Spectral equivalence of matrix polynomials, the Index Sum Theorem and consequences

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Outline

1. Spectral structure and equivalence relations on matrix polynomials.
   - Unimodular equivalence. Linearizations.
   - New equivalence relations. ℓ-ifications.
2. Companion forms
3. The Index Sum Theorem
4. Consequences of the Index Sum Theorem

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Basic notions

Matrix polynomial:

$$P(\lambda) = \lambda^k A_k + \lambda^{k-1} A_{k-1} + \cdots + \lambda A_1 + A_0, \quad A_i \in \mathbb{F}^{m \times n}$$

$\mathbb{F}$ and **arbitrary** field

$k = 1$: matrix "pencil"

k is the grade of P (non unique!!)

$A_k \neq 0$: P has degree k

P is regular if $m = n$ and $\det P \not\equiv 0$, and it is singular otherwise

P is unimodular if $\det P(\lambda) = c \neq 0 \qquad (m = n)$

$P(\lambda) \in \mathbb{F}[\lambda]^{m \times n} \subseteq \mathbb{F}(\lambda)^{m \times n}$, where

$\mathbb{F}(\lambda) = \left\{ \frac{p(\lambda)}{q(\lambda)} : p, q \in \mathbb{F}[\lambda] \right\}$, field of rational functions

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Spectral structure (scalar)

Regular structure:

- **Finite elementary divisors (e.d)** of P:
  $(\lambda-\lambda_i)^{\alpha_{i,1}},\ldots,(\lambda-\lambda_i)^{\alpha_{i,g_i}},\ \ \alpha_{i,j}>0,\ \ i=1,\ldots,s$
  $\lambda_1,\ldots,\lambda_s\in\overline{\mathbb{F}}\rightsquigarrow$ eigenvalues of P ($\overline{\mathbb{F}}$: algebraic closure of $\mathbb{F}$)
  $\alpha_{i,j},\ j=1:g_i\rightsquigarrow$ partial multiplicities (p.m.) at λ_i
- **Infinite elementary divisors** of P: Elementary divisors at 0 of
  $\operatorname{rev}P:=\lambda^kP(1/\lambda)$ Depend on the grade!

Singular structure:

- **Right minimal basis** of P: Polynomial basis, $\{x_1(\lambda),\ldots,x_p(\lambda)\}$, of

$$\mathcal{N}_r(P)=\{x(\lambda)\in\mathbb{F}(\lambda)^n:\ P(\lambda)x(\lambda)\equiv 0\}$$

  whose sum of degrees is minimal among all polynomial bases.
- **Right minimal indices (m. i.)** of P: Sequence of degrees of a minimal basis. **Invariant of P !** (*Forney*, 1975)
- **Left minimal basis / indices**: Same for

$$\mathcal{N}_l(P)=\{y(\lambda)\in\mathbb{F}(\lambda)^m:\ P(\lambda)^Ty(\lambda)\equiv 0)\}.$$

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Linearizations

Associated Differential-Algebraic Equation:

$$P\left(\frac{d}{dt}\right)x(t) = \left(A_k\left(\frac{d}{dt}\right)^k + A_{k-1}\left(\frac{d}{dt}\right)^{k-1} + \cdots + A_1\frac{d}{dt} + A_0\right)x(t) = f(t), \quad A_i \in \mathbb{F}^{m\times n}$$

Spectral structure: Relevant to know about the solvability and the behavior of the solutions (if any).

Question: Computation of the spectral structure?

Standard way: Using **Linearizations**

Definition (classical linearization)

$P(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ of degree k. A **pencil** $L(\lambda) \in \mathbb{F}[\lambda]^{nk\times nk}$ is a linearization of P if

$$U(\lambda)L(\lambda)V(\lambda) = \operatorname{diag}(P(\lambda), I_{(k-1)n}),$$

for some $U(\lambda), V(\lambda)$ unimodular.
L is a strong linearization if, in addition, $\operatorname{rev} L$ is a linearization of $\operatorname{rev} P$.

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Properties of linearizations

Advantages:

- Preserve part of the spectral structure: finite and **infinite** (**strong**) e. d.
- Numerical methods available to compute the eigenvalues of pencils (GEP).

