On bundles of matrix pencils under strict equivalence

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Outline

1. Orbits closures of matrix pencils under strict equivalence
2. Bundles come into play
3. Conclusions and open questions

Outline

Orbit: definition

Matrix pencil: $A+\lambda B$, with $A, B \in \mathbb{C}^{m\times n}$ (or matrix pairs (A, B)).

$A+\lambda B$ and $A'+\lambda B'$ are strictly equivalent if:

$$A' = PAQ, \; B' = PBQ, \qquad \text{for some } P, Q \text{ invertible.}$$

(namely, $A'+\lambda B' = P(A+\lambda B)Q$).

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(namely, $A'+\lambda B'=P(A+\lambda B)Q$).

Definition (orbit)

$\mathscr{O}(A+\lambda B):=\{P(A+\lambda B)Q:\ P,Q \text{ invertible}\}$.

(i.e.: it's the set of pencils which are strictly equivalent to $A+\lambda B$).

Orbit and the KCF

Theorem (Kronecker Canonical Form, KCF)

Every pencil is strictly equivalent to a direct sum, uniquely determined (up to permutation), of blocks:

- **Blocks associated with finite evals** (μ): $J_k(\mu) := \begin{bmatrix} \lambda-\mu & 1 & & \\ & \ddots & \ddots & \\ & & \lambda-\mu & 1 \\ & & & \lambda-\mu \end{bmatrix}_{k\times k}$ $(k \geq 1)$.
- **Blocks associated with the ∞ eval**: $J_k(\infty) := \begin{bmatrix} 1 & \lambda & & \\ & \ddots & \ddots & \\ & & 1 & \lambda \\ & & & 1 \end{bmatrix}_{k\times k}$ $(k \geq 1)$.
- **Right singular blocks**: $R_k(\lambda) =: \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \end{bmatrix}_{k\times(k+1)}$ $(k \geq 0)$.
- **Left singular blocks**: $R_k(\lambda)^\top$ $(k \geq 0)$.

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- **Left singular blocks**: $R_k(\lambda)^\top$ $(k \geq 0)$.

☞ Every orbit is (uniquely) determined by the KCF.

Orbit closures

L, L_1, L_2 are $m \times n$ pencils.

$\overline{\mathscr{O}}(L)$: closure of $\mathscr{O}(L)$ (in the standard topology of $\mathbb{C}^{m\times n} \times \mathbb{C}^{m\times n}$).

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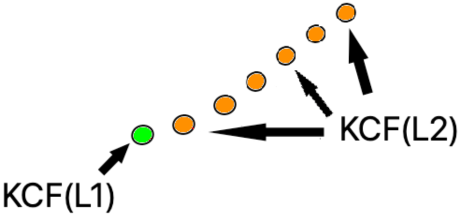
Motivation: When computing KCF(L_1), if $L_1 \in \overline{\mathscr{O}}(L_2)$, there are arbitrarily small perturbations, $L_1 + L_\varepsilon$, s. t. KCF($L_1 + L_\varepsilon$) =KCF(L_2).



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Lemma

$L_1 \in \overline{\mathscr{O}}(L_2) \Leftrightarrow \mathscr{O}(L_1) \subseteq \overline{\mathscr{O}}(L_2) \Leftrightarrow \overline{\mathscr{O}}(L_1) \subseteq \overline{\mathscr{O}}(L_2)$.

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Proof: If $P_n L_2 Q_n \to L_1$, then $(PP_n)L_2(Q_n Q) \to PL_1 Q$. □

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☞ The inclusion relationship between orbit closures allows us to classify the KCFs according to their "genericity".

Orbit closures: domination rules

(Recall: Weyr characteristic of $N = (n_1, n_2, \ldots, n_k, 0, 0, \ldots)$, is $W(N) := (w_1(N), w_2(N), \ldots)$ with $w_i(N) = \#\{n_j : \; n_j \geq i\}$).

$r(L)$: Weyr characteristic of the sizes of right singular blocks in KCF(L).

