Eigenvector recovery of linearizations and the condition number of eigenvalues of matrix polynomials

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Outline

1. Eigenvectors of matrix polynomials

2. Fiedler-like pencils
   - Fiedler pencils (FP)
   - Generalized Fiedler pencils (GFP)
   - Fiedler pencils with repetition (FPR)

3. Formulas for the eigenvectors

4. Conditioning of eigenvalues

Outline

Linearizations of matrix polynomials

For an $m \times n$ **matrix polynomial** of degree k

$$P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i = \lambda^k A_k + \cdots + \lambda A_1 + A_0 \,, \quad A_i \in \mathbb{C}^{n\times n}, \quad A_k \neq 0,$$

a linearization for $P(\lambda)$ is an $\ell \times \ell$ pencil $L(\lambda) = \lambda X + Y$ such that

$$U(\lambda)L(\lambda)V(\lambda) = \mathrm{diag}(I, P(\lambda)) \quad (U(\lambda), V(\lambda) \text{ unimodular}).$$

$L(\lambda)$ is "strong" if, in addition, $\mathrm{rev}\, L(\lambda)$ is a linearization for $\mathrm{rev}\, P(\lambda)$ ($\mathrm{rev}\, P(\lambda) = \lambda^k A_0 + \ldots + \lambda A_{k-1} + A_k$).

Why linearizations?:

- Strong linearizations preserve the finite and infinite elementary divisors.
- Well developed theory (numerical methods, software, analysis of errors,...) for the GEP (square) and for the computation of elementary divisors of pencils.

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Families of linearizations

1. $\mathbb{L}_1(P), \mathbb{L}_2(P)$ and $\mathbb{DL}(P)$ $\rightsquigarrow$ [Mackey, Mackey, Mehl, Merhrmann, SIMAX 28 (2006)]
2. **Fiedler** pencils $\rightsquigarrow$ [Antoniou, Vologiannidis, ELA 11 (2004)], [DT, Dopico, Mackey, SIMAX 31 (2010)]
3. **Generalized Fiedler** pencils $\rightsquigarrow$ [Antoniou, Vologiannidis, ELA 11 (2004)], [Bueno, DT, Dopico, SIMAX 32 (2011)]
4. Linearizations in other “polynomial bases" $\rightsquigarrow$ [Amiraslani, Corless, Lancaster, TCS 381 (2007)]
5. **Fiedler** pencils **with repetition** $\rightsquigarrow$ [Vologiannidis, Antoniou, MCSS 22 (2011)]

Linearizations and eigenvectors

$P(\lambda) = \sum_{i=0}^{k} A_i \lambda^i$ regular.

Eigenvectors are not preserved by linearization.

Actually: $L(\lambda)$ linearization of $P(\lambda)$, and

$$\left.\begin{array}{ll} v & \text{a (right) e-vec of } L(\lambda) \\ x & \text{a (right) e-vec of } P(\lambda) \end{array}\right\} \Rightarrow v \in \mathbb{C}^{nk},\ x \in \mathbb{C}^{n}.$$

Example: $C_1(\lambda) = \begin{bmatrix} A_{k-1} + \lambda A_k & A_{k-2} & \cdots & A_0 \\ -I & \lambda I & & 0 \\ & \ddots & \ddots & \\ 0 & & -I & \lambda I \end{bmatrix}$, the first companion form.

Then, if x, v are associated with λ_0: $\quad \mathbf{v} = \begin{bmatrix} \lambda_0^{k-1} x \\ \vdots \\ \lambda_0 x \\ x \end{bmatrix} \quad (= \Lambda \otimes x)$

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Obtain formulas for the eigenvectors in all know families of linearizations.

- We will focus on the families of **Fiedler-like** pencils.

Already done for the families:

- $\mathbb{L}_1(P), \mathbb{L}_2(P)$ and $\mathbb{DL}(P)$ in [Mackey, Mackey, Mehl, Merhrmann, SIMAX 28 (2006)]
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Motivation: condition number

$\lambda \neq 0$ finite (simple) e-val of $P(\lambda) = \sum_{i=0}^{k} A_i \lambda^i$.

