\lambda)$.

Exploit the structure!

☟ The Frobenius companion forms do not preserve any of the standard symmetry structures of matrix polynomials arising in applications:

- $\star$-Symmetric: $P^\star(\lambda) = P(\lambda)$.
- $\star$-Skew-symmetric: $P^\star(\lambda) = -P(\lambda)$.
- $\star$-palindromic: $P^\star(\lambda) = \mathrm{rev}\, P(\lambda)$.
- $\star$-alternating (odd/even): $P^\star(\lambda) = \pm P(-\lambda)$.

☞ The family of block-Kronecker pencils allows to create **structured** pencils for **all (possible) structures above**.

Exploit the structure!

☟ The Frobenius companion forms do not preserve any of the standard symmetry structures of matrix polynomials arising in applications:

- $\star$-Symmetric: $P^\star(\lambda) = P(\lambda)$.
- $\star$-Skew-symmetric: $P^\star(\lambda) = -P(\lambda)$.
- $\star$-palindromic: $P^\star(\lambda) = \mathrm{rev}\,P(\lambda)$.
- $\star$-alternating (odd/even): $P^\star(\lambda) = \pm P(-\lambda)$.

☞ The family of block-Kronecker pencils allows to create **structured** pencils for **all (possible) structures above**.

☞ There cannot be **structured** companion linearizations for matrix polynomials of even degree $d \geq 2$ for the $\star$-symmetric, $\star$-alternating, and $\star$-palindromic structures.

FDT, F. M. Dopico, D. S. Mackey.
Spectral equivalence of matrix polynomials and the Index Sum theorem.
LAA 459 (2014) 264–333.

Look for structured ℓ-ifications!

Companion structured ℓ-ifications for $\star$-symmetric, $\star$-alternating, and $\star$-palindromic matrix polynomials with degree d, are known for $d = (2s+1)\ell$:

FDT, C. Hernando, J. Pérez.
Structured strong ℓ-ifications for structured matrix polynomials in the monomial basis.
ELA 37 (2021) 35–71.

Look for structured ℓ-ifications!

Companion structured ℓ-ifications for $\star$-symmetric, $\star$-alternating, and $\star$-palindromic matrix polynomials with degree d, are known for $d = (2s+1)\ell$:

FDT, C. Hernando, J. Pérez.
Structured strong ℓ-ifications for structured matrix polynomials in the monomial basis.
ELA 37 (2021) 35–71.

☞ In particular, quadratifications of $(2s+1)\ell$-degree matrix polynomials.

Example ($\star$-palindromic quadratification of degree-10 matrix polynomials):

$$\left[\begin{array}{ccc|cc} P_4+\lambda P_5+\lambda^2 P_6 & \lambda P_3/2 & P_0+\lambda P_1+\lambda^2 P_2 & -\lambda^2 I & 0 \\ \lambda P_7/2 & 0 & \lambda P_3/2 & I & -\lambda^2 I \\ P_8+\lambda P_9+\lambda^2 P_{10} & \lambda P_7/2 & 0 & 0 & I \\ \hline -I & \lambda^2 I & 0 & 0 & 0 \\ 0 & -I & \lambda^2 I & 0 & 0 \end{array}\right]$$

Look for structured ℓ-ifications!

Companion structured ℓ-ifications for $\star$-symmetric, $\star$-alternating, and $\star$-palindromic matrix polynomials with degree d, are known for $d = (2s+1)\ell$:

FDT, C. Hernando, J. Pérez.
Structured strong ℓ-ifications for structured matrix polynomials in the monomial basis.
ELA 37 (2021) 35–71.

☞ In particular, quadratifications of $(2s+1)\ell$-degree matrix polynomials.

Example ($\star$-palindromic quadratification of degree-10 matrix polynomials):

$$\left[\begin{array}{ccc|cc} P_4+\lambda P_5+\lambda^2 P_6 & \lambda P_3/2 & P_0+\lambda P_1+\lambda^2 P_2 & -\lambda^2 I & 0 \\ \lambda P_7/2 & 0 & \lambda P_3/2 & I & -\lambda^2 I \\ P_8+\lambda P_9+\lambda^2 P_{10} & \lambda P_7/2 & 0 & 0 & I \\ \hline -I & \lambda^2 I & 0 & 0 & 0 \\ 0 & -I & \lambda^2 I & 0 & 0 \end{array}\right]$$

☞ No structured quadratifications for degree-4 matrix polynomials exist.