Drawbacks:

- Classical linearizations (Frobenius companion forms) do not preserve any of the structure that the polynomial may have (symmetric, palindromic, hermitian, etc.), which arise frequently in applications.
- The size of the problem increases too much! ($n \times n \longrightarrow nk \times nk$).
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Then...

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Definition

P, Q with the same size $m \times n$. Then $P \sim_{ue} Q$ (**unimodularly equivalent**) if:

$$Q(\lambda) = U(\lambda)P(\lambda)V(\lambda),$$

U, V unimodular.

Hence:

L linearization of $P \Leftrightarrow L \sim_{ue} \text{diag}(P, I_{(k-1)n})$.
L strong linearization if, in addition, $\text{rev}\, L \sim_{ue} \text{diag}(\text{rev}\, P, I_{(k-1)n})$.

Theorem

$P \sim_{ue} Q \Leftrightarrow P, Q$ have the same finite e. d. and the same rank. (**Smith form**)

$\sim_{ue}$: Preserves **finite e.d.** + **rank** + **size**

Different size? $\rightsquigarrow$ Linearizations!!

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$P \in \mathbb{F}[\lambda]^{m\times n}, Q \in \mathbb{F}[\lambda]^{p\times q}$ are:

- $P \smile Q \Leftrightarrow \mathrm{diag}(P, I_r) \sim_{ue} \mathrm{diag}(Q, I_s)$, for some $r, s \geq 0$ — **extended unimodularly equivalent**
- $P \asymp Q \Leftrightarrow P \smile Q$ and $\mathrm{rev}\, P \smile \mathrm{rev}\, Q$ — **spectrally equivalent**

▶ Allow for different size and degree of P and Q!... but $m+q=n+p$

If $\deg Q = \ell$, then Q is an **ℓ-ification** (resp. **strong ℓ-ification**) of P if $P \smile Q$ (resp. $P \asymp Q$).

Particular cases:

- $\ell = 1$ (pencil) — (**strong**) **linearization** of P
- $\ell = 2$ (quadratic) — (**strong**) **quadratification** of P

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Spectral characterization of $P \smile Q$ and $P \asymp Q$

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$P \smile Q \Leftrightarrow \begin{cases} \text{(a) } P, Q \text{ have the } \textbf{same finite e. d.} \\ \text{(b)} \dim \mathcal{N}_r(P) = \dim \mathcal{N}_r(Q) \text{ and } \dim \mathcal{N}_l(P) = \dim \mathcal{N}_l(Q) \end{cases}$

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Note: (b) $\not\Leftarrow$ rank(P) = rank(Q), since P, Q may have **different size**!

$\asymp$ preserves: regular spectral structure + little bit of singular structure

($\dim \mathcal{N}_r = \#$ right m. i. $\dim \mathcal{N}_l = \#$ left m. i.)

Minimal indices are **not** preserved!!

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Definition and basic examples

$\mathscr{P}(k, m\times n)$: Class of all matrix polynomials of fixed grade k and size $m\times n$.

$P(\lambda)=\sum_{i=0}^{k}\lambda^i A_i \in \mathscr{P}(k, m\times n)$

Companion form: Uniform template for building a pencil $\mathscr{C}_P(\lambda)$ **directly from** A_i, **without any matrix operations**, such that $\mathscr{C}_P(\lambda)$ is a strong linearization for **every** $P\in\mathscr{P}(k, m\times n)$, regular or singular, over any field.

Example: Fiedler pencils:

First Frobenius companion form:

$$C_1(\lambda)=\lambda\begin{bmatrix} A_k & & & \\ & I_n & & \\ & & \ddots & \\ & & & I_n \end{bmatrix}+\begin{bmatrix} A_{k-1} & A_{k-2} & \cdots & A_0 \\ -I_n & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & -I_n & 0 \end{bmatrix}$$

Another Fiedler:

$$F(\lambda)=\lambda\begin{bmatrix} A_k & & & \\ & I_n & & \\ & & \ddots & \\ & & & I_m \end{bmatrix}+\begin{bmatrix} A_{k-1} & A_{k-2} & \cdots & -I_m \\ -I_n & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & A_0 & 0 \end{bmatrix}$$

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$P(\lambda)=\sum_{i=0}^{k}\lambda^i A_i \in \mathscr{P}(k, m\times n)$

Companion form: Uniform template for building a pencil $\mathscr{C}_P(\lambda)$ **directly from** A_i, **without any matrix operations**, such that $\mathscr{C}_P(\lambda)$ is a strong linearization for **every** $P\in\mathscr{P}(k, m\times n)$, regular or singular, over any field.