Condition number (normwise) of λ:

$$K_P(\lambda) = \lim_{\varepsilon \to 0} \sup \left\{ \frac{|\Delta\lambda|}{\varepsilon|\lambda|} : (P(\lambda+\Delta\lambda) + \Delta(\lambda+\Delta\lambda))(x+\Delta x) = 0, \right. \\ \left. \| \Delta A_i \|_2 \leq \varepsilon \omega_i, i = 0 : k \right\}$$

Theorem (Tisseur, 2000)

x right e-vec, y left e-vec of λ. Then:

$$K_P(\lambda) = \frac{\left(\sum_{i=0}^{k} |\lambda|^i \| A_i \|\right) \| y \|_2 \| x \|_2}{|\lambda| |y^* P'(\lambda) x|}.$$

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Outline

Basic notation: ordered products and reverse permutation

Let $k \in \mathbb{N}$, and an **ordered** tuple $S = (i_1, \dots, i_s)$ (*index tuple*), with $0 \le i_1, \dots, i_s \le k$, and matrices $M_0, M_1, \dots, M_k \in \mathbb{C}^{\ell \times \ell}$, we set

$$\prod_S M_i := M_{i_1} \cdots M_{i_s}$$

for the product of $M_{i_1}, \dots, M_{i_s}$ **in the order given by S**.

In particular, for a permutation, $\sigma = (j_1, \dots, j_s)$:

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Definition (Fiedler, 2003–Antoniou & Vologiannidis, 2004)

Let $P(\lambda) = \lambda^k A_k + \cdots + \lambda A_1 + A_0\,,\ \ A_i \in \mathbb{C}^{n\times n}$. We define $nk \times nk$ matrices:

$$M_j := \begin{bmatrix} I_{n(k-j-1)} & & & \\ & -A_j & I_n & \\ & I_n & 0 & \\ & & & I_{n(j-1)} \end{bmatrix}, \quad j = 1, \ldots, k-1\,,$$

$$M_0 := \begin{bmatrix} I_{n(k-1)} & \\ & -A_0 \end{bmatrix}, \quad M_k := \begin{bmatrix} A_k & \\ & I_{n(k-1)} \end{bmatrix}.$$

Given any **permutation** σ of $(0, 1, \ldots, k-1)$, the **Fiedler pencil associated with** σ is

$$F_\sigma(\lambda) = \lambda M_k - \prod_\sigma M_i$$

Examples: Companion forms-Pentadiagonal Fiedler pencils

$$C_1(\lambda) = \lambda M_k - M_{k-1} \cdots M_1 M_0$$

$$C_2(\lambda) = \lambda M_k - M_0 M_1 \cdots M_{k-1}$$

$$T(\lambda) = \lambda M_k - (M_1 M_3 M_5 \cdots)(M_2 M_4 M_6 \cdots) M_0$$

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Definition (Fiedler (2003), Antoniou-Vologiannidis (2004))

Set:

- $\{S_0, S_1\}$ a partition of $\{0, 1, \ldots, k\}$, with $0 \in S_0, k \in S_1$.
- σ_0 a permutation of S_0; σ_1 a permutation of S_1.

Then the **(proper) generalized Fiedler pencil (GFP)** for $P(\lambda)$ associated with σ_0 and σ_1 is:

$$F_{(\sigma_0,\sigma_1)} = \lambda \prod_{\sigma_1} \widetilde{M}_i - \prod_{\sigma_0} M_j \,,$$

where

$$\widetilde{M}_i := \begin{cases} M_i^{-1}, & \text{if } i \neq k \\ M_k\,, & \text{if } i = k \end{cases} .$$

Remarks

Well-defined: Given $P(\lambda) = \lambda^k A_k + \cdots + \lambda A_1 + A_0\,, \;\; A_i \in \mathbb{C}^{n\times n}$, recall the matrices:

$$M_j := \begin{bmatrix} I_{n(k-j-1)} & & & \\ & -A_j & I_n & \\ & I_n & 0 & \\ & & & I_{n(j-1)} \end{bmatrix} \in \mathbb{C}^{nk\times nk}, \quad j = 1, \ldots, k-1\,,$$

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and note that $M_1, M_2, \ldots, M_{k-1}$ **are always invertible**.