An interesting example of structured companion pencil

The symmetric companion pencil:

$$\begin{bmatrix} A_{d-1}+\lambda A_d & -I & & & & & & & 0 \\ -I & 0 & \lambda I & & & & & & \\ & \lambda I & A_{d-3}+\lambda A_{d-2} & -I & & & & & \\ & & -I & 0 & & & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & & & \lambda I & A_2+\lambda A_3 & -I & & \\ & & & & & -I & 0 & \lambda I & \\ 0 & & & & & & \lambda I & A_0+\lambda A_1 \end{bmatrix}$$

presents better numerical behavior (b'err & conditioning) than any other symmetric linearization know so far.

M. I. Bueno, F. M. Dopico, S. Furtado, L. Medina.
A block-symmetric linearization of odd degree matrix polynomials with optimal eigenvalue condition number and backward error.
Calcolo 55 (2018) 1–32:43.

What is a companion form? (revisited)

Let $\mathbb{F}$ be a field.

A companion pencil for **scalar** polynomials $p(\lambda) = \sum_{i=0}^{d} \lambda^i a_i$, with $a_i \in \mathbb{F}$, is a pencil $A + \lambda B$, with $A, B \in \mathbb{F}[a_0, \ldots, a_{d-1}, a_d]^{n \times n}$ such that

$$\det(A + \lambda B) = \alpha \cdot p(\lambda) \qquad (\alpha \in \mathbb{F}).$$

FDT, C. Hernando.
A note on generalized companion pencils.
RACSAM 114 (2020) Article number: 8.

Are they “strong linearizations"?

($L(\lambda) = A + \lambda B$ Companion pencil: $\det L(\lambda) = \alpha \cdot p(\lambda), \alpha \in \mathbb{F}$).

Question

$$L(\lambda) \text{ companion pencil} \Rightarrow \begin{cases} L(\lambda) \sim_{ue} \begin{bmatrix} p(\lambda) & \\ & I \end{bmatrix} \\ \operatorname{rev} L(\lambda) \sim_{ue} \begin{bmatrix} \operatorname{rev} p(\lambda) & \\ & I \end{bmatrix} \end{cases} \quad \textbf{?}$$

Are they “strong linearizations"?

($L(\lambda) = A + \lambda B$ Companion pencil: $\det L(\lambda) = \alpha \cdot p(\lambda), \alpha \in \mathbb{F}$).

Question

$$L(\lambda) \text{ companion pencil} \Rightarrow \begin{cases} L(\lambda) \sim_{ue} \begin{bmatrix} p(\lambda) & \\ & I \end{bmatrix} \\ \operatorname{rev} L(\lambda) \sim_{ue} \begin{bmatrix} \operatorname{rev} p(\lambda) & \\ & I \end{bmatrix} \end{cases} \quad \textbf{?}$$

Over which ring?

Are they “strong linearizations"?

($L(\lambda)=A+\lambda B$ Companion pencil: $\det L(\lambda)=\alpha\cdot p(\lambda), \alpha\in\mathbb{F}$).

Question

$$L(\lambda) \text{ companion pencil} \Rightarrow \begin{cases} L(\lambda) \sim_{ue} \begin{bmatrix} p(\lambda) & \\ & I \end{bmatrix} \\ \operatorname{rev} L(\lambda) \sim_{ue} \begin{bmatrix} \operatorname{rev} p(\lambda) & \\ & I \end{bmatrix} \end{cases} \quad \textbf{?}$$

Over which ring?

☞ We know: $L(\lambda) \sim_{ue} \begin{bmatrix} p(\lambda) & \\ & I \end{bmatrix}$ over $\mathbb{F}(a_0,\ldots,a_d)[\lambda]$ (Smith form over $\mathbb{F}(a_0,\ldots,a_d)$).

Are they "strong linearizations"?