Example: Fiedler pencils:

First Frobenius companion form:

$$C_1(\lambda)=\lambda\begin{bmatrix} A_k & & & \\ & I_n & & \\ & & \ddots & \\ & & & I_n \end{bmatrix}+\begin{bmatrix} A_{k-1} & A_{k-2} & \cdots & A_0 \\ -I_n & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & -I_n & 0 \end{bmatrix}$$

Another Fiedler:

$$F(\lambda)=\lambda\begin{bmatrix} A_k & & & \\ & I_n & & \\ & & \ddots & \\ & & & I_m \end{bmatrix}+\begin{bmatrix} A_{k-1} & A_{k-2} & \cdots & -I_m \\ -I_n & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & A_0 & 0 \end{bmatrix}$$

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Structured companion forms

Structured companion form: Companion form that shares some particular structure of P (symmetric, palindromic, alternating, etc.) $\rightsquigarrow$ **symmetric / palindromic / alternating companion form**

Fiedler pencils are not structured, but they are the source of structured companion forms.

Known structured companion forms for:

- Symmetric (odd degree; even degree: nonsingular leading coeff.) ($A_i = A_i^T$, $i = 0 : k$)
- Skew-symmetric (odd degree) ($A_i = -A_i^T$, $i = 0 : k$)
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What happens for arbitrary degree/grade???

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ℓ-ifications

Similar templates for ℓ-ifications ("Companion ℓ-ifications") ???

$$C_1^\ell(\lambda) := \begin{bmatrix} P_\ell(\lambda) & A_{k-\ell-1} & \cdots & A_0 \\ -I_n & \lambda I_n & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & -I_n & \lambda I_n \end{bmatrix}$$

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are ℓ-ifications (for all $\ell \le k$) of $P = \sum_{i=0}^k \lambda^i A_i$ (even for P rectangular).
$P_\ell(\lambda) = \lambda^\ell A_k + \lambda^{\ell-1} A_{k-1} + \cdots + A_{k-\ell}$ (ℓ th Horner shift of P).

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Outline

Size, rank and spectral structure

$\delta_{\text{fin}}(P) = \sum$ (finite partial multiplicities), $\delta_\infty(P) = \sum$ (infinite partial multiplicities)
$\mu(P) := \sum$(left+right minimal indices)

Index Sum Theorem [Praagman 1991]

$\delta_{\text{fin}}(P) + \delta_\infty(P) + \mu(P) = \text{grade}(P) \cdot \text{rank}(P)$

Proof: Let C_1 be the first Frobenius companion form of P. Set $k = \text{grade}(P)$. Then

$$\begin{aligned} \text{rank}(C_1) &= \delta_{\text{fin}}(C_1) + \delta_\infty(C_1) + \mu(C_1) \\ &= \delta_{\text{fin}}(P) + \delta_\infty(P) + \mu(P) + (n - \text{rank}(P))(k-1). \end{aligned}$$

Since $C_1 \smile \text{diag}(P, I_{n(k-1)})$:

$$\text{rank}(C_1) = \text{rank}(P) + n(k-1).$$

Equating:

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Outline

1. Spectral structure and equivalence relations on matrix polynomials.
   - Unimodular equivalence. Linearizations.
   - New equivalence relations. ℓ-ifications.
2. Companion forms
3. The Index Sum Theorem
4. Consequences of the Index Sum Theorem

Companion forms and ℓ-ifications

Index Sum Th.: $\boxed{\delta_{\text{fin}}(P)+\delta_\infty(P)+\mu(P)=\text{grade}(P)\cdot\text{rank}(P)}$

Consequences on companion forms ($P\in\mathscr{P}(k,n\times n)$):

1. P is regular $\Leftrightarrow \delta_{\text{fin}}(P)+\delta_\infty(P)=kn$
2. Any companion form must have size $kn\times kn$.
3. If $L\asymp P$ and L is $nk\times nk$, then $\mu(L)-\mu(P)=(k-1)(n-\text{rank}(P))$.
4. No companion form can preserve all minimal indices of P.