Remark

If A_k and/or A_0 in $P(\lambda) = \lambda^k A_k + \cdots + \lambda A_1 + A_0$ are nonsingular, it is possible to multiply $F_\sigma(\lambda)$ by M_k^{-1} and/or M_0^{-1} and construct a **wider class of GFPs**, which

- contains symmetry-preserving linearizations of **even degree** polys.
- Not easy to construct: A_k^{-1} and/or A_0^{-1} are required.

For brevity, in this talk, we do not consider these pencils.

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Definition

Definition: An ordered tuple $(i_1,\ldots,i_t)$ of integers satisfies the **SIP** if for every pair i_a, i_b with $1 \leq a < b \leq t$ and $i_a = i_b$, there is $a < c < b$ such that $i_c = i_a + 1 = i_b + 1$.

SIP: Avoids products between $A_i's$.

Definition (Vologiannidis, Antoniou, 2011)

$P(\lambda)$ with degree $k \geq 2$, and $0 \leq h \leq k-1$. Let σ_0, σ_1 be permutations of $\{0,1,\ldots,h\}$ and $\{h+1,\ldots,k\}$, respectively. Let S be an index tuple with elements in $\{1,\ldots,h-1\}$ s.t. (σ_0, S) **satisfies the SIP**. Then

$$F_{(\sigma_0,\sigma_1,S)}(\lambda) = \left(\lambda \prod_{\sigma_1} \widetilde{M}_i - \prod_{\sigma_0} M_j\right) \cdot \prod_S M_\ell$$

is the **Fiedler pencil with repetition (FPR)** for $P(\lambda)$ assoc. to (σ_0, σ_1, S).

Remark: Can be extended considering products of $\widetilde{M}_i$ matrices, and even with products to the left, but we only consider this case for simplicity.

Examples (FP, GFP, FPR)

$$P(\lambda) = \lambda^3 A_3 + \lambda^2 A_2 + \lambda A_1 + A_0$$

FP: $$\lambda M_3 - M_1 M_2 M_0 = \lambda \begin{bmatrix} A_3 & & \\ & I & \\ & & I \end{bmatrix} + \begin{bmatrix} A_2 & -I & \\ A_1 & 0 & A_0 \\ -I & & \end{bmatrix}.$$

GFP: $$\lambda M_1^{-1} M_3 - M_2 M_0 = \lambda \begin{bmatrix} A_3 & & \\ & 0 & I \\ & I & A_1 \end{bmatrix} + \begin{bmatrix} A_2 & -I & \\ -I & 0 & \\ & & A_0 \end{bmatrix}.$$

FPR: $$(\lambda M_3 - M_1 M_2 M_0) M_1 = \lambda \begin{bmatrix} A_3 & & \\ & -A_1 & I \\ & I & 0 \end{bmatrix} + \begin{bmatrix} A_2 & A_1 & -I \\ A_1 & A_0 & \\ -I & & \end{bmatrix}.$$

Strong linearizations

Theorem (Antoniou-Volgiannidis, DT-Dopico-Mackey, Bueno-DT-Dopico)

All **FP**, **GFP** and **FPR** are **strong linearizations** for both regular and square singular matrix polynomials.

Consequence: Same eigenvalues as the polynomial.

Formulas for the eigenvectors?