($L(\lambda) = A + \lambda B$ Companion pencil: $\det L(\lambda) = \alpha \cdot p(\lambda), \alpha \in \mathbb{F}$).

Question

$$L(\lambda) \text{ companion pencil} \Rightarrow \begin{cases} L(\lambda) \sim_{ue} \begin{bmatrix} p(\lambda) & \\ & I \end{bmatrix} \\ \operatorname{rev} L(\lambda) \sim_{ue} \begin{bmatrix} \operatorname{rev} p(\lambda) & \\ & I \end{bmatrix} \end{cases} \quad \textbf{?}$$

Over which ring?

☞ We know: $L(\lambda) \sim_{ue} \begin{bmatrix} p(\lambda) & \\ & I \end{bmatrix}$ over $\mathbb{F}(a_0, \ldots, a_d)[\lambda]$ (Smith form over $\mathbb{F}(a_0, \ldots, a_d)$).

☟ This may involve dividing by some a_i!

U. e. over $\mathbb{F}[a_0, \ldots, a_d, \lambda]$ and $\mathbb{F}(a_0, \ldots, a_d)[\lambda]$

u. e. over $\mathbb{F}(a_0, \ldots, a_d)[\lambda] \not\Rightarrow$ u. e. over $\mathbb{F}[a_0, \ldots, a_d, \lambda]$:

uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

U. e. over $\mathbb{F}[a_0, \ldots, a_d, \lambda]$ and $\mathbb{F}(a_0, \ldots, a_d)[\lambda]$

u. e. over $\mathbb{F}(a_0, \ldots, a_d)[\lambda] \not\Rightarrow$ u. e. over $\mathbb{F}[a_0, \ldots, a_d, \lambda]$:

Counterexample: $\begin{bmatrix} y & \lambda \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are:

- u. e. over $\mathbb{F}(y)[\lambda]$: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/y & -\lambda/y \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y & \lambda \\ 0 & 1 \end{bmatrix}$
- not u. e. over $\mathbb{F}[y, \lambda]$: $\det \begin{bmatrix} y & \lambda \\ 0 & 1 \end{bmatrix} = y \neq 1 = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

U. e. over $\mathbb{F}[a_0,\dots,a_d,\lambda]$ and $\mathbb{F}(a_0,\dots,a_d)[\lambda]$

u. e. over $\mathbb{F}(a_0,\dots,a_d)[\lambda] \not\Rightarrow$ u. e. over $\mathbb{F}[a_0,\dots,a_d,\lambda]$:

Counterexample: $\left[\begin{smallmatrix} y & \lambda \\ 0 & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ are:

- u. e. over $\mathbb{F}(y)[\lambda]$: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/y & -\lambda/y \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y & \lambda \\ 0 & 1 \end{bmatrix}$
- not u. e. over $\mathbb{F}[y,\lambda]$: $\det \begin{bmatrix} y & \lambda \\ 0 & 1 \end{bmatrix} = y \neq 1 = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The Frobenius companion pencils are u. e. over $\mathbb{F}[a_0,\dots,a_d,\lambda]$ to $\operatorname{diag}(I_{d-1}, p(\lambda))$:

$$\begin{bmatrix} a_{d-1} + \lambda a_d & a_{d-2} & \cdots & a_0 \\ -1 & \lambda & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & -1 & \lambda \end{bmatrix}$$

U. e. over $\mathbb{F}[a_0,\ldots,a_d,\lambda]$ and $\mathbb{F}(a_0,\ldots,a_d)[\lambda]$

u. e. over $\mathbb{F}(a_0,\ldots,a_d)[\lambda] \not\Rightarrow$ u. e. over $\mathbb{F}[a_0,\ldots,a_d,\lambda]$:

Counterexample: $\begin{bmatrix} y & \lambda \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are:

- u. e. over $\mathbb{F}(y)[\lambda]$: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/y & -\lambda/y \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y & \lambda \\ 0 & 1 \end{bmatrix}$
- not u. e. over $\mathbb{F}[y,\lambda]$: $\det \begin{bmatrix} y & \lambda \\ 0 & 1 \end{bmatrix} = y \neq 1 = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The Frobenius companion pencils are u. e. over $\mathbb{F}[a_0,\ldots,a_d,\lambda]$ to $\operatorname{diag}(p(\lambda), I_{d-1})$:

$$\begin{bmatrix} a_{d-1}+\lambda a_d & a_{d-2}+\lambda a_{d-1}+\lambda^2 a_d & \cdots & a_0 \\ -1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & -1 & \lambda \end{bmatrix}$$

U. e. over $\mathbb{F}[a_0,\ldots,a_d,\lambda]$ and $\mathbb{F}(a_0,\ldots,a_d)[\lambda]$

u. e. over $\mathbb{F}(a_0,\ldots,a_d)[\lambda] \not\Rightarrow$ u. e. over $\mathbb{F}[a_0,\ldots,a_d,\lambda]$:

Counterexample: $\begin{bmatrix} y & \lambda \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are:

- u. e. over $\mathbb{F}(y)[\lambda]$: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/y & -\lambda/y \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y & \lambda \\ 0 & 1 \end{bmatrix}$
- not u. e. over $\mathbb{F}[y,\lambda]$: $\det \begin{bmatrix} y & \lambda \\ 0 & 1 \end{bmatrix} = y \neq 1 = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The Frobenius companion pencils are u. e. over $\mathbb{F}[a_0,\ldots,a_d,\lambda]$ to $\mathrm{diag}(p(\lambda), I_{d-1})$:

$$\begin{bmatrix} a_{d-1}+\lambda a_d & a_{d-2}+\lambda a_{d-1}+\lambda^2 a_d & \cdots & a_0 \\ -1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & -1 & \lambda \end{bmatrix}$$

U. e. over $\mathbb{F}[a_0,\ldots,a_d,\lambda]$ and $\mathbb{F}(a_0,\ldots,a_d)[\lambda]$

u. e. over $\mathbb{F}(a_0,\ldots,a_d)[\lambda] \not\Rightarrow$ u. e. over $\mathbb{F}[a_0,\ldots,a_d,\lambda]$:

Counterexample: $\left[\begin{smallmatrix} y & \lambda \\ 0 & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ are:

- u. e. over $\mathbb{F}(y)[\lambda]$: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/y & -\lambda/y \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y & \lambda \\ 0 & 1 \end{bmatrix}$
- not u. e. over $\mathbb{F}[y,\lambda]$: $\det \begin{bmatrix} y & \lambda \\ 0 & 1 \end{bmatrix} = y \neq 1 = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The Frobenius companion pencils are u. e. over $\mathbb{F}[a_0,\ldots,a_d,\lambda]$ to $\operatorname{diag}(p(\lambda), I_{d-1})$:

$$\begin{bmatrix} a_{d-1}+\lambda a_d & a_{d-2}+\lambda a_{d-1}+\lambda^2 a_d & \cdots & a_0+\lambda a_1+\cdots+\lambda^d a_d \\ -1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & -1 & 0 \end{bmatrix}$$

U. e. over $\mathbb{F}[a_0, \ldots, a_d, \lambda]$ and $\mathbb{F}(a_0, \ldots, a_d)[\lambda]$

u. e. over $\mathbb{F}(a_0, \ldots, a_d)[\lambda] \not\Rightarrow$ u. e. over $\mathbb{F}[a_0, \ldots, a_d, \lambda]$:

Counterexample: $\begin{bmatrix} y & \lambda \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are:

- u. e. over $\mathbb{F}(y)[\lambda]$: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/y & -\lambda/y \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y & \lambda \\ 0 & 1 \end{bmatrix}$
- not u. e. over $\mathbb{F}[y, \lambda]$: $\det \begin{bmatrix} y & \lambda \\ 0 & 1 \end{bmatrix} = y \neq 1 = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The Frobenius companion pencils are u. e. over $\mathbb{F}[a_0, \ldots, a_d, \lambda]$ to $\operatorname{diag}(p(\lambda), I_{d-1})$:

$$\begin{bmatrix} a_{d-1} + \lambda a_d & a_{d-2} + \lambda a_{d-1} + \lambda^2 a_d & \cdots & a_0 + \lambda a_1 + \cdots + \lambda^d a_d \\ -1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & -1 & 0 \end{bmatrix}$$