$\ell(L)$: Weyr characteristic of the sizes of left singular blocks in KCF(L).

$W(\mu, L)$: Weyr characteristic of the sizes of Jordan blocks of μ in KCF(L).

Definition: $(m_1, m_2, \ldots) \prec (n_1, n_2, \ldots) \Leftrightarrow \sum_{i=1}^k m_i \leq \sum_{i=1}^k n_i$, for all $k \geq 1$.

Theorem [Pokrzywa'86]

$\overline{\mathscr{O}}(L_1) \subseteq \overline{\mathscr{O}}(L_2)$ iff:

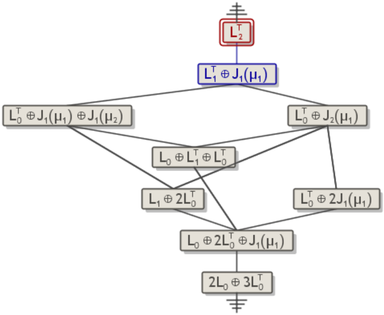
(i) $r(L_1) \prec r(L_2) + (h, h, \ldots)$,

(ii) $\ell(L_1) \prec \ell(L_2) + (h, h, \ldots)$,

(iii) $W(\mu, L_2) \prec W(\mu, L_1) + (h, h, \ldots), \; \forall \mu \in \overline{\mathbb{C}}$,

where $h := \operatorname{rank} L_2 - \operatorname{rank} L_1$.

Domination rules: Visualization



Stratification of closure orbits of 3×2 pencils

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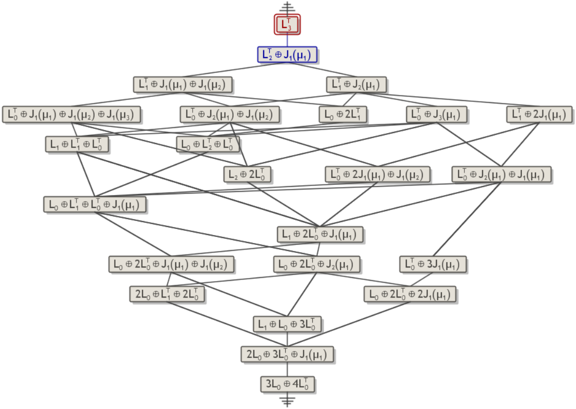

Stratification of closure orbits of 4×3 pencils

Made with `Stratigraph`:

`https://www.umu.se/en/research/projects/stratigraph-and-mcs-toolbox/`

Orbit closures: summary

- A characterization for the inclusion is known.
- This allows us to known whether a given KCF can be obtained after an arbitrarily small perturbation of another one...
- ...and to to classify the KCFs according to their “likelihood".

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☟ However...the eigenvalues must be fixed!

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Definition (bundle)

The bundle of L, $\mathscr{B}(L)$, is the set of matrix pencils with the same KCF as L, up to the specific values of the evals.

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Example: If

$$L = R_1(\lambda) \oplus J_1(\mu) = \left[\begin{array}{cc|c} \lambda & 1 & 0 \\ \hline 0 & 0 & \lambda - \mu \end{array}\right],$$

then

$$\begin{aligned} \mathscr{B}(L) &= \left\{ P \left[\begin{array}{cc|c} \lambda & 1 & 0 \\ \hline 0 & 0 & \lambda - \alpha \end{array}\right] Q : \ P, Q \text{ invertible}, \ \alpha \in \mathbb{C} \right\} \\ &\cup \left\{ P \left[\begin{array}{cc|c} \lambda & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] Q : \ P, Q \text{ invertible} \right\}. \end{aligned}$$

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☞ A bundle is a union of orbits (infinite, provided that there are eigenvalues).

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then $\mathscr{B}(L) = P(J_2(\alpha) \oplus J_1(\beta))Q$ with $\alpha \neq \beta$ (one of α, β can be ∞).