We will focus on **right** eigenvectors (similar formulas for the left ones)

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Consecutions and inversions. Horner shifts

Let $\sigma = (j_0, j_1, \ldots, j_s)$ be a permutation of $S = \{i_1, i_2, \ldots, i_s\}$.
For $i = i_1, i_2, \ldots i_s$, we say that σ has a

- **consecution at** i, if $i, i+1 \in S$ and

$$\sigma = (\ldots, i, \ldots, i+1, \ldots)$$

- **inversion at** i, if $i, i+1 \in S$ and

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The d**th Horner shift** of $P(\lambda) = \lambda^k A_k + \cdots + \lambda A_1 + A_0$ is:

$$P_d(\lambda) := \lambda^d A_k + \cdots + \lambda A_{k-d+1} + A_{k-d}, \qquad 0 \leq d \leq k$$

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- **inversion at i**, if $i, i+1 \in S$ and
$$\sigma = (\ldots, i+1, \ldots, i, \ldots)$$

The d**th Horner shift** of $P(\lambda) = \lambda^k A_k + \cdots + \lambda A_1 + A_0$ is:

$$P_d(\lambda) := \lambda^d A_k + \cdots + \lambda A_{k-d+1} + A_{k-d}, \qquad 0 \le d \le k$$

E-vecs of Fiedler pencils

Theorem

$P(\lambda)$ **regular** $n \times n$ of degree $k \geq 2$ and $F_\sigma(\lambda)$ the Fiedler pencil associated to a bijection σ of $\{0, 1, \ldots, k-1\}$. Then $v \in \mathbb{C}^{nk \times nk}$ is an **eigenvector** of $F_\sigma(\lambda)$ associated with λ_0 iff:

$$v = \begin{bmatrix} B_0 \\ \vdots \\ B_{k-1} \end{bmatrix} x,$$

where, for $i = 0, \ldots, k-1$,

$$\mathbf{B}_i = \begin{cases} \lambda^j P_i, & \text{if } i \geq 1 \text{ and } \sigma \text{ has a consecution at } k-i-1 \\ \lambda^j I, & \text{if } i = 0 \text{ or } \sigma \text{ has an inversion at } k-i-1 \end{cases}$$

with $j =$ *number of inversions of* σ *from* 0 *to* $k-i-2$, and $x \in \mathbb{C}^n$ eigenvector of $P(\lambda)$ associated with λ_0.

E-vecs of Generalized Fiedler pencils

Theorem

$P(\lambda)$ **regular** $n \times n$ of degree $k \geq 2$ and $F_{(\sigma_0,\sigma_1)}(\lambda) = \lambda \widetilde{M}_{\sigma_1} - M_{\sigma_0}$ the GFP associated to σ_0 and σ_1. Write

$$\prod_{\sigma_1} \widetilde{M}_i = \prod_{\widehat{\sigma}_1} \widetilde{M}_i \prod_{(k,k-1,\ldots,k-\iota_k)} \widetilde{M}_i.$$

Then $v \in \mathbb{C}^{nk \times nk}$ is an **eigenvector** of $F_{(\sigma_0,\sigma_1)}(\lambda)$ associated with λ_0 iff:

$$v = \left[\begin{array}{cccc|ccc} \lambda^s P_0 & \lambda^s P_1 & \ldots & \lambda^s P_{\iota_k - 1} & B_0 & \ldots & B_{k-\iota_k-1} \end{array} \right]^{\mathscr{B}} x,$$

where $\mathbf{s} = 1 +$ *number of inversions of* $(\mathrm{rev}\, \widehat{\sigma}_1, \sigma_0)$, and, for $i = 0, \ldots, k - \iota_k - 1$,

$$\mathbf{B}_i = \begin{cases} \lambda^j P_i, & \text{if } i \geq 1 \text{ and } (\mathrm{rev}\, \widehat{\sigma}_1, \sigma_0) \text{ has a consecution at } k - \iota_k - i - 1 \\ \lambda^j I, & \text{if } i = 0 \text{ or } (\mathrm{rev}\, \widehat{\sigma}_1, \sigma_0) \text{ has an inversion at } k - \iota_k - i - 1 \end{cases},$$

with $j =$ *number of inversions of* $(\mathrm{rev}\, \widehat{\sigma}_1, \sigma_0)$ *from* 0 *to* $k - \iota_k - i - 2$, and $x \in \mathbb{C}^n$ eigenvector of $P(\lambda)$ associated with λ_0.