U. e. over $\mathbb{F}[a_0, \ldots, a_d, \lambda]$ and $\mathbb{F}(a_0, \ldots, a_d)[\lambda]$

u. e. over $\mathbb{F}(a_0, \ldots, a_d)[\lambda] \not\Rightarrow$ u. e. over $\mathbb{F}[a_0, \ldots, a_d, \lambda]$:

Counterexample: $\left[\begin{smallmatrix} y & \lambda \\ 0 & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ are:

- u. e. over $\mathbb{F}(y)[\lambda]$: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/y & -\lambda/y \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y & \lambda \\ 0 & 1 \end{bmatrix}$
- not u. e. over $\mathbb{F}[y, \lambda]$: $\det \begin{bmatrix} y & \lambda \\ 0 & 1 \end{bmatrix} = y \neq 1 = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The Frobenius companion pencils are u. e. over $\mathbb{F}[a_0, \ldots, a_d, \lambda]$ to $\mathrm{diag}(p(\lambda), I_{d-1})$:

$$\begin{bmatrix} 0 & 0 & \cdots & p(\lambda) \\ -1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & -1 & 0 \end{bmatrix}$$

U. e. over $\mathbb{F}[a_0,\ldots,a_d,\lambda]$ and $\mathbb{F}(a_0,\ldots,a_d)[\lambda]$

u. e. over $\mathbb{F}(a_0,\ldots,a_d)[\lambda] \not\Rightarrow$ u. e. over $\mathbb{F}[a_0,\ldots,a_d,\lambda]$:

Counterexample: $\begin{bmatrix} y & \lambda \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are:

- u. e. over $\mathbb{F}(y)[\lambda]$: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/y & -\lambda/y \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y & \lambda \\ 0 & 1 \end{bmatrix}$
- not u. e. over $\mathbb{F}[y,\lambda]$: $\det \begin{bmatrix} y & \lambda \\ 0 & 1 \end{bmatrix} = y \neq 1 = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The Frobenius companion pencils are u. e. over $\mathbb{F}[a_0,\ldots,a_d,\lambda]$ to $\operatorname{diag}(p(\lambda), I_{d-1})$:

$$\begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & p(\lambda) \end{bmatrix}$$

U. e. over $\mathbb{F}[a_0,\ldots,a_d,\lambda]$ and $\mathbb{F}(a_0,\ldots,a_d)[\lambda]$

u. e. over $\mathbb{F}(a_0,\ldots,a_d)[\lambda] \not\Rightarrow$ u. e. over $\mathbb{F}[a_0,\ldots,a_d,\lambda]$:

Counterexample: $\left[\begin{smallmatrix} y & \lambda \\ 0 & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ are:

- u. e. over $\mathbb{F}(y)[\lambda]$: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/y & -\lambda/y \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y & \lambda \\ 0 & 1 \end{bmatrix}$
- not u. e. over $\mathbb{F}[y,\lambda]$: $\det \begin{bmatrix} y & \lambda \\ 0 & 1 \end{bmatrix} = y \neq 1 = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The Frobenius companion pencils are u. e. over $\mathbb{F}[a_0,\ldots,a_d,\lambda]$ to $\mathrm{diag}(p(\lambda), I_{d-1})$:

$$\begin{bmatrix} p(\lambda) & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

All transformations belong to $\mathbb{F}[a_0,\ldots,a_d,\lambda]$!

uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

U. e. over $\mathbb{F}[a_0,\ldots,a_d,\lambda]$ and $\mathbb{F}(a_0,\ldots,a_d)[\lambda]$

u. e. over $\mathbb{F}(a_0,\ldots,a_d)[\lambda] \not\Rightarrow$ u. e. over $\mathbb{F}[a_0,\ldots,a_d,\lambda]$:

Counterexample: $\begin{bmatrix} y & \lambda \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are:

- u. e. over $\mathbb{F}(y)[\lambda]$: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/y & -\lambda/y \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y & \lambda \\ 0 & 1 \end{bmatrix}$
- not u. e. over $\mathbb{F}[y,\lambda]$: $\det \begin{bmatrix} y & \lambda \\ 0 & 1 \end{bmatrix} = y \neq 1 = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Q: When is a companion pencil u. e. over $\mathbb{F}[a_0,\ldots,a_d,\lambda]$ to its Smith form?