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Therefore:

$$J_2(0) \oplus J_1(0) = \left[\begin{array}{cc|c} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ \hline 0 & 0 & \lambda \end{array}\right] \notin \mathscr{B}(L).$$

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Problem: Characterize the inclusion $L_1 \in \overline{\mathscr{B}}(L_2)$.

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\hline
& & & \lambda-1 & 1 & & & & & & & \\
& & & & \lambda-1 & & & & & & & \\
\hline
& & & & & \lambda & 1 & & & & & \\
& & & & & & \lambda & & & & & \\
\hline
& & & & & & & \lambda & & & & \\
\hline
& & & & & & & & \lambda-2 & 1 & & \\
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Rules for the inclusion of bundle closures

Theorem (domination rules for bundle closures)

$\overline{\mathscr{B}}(\widetilde{L}) \subseteq \overline{\mathscr{B}}(L)$ if and only if $\mathrm{KCF}(\widetilde{L})$ is obtained from $\mathrm{KCF}(L)$ after coalescing eigenvalues and applying the dominance rules for closure orbit inclusion.

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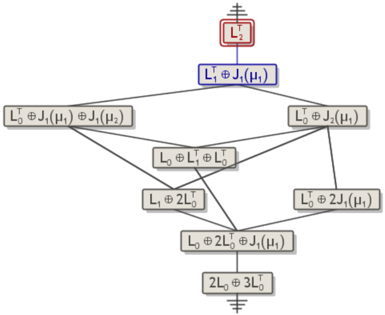
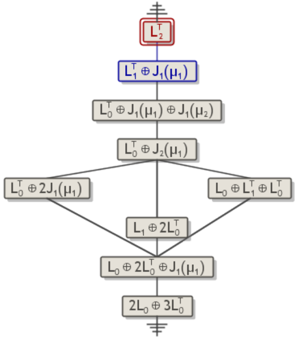
However, no formal definition of coalescence is provided in that reference.

Stratigraph again

Let's compare bundles and orbits:





Orbits (left) and bundles (right) of 3×2 pencils

Stratigraph again

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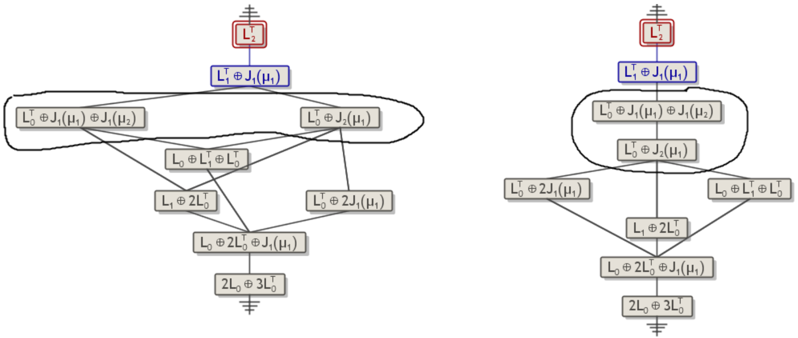

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Some aside results

- $\widetilde{L} \in \overline{\mathscr{B}}(L) \Rightarrow \mathscr{B}(\widetilde{L}) \subseteq \overline{\mathscr{B}}(L)$.
- Bundle closures are "stratified manifolds" (namely, the union of the bundle itself with other bundles of smaller dimension).
- Bundles are open in their closure.

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Outline

Our contribution

- We have provided a formal notion of coalescence of eigenvalues.
- We have provided necessary and sufficient conditions for the inclusion of bundle closures.
- We have proved that bundles are open in their closure.

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Open questions

- For matrix polynomials, P_1, P_2: provide necessary and sufficient conditions for $\overline{\mathscr{B}}(P_1) \subseteq \overline{\mathscr{B}}(P_2)$.
- For structured pencils (alternating, symmetric, Hermitian, palindromic...), L_1, L_2: provide necessary and sufficient conditions for $\overline{\mathscr{B}}(L_1) \subseteq \overline{\mathscr{B}}(L_2)$.
- For matrix polynomials and structured pencils: Are bundles open in their closure?

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THANK YOU!
GRACIAS!