FPR of type 1

Definition

Let σ be a permutation of $\{0,1,\ldots,h\}$, and $0 < s < h$. Then s is a **type 1** index for σ if σ has an inversion at $s-1$ and a consecution at s.
In this case, $\sigma = (\ldots, s, \ldots, s+1, \ldots, s-1, \ldots)$, and the **associated permutation** to (σ, s) is $\tau(\sigma, s) = (\ldots, s+1, \ldots, s-1, s, \ldots)$.

Definition

Let $F_{(\sigma_0,\sigma_1,S)}(\lambda)$ be the FPR for $P(\lambda)$ assoc. to (σ_0, σ_1, S), with $S = (s_1, \ldots, s_r)$. Then it is a **type 1 FPR** if, for each $j = 1, \ldots, r$, s_j is a type 1 index associated to $\tau(\sigma_0, \mathbf{s_1}, \ldots, \mathbf{s_{j-1}}) = \tau(\tau(\sigma_0, \mathbf{s_1}, \ldots, \mathbf{s_{j-2}}), \mathbf{s_{j-1}})$.

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GF associated to a type 1 FPR

Definition

Given a type 1 FPR, $F_{(\sigma_0,\sigma_1,S)}(\lambda)$, the **GF pencil associated to** $F_{(\sigma_0,\sigma_1,S)}(\lambda)$ is $F_{(\tau(\sigma_0,S),\sigma_1)}(\lambda)$.

Example: $k=9, h=6$ and
$\sigma_0=(3,4,5,6,1,2,0)$, $\sigma_1=(7,9,8)$, $S=(3)$, then

$$F_{(\sigma_0,\sigma_1,S)}=\left(\lambda M_7^{-1}M_9M_8^{-1}-M_3M_4M_5M_6M_1M_2M_0\right)M_3,$$

and

$$F_{(\tau(\sigma_0,3),\sigma_1)}=\lambda M_7^{-1}M_9M_8^{-1}-M_4M_5M_6M_1M_2M_3M_0.$$

GF associated to a type 1 FPR

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Given a type 1 FPR, $F_{(\sigma_0,\sigma_1,S)}(\lambda)$, the **GF pencil associated to** $F_{(\sigma_0,\sigma_1,S)}(\lambda)$ is $F_{(\tau(\sigma_0,S),\sigma_1)}(\lambda)$.

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and

$$F_{(\tau(\sigma_0,3),\sigma_1)}=\lambda M_7^{-1}M_9M_8^{-1}-M_4M_5M_6M_1M_2M_3M_0.$$

E-vecs of (type 1) Fiedler pencils with repetition

Theorem

- $P(\lambda)$ **regular** $n \times n$ of degree $k \geq 2$.
- $F_{(\sigma_0,\sigma_1,S)}(\lambda)$ the type 1 FPR associated to σ_0, σ_1 and S.
- $F_{(\tau(\sigma_0,S),\sigma_1)}(\lambda)$ be the GF pencil associated to $F_{(\sigma_0,\sigma_1,S)}(\lambda)$.

Then a **right eigenvector of** $F_{(\sigma_0,\sigma_1,S)}(\lambda)$ is a **right eigenvector of** $F_{(\tau(\sigma_0,S),\sigma_1)}(\lambda)$.

Example

Let $k=9, h=6$ and $\sigma_0=(3,4,5,6,1,2,0), \sigma_1=(7,9,8), S=(3)$. Then, a right e-vec of the **type 1 FPR:** $F_{(\sigma_0,\sigma_1,S)}$ is an e-vec of $F_{(\tau(\sigma_0,S),\sigma_1)} = \lambda M_7^{-1} M_9 M_8^{-1} - M_4 M_5 M_6 M_1 M_2 M_3 M_0$. This is:

$$v = \begin{bmatrix} \\ \\ \\ \end{bmatrix} x \qquad (x \text{ e-vec of } P(\lambda))$$

$\widehat{\sigma}_1=(7) \Rightarrow (\mathrm{rev}\,\widehat{\sigma}_1, \tau(\sigma_0,3)) = (7,4,5,6,1,2,3,0) \Rightarrow$ *1+number of inversions of* $(\mathrm{rev}\,\widehat{\sigma}_1, \tau(\sigma_0,3)) = 4$.