The result by Li, Liu & Chu

Theorem [Li, Liu & Chu'2020]

Let $P(z_1,\ldots,z_m) \in \mathbb{F}[z_1,\ldots,z_m]^{n\times n}$. If

$$\det P(z_1,\ldots,z_m) = z_1 - f(z_2,\ldots,z_m)$$

then $P(z_1,\ldots,z_m)$ is u. e. over $\mathbb{F}[z_1,\ldots,z_m]$ to its Smith form.

D. Li, J. Liu, D. Chu.
The Smith form of a multivariate polynomial matrix over an arbitrary coefficient field.
LAMA 70 (2020) 366–379.

Relies on the Quillen-Suslin Theorem in

D. Quillen
Projective modules over polynomial rings.
Invent. Math. 36 (1976) 167–171.

The result by Li, Liu & Chu

Theorem [Li, Liu & Chu'2020]

Let $P(z_1, \ldots, z_m) \in \mathbb{F}[z_1, \ldots, z_m]^{n \times n}$. If

$$\det P(z_1, \ldots, z_m) = z_1 - f(z_2, \ldots, z_m)$$

then $P(z_1, \ldots, z_m)$ is u. e. over $\mathbb{F}[z_1, \ldots, z_m]$ to its Smith form.

For companion pencils, we can set: $z_1 = a_0, \ldots, z_{m-1} = a_d, z_m = z$, so that $P(z_1, \ldots, z_m) = A(a_0, \ldots, a_d) + zB(a_0, \ldots, a_d)$ and

$$\det(A(a_0, \ldots, a_d) + zB(a_0, \ldots, a_d)) = a_0 - (-za_1 - \cdots - z^n a_d).$$

The result by Li, Liu & Chu

Theorem [Li, Liu & Chu'2020]

Let $P(z_1,\ldots,z_m) \in \mathbb{F}[z_1,\ldots,z_m]^{n\times n}$. If

$$\det P(z_1,\ldots,z_m) = z_1 - f(z_2,\ldots,z_m)$$

then $P(z_1,\ldots,z_m)$ is u. e. over $\mathbb{F}[z_1,\ldots,z_m]$ to its Smith form.

For companion pencils, we can set: $z_1 = a_0,\ldots,z_{m-1} = a_d, z_m = z$, so that $P(z_1,\ldots,z_m) = A(a_0,\ldots,a_d) + zB(a_0,\ldots,a_d)$ and

$$\det(A(a_0,\ldots,a_d) + zB(a_0,\ldots,a_d)) = a_0 - (-za_1 - \cdots - z^n a_d).$$

Theorem

Every companion pencil is u. e. over $\mathbb{F}[a_0,\ldots,a_d,z]$ to $\begin{bmatrix} I_{d-1} & \\ & p(z) \end{bmatrix}$.

Also, the reversal is u. e. over $\mathbb{F}[a_0,\ldots,a_d,z]$ to $\begin{bmatrix} I_{d-1} & \\ & \mathrm{rev}\,p(z) \end{bmatrix}$.

Can this be extended to matrix polynomials?

Let us consider the following companion pencil:

$$L(a_0, a_1, a_2, \lambda) := \begin{bmatrix} 1 - \lambda a_1 & a_0 a_1 + a_1 + \lambda a_2 \\ -\lambda & a_0 \end{bmatrix},$$

with $\det L(a_0, a_1, a_2, \lambda) = a_0 + \lambda a_1 + \lambda^2 a_2$.

Can this be extended to matrix polynomials?

Let us consider the following companion pencil:

$$L(a_0,a_1,a_2,\lambda) := \begin{bmatrix} 1-\lambda a_1 & a_0 a_1 + a_1 + \lambda a_2 \\ -\lambda & a_0 \end{bmatrix},$$

with $\det L(a_0,a_1,a_2,\lambda) = a_0 + \lambda a_1 + \lambda^2 a_2$.