P **regular** with grade k, Q a strong ℓ-ification of $P \Rightarrow \boxed{\ell\,\text{rank}\,Q=kn}$. Then:

1. ℓ divides kn.
2. For k,n **odd** there are no strong quadratifications of P.
3. "Companion" strong ℓ-ifications do **not always** exist (for all k)!!

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Strong companion ℓ-ifications (when ℓ divides k)

$P(\lambda)=\sum_{i=0}^{k}\lambda^i A_i$. Let ℓ be a **divisor** of k ($k=\ell s$). Define:

(1) $$B_1(\lambda) := \lambda^\ell A_\ell + \lambda^{\ell-1}A_{\ell-1}+\cdots+\lambda A_1 + A_0\,,$$

(2) $$B_j(\lambda) := \lambda^\ell A_{\ell j} + \lambda^{\ell-1}A_{\ell j-1}+\cdots+\lambda A_{\ell(j-1)+1}, \qquad j=2,\dots,s.$$

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$$C_1^\ell(\lambda) := \begin{bmatrix} B_s(\lambda) & B_{s-1}(\lambda) & B_{s-2}(\lambda) & \cdots & B_1(\lambda) \\ -I_n & \lambda^\ell I_n & 0 & \cdots & 0 \\ & -I_n & \lambda^\ell I_n & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ & & & -I_n & \lambda^\ell I_n \end{bmatrix} \in \mathbb{F}[\lambda]^{(m+(s-1)n)\times sn}$$

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$P(\lambda)=\sum_{i=0}^{k}\lambda^i A_i$ ($n\times n$), then $P^\star(\lambda)=\sum_{i=0}^{k}\lambda^i A_i^\star$. P is said to be:

(a) $\star$-symmetric if $P^\star(\lambda)=P(\lambda)$.

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P (singular) with **any of these structures**, and $\varepsilon_1\leq\ldots\leq\varepsilon_p$, $\eta_1\leq\ldots\leq\eta_p$ its right / left m. i. Then $\boxed{\varepsilon_1=\eta_1,\ldots,\varepsilon_p=\eta_p}$

Corollary (of Index Sum Theorem)

P is $n\times n$ with **even grade** whose left and right m. i. coincide, and having an $nk\times nk$ **linearization** whose left and right m. i. also coincide. Then P has an **even number** of **left m. i.** and also an **even number** of **right m. i.**

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$\mathscr{S}$: any of the previous structures (symmetric, skew-symmetric, alternating, palindromic)

$\mathscr{P}_{\rm sing}(k, n \times n, \mathscr{S})$: Set of $n \times n$ singular polynomials of grade k with the structure $\mathscr{S}$

Corollary

For k even, the set of singular matrix polynomials in $\mathscr{P}(k, n \times n, \mathscr{S})$ having a structured linearization is contained in:

$$\mathscr{P}_{\rm sing}(k, n \times n, \mathscr{S}) \cap \{P \in \mathscr{P}(k, n \times n) : \operatorname{rank} P \leq n-2\}$$

("Generically", there are no structured linearizations for structured **singular** matrix polynomials of even grade!)

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There are no structured companion forms for polynomials with even grade.

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Conclusions

- We have introduced two equivalence relations on matrix polynomials ($\smile$ and $\asymp$) that allow for different sizes and degrees.
- The invariants of $\asymp$ are: regular spectral structure **+** dimension of the right and left null spaces.
- We have introduced the **Index Sum Theorem**, an elementary relation between partial multiplicites (finite and infinite), sum of minimal indices, and the grade and rank of an arbitrary matrix polynomial.
- We have reviewed some strong consequences of the Index Sum Theorem, related to the size of companion forms, and to the existence of strong ℓ-ifications and structured companion forms.
- We have presented strong companion ℓ-ifications of a matrix polynomial of degree k. For **all possible** ℓ, namely, for every divisor ℓ of k.