If we consider the matrix extension:

$$L(A_0,A_1,A_2,\lambda) := \begin{bmatrix} I-\lambda A_1 & A_0 A_1 + A_1 + \lambda A_2 \\ -\lambda I & A_0 \end{bmatrix},$$

then, by elementary row operations

$$L(A_0,A_1,A_2,\lambda) \sim_{ue} \begin{bmatrix} I & 0 \\ 0 & P(\lambda) + A_0 A_1 - A_1 A_0 \end{bmatrix} \neq \begin{bmatrix} I & 0 \\ 0 & P(\lambda) \end{bmatrix}.$$

Can this be extended to matrix polynomials?

Let us consider the following companion pencil:

$$L(a_0,a_1,a_2,\lambda) := \begin{bmatrix} 1-\lambda a_1 & a_0 a_1 + a_1 + \lambda a_2 \\ -\lambda & a_0 \end{bmatrix},$$

with $\det L(a_0,a_1,a_2,\lambda) = a_0 + \lambda a_1 + \lambda^2 a_2$.

If we consider the matrix extension:

$$L(A_0,A_1,A_2,\lambda) := \begin{bmatrix} I-\lambda A_1 & A_0 A_1 + A_1 + \lambda A_2 \\ -\lambda I & A_0 \end{bmatrix},$$

then, by elementary row operations

$$L(A_0,A_1,A_2,\lambda) \sim_{ue} \begin{bmatrix} I & 0 \\ 0 & P(\lambda) + A_0 A_1 - A_1 A_0 \end{bmatrix} \neq \begin{bmatrix} I & 0 \\ 0 & P(\lambda) \end{bmatrix}.$$

Q: Conditions to guarantee that the extension to matrix polynomials provide strong linearizations?

Are unimodular matrices a product of elementary matrices?

Elementary matrix (over $\mathscr{R}$): $I+E_{ij}(\alpha)$, with $E_{ij}(\alpha)$ being zero except for the (i,j) entry (which is $\alpha \in \mathscr{R}$).

Are unimodular matrices a product of elementary matrices?

Elementary matrix (over $\mathscr{R}$): $I+E_{ij}(\alpha)$, with $E_{ij}(\alpha)$ being zero except for the (i,j) entry (which is $\alpha\in\mathscr{R}$).

☞ The answer is **No** for arbitrary $\mathscr{R}$ (commutative).

Are unimodular matrices a product of elementary matrices?

Elementary matrix (over $\mathscr{R}$): $I + E_{ij}(\alpha)$, with $E_{ij}(\alpha)$ being zero except for the (i,j) entry (which is $\alpha \in \mathscr{R}$).

☞ The answer is **No** for arbitrary $\mathscr{R}$ (commutative).

☞ Not even when $\mathscr{R}$ is a polynomial ring.

Counterexample: $\begin{bmatrix} 1+xy & x^2 \\ -y^2 & 1-xy \end{bmatrix}$.

uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

Are unimodular matrices a product of elementary matrices?

Elementary matrix (over $\mathscr{R}$): $I + E_{ij}(\alpha)$, with $E_{ij}(\alpha)$ being zero except for the (i,j) entry (which is $\alpha \in \mathscr{R}$).

☞ The answer is **No** for arbitrary $\mathscr{R}$ (commutative).

☞ Not even when $\mathscr{R}$ is a polynomial ring.

Counterexample: $\begin{bmatrix} 1+xy & x^2 \\ -y^2 & 1-xy \end{bmatrix}$.

Theorem

If $\mathscr{R} = \mathbb{F}[x_1, \ldots, x_n]$ and $r \geq 3$, every $r \times r$ unimodular matrix over $\mathscr{R}$ (with det=1) is a product of elementary matrices.

D. Suslin.
On the structure of the special linear group over polynomial rings.
Math. USSR Izvestija, 11 (1977) no. 2.

On the sparsity

Q: Which is the smallest possible number of nonzero entries in a companion pencil?

On the sparsity

Q: Which is the smallest possible number of nonzero entries in a companion pencil?

For companion matrices:

Theorem [Ma & Zhan'2013]

If $A(a_0, \ldots, a_{d-1})$ is a companion matrix, then it has, at least, $2d - 1$ nonzero entries.

C. Ma, X. Zhan.
Extremal sparsity of the companion matrix of a polynomial.
LAA 438 (2013) 621–625.

On the sparsity

Q: Which is the smallest possible number of nonzero entries in a companion pencil?

For companion matrices:

Theorem [Ma & Zhan'2013]

If $A(a_0, \dots, a_{d-1})$ is a companion matrix, then it has, at least, $2d-1$ nonzero entries.

This is the case of the Frobenius companion matrices

$$C_1 = \begin{bmatrix} -a_{d-1} & -a_{d-2} & \cdots & -a_0 \\ 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & 1 & 0 \end{bmatrix}, \qquad (C_2 = C_1^\top).$$

Our conjecture

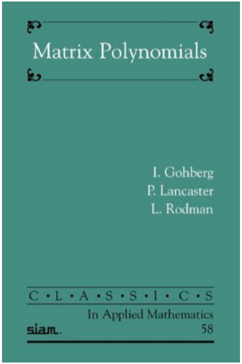
Conjecture: The smallest number of nonzero entries in a companion pencil is: $2d-1+\left\lfloor \frac{d}{2} \right\rfloor$.

It is the number of nonzero entries, for instance, in:

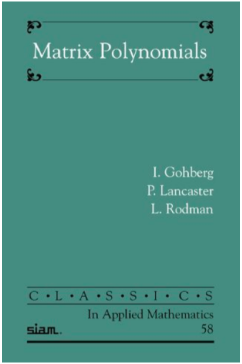
$$\left[\begin{array}{cccc|ccc}
a_{d-1}+\lambda a_d & 0 & \cdots & 0 & -1 & & \\
0 & a_{d-3}+\lambda a_{d-2} & \ddots & \vdots & \lambda & \ddots & \\
\vdots & \ddots & \ddots & 0 & & \ddots & -1 \\
0 & \cdots & 0 & a_0+\lambda a_1 & & & \lambda \\
\hline
-1 & \lambda & & & & & \\
 & \ddots & \ddots & & & & \\
 & & -1 & \lambda & & &
\end{array}\right] \quad (d \text{ odd})$$

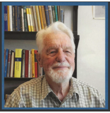
$$\left[\begin{array}{cccc|cccc}
\lambda a_d & 0 & \cdots & 0 & -1 & & & \\
a_{d-2}+\lambda a_{d-1} & 0 & \ddots & \vdots & \lambda & -1 & & \\
0 & \ddots & \ddots & 0 & & \ddots & \ddots & \\
\vdots & \ddots & a_2+\lambda a_3 & 0 & & & \lambda & -1 \\
0 & \cdots & 0 & a_0+\lambda a_1 & & & & \lambda \\
\hline
-1 & \lambda & & & & & & \\
 & \ddots & \ddots & & & & & \\
 & & -1 & \lambda & & & &
\end{array}\right] \quad (d \text{ even})$$

Peter Lancaster is turning 95!



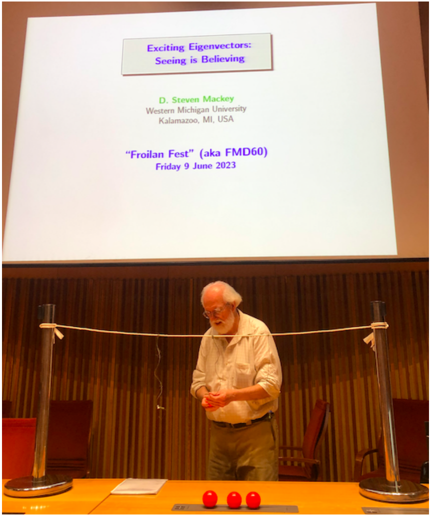
Peter Lancaster is turning 95!





Special Western Canadian Linear Algebra Meeting
Honouring Professor Peter Lancaster at 95
May 25-26, 2024
University of Calgary

And Steve is turning 70!



Happy birthday!