Example

Let $k = 9, h = 6$ and $\sigma_0 = (3,4,5,6,1,2,0), \sigma_1 = (7,9,8), S = (3)$. Then, a right e-vec of the **type 1 FPR:** $F_{(\sigma_0,\sigma_1,S)}$ is an e-vec of $F_{(\tau(\sigma_0,S),\sigma_1)} = \lambda M_7^{-1} M_9 M_8^{-1} - M_4 M_5 M_6 M_1 M_2 M_3 M_0$. This is:

$$v = \begin{bmatrix} \\ \\ \\ \end{bmatrix} x \qquad (x \text{ e-vec of } P(\lambda))$$

$\widehat{\sigma}_1 = (7) \Rightarrow (\mathrm{rev}\,\widehat{\sigma}_1, \tau(\sigma_0,3)) = (7,4,5,6,1,2,3,0) \Rightarrow$ *1+number of inversions of* $(\mathrm{rev}\,\widehat{\sigma}_1, \tau(\sigma_0,3)) = 4$.

Example

Let $k = 9, h = 6$ and $\sigma_0 = (3,4,5,6,1,2,0), \sigma_1 = (7,9,8), S = (3)$. Then, a right e-vec of the **type 1 FPR**: $F_{(\sigma_0,\sigma_1,S)}$ is an e-vec of $F_{(\tau(\sigma_0,S),\sigma_1)} = \lambda M_7^{-1} M_9 M_8^{-1} - M_4 M_5 M_6 M_1 M_2 M_3 M_0$. This is:

$$v = \begin{bmatrix} \lambda^4 A_9 \\ \\ \\ \\ \\ \end{bmatrix} x \qquad (x \text{ e-vec of } P(\lambda))$$

$\widehat{\sigma}_1 = (7) \Rightarrow (\operatorname{rev} \widehat{\sigma}_1, \tau(\sigma_0, 3)) = (7,4,5,6,1,2,3,0) \Rightarrow$ *1+number of inversions of* $(\operatorname{rev} \widehat{\sigma}_1, \tau(\sigma_0, 3)) = 4$.

Example

Let $k = 9, h = 6$ and $\sigma_0 = (3,4,5,6,1,2,0), \sigma_1 = (7,9,8), S = (3)$. Then, a right e-vec of the **type 1 FPR**: $F_{(\sigma_0,\sigma_1,S)}$ is an e-vec of $F_{(\tau(\sigma_0,S),\sigma_1)} = \lambda M_7^{-1} M_9 M_8^{-1} - M_4 M_5 M_6 M_1 M_2 M_3 M_0$. This is:

$$v = \begin{bmatrix} \lambda^4 A_9 \\ \\ \\ \\ \\ \end{bmatrix} x \qquad (x \text{ e-vec of } P(\lambda))$$

$\imath_k = 1, (\mathrm{rev}\,\widehat{\sigma}_1, \tau(\sigma_0,3)) = (7,4,5,6,1,2,3,0) \Rightarrow$ *number of inversions of* $(\mathrm{rev}\,\widehat{\sigma}_1, \tau(\sigma_0,3))$ *from 0 to* $k - \imath_k - 0 - 2 = 3$.

Example

Let $k = 9, h = 6$ and $\sigma_0 = (3,4,5,6,1,2,0), \sigma_1 = (7,9,8), S = (3)$. Then, a right e-vec of the **type 1 FPR**: $F_{(\sigma_0,\sigma_1,S)}$ is an e-vec of $F_{(\tau(\sigma_0,S),\sigma_1)} = \lambda M_7^{-1} M_9 M_8^{-1} - M_4 M_5 M_6 M_1 M_2 M_3 M_0$. This is:

$$v = \begin{bmatrix} \lambda^4 A_9 \\ \hline \lambda^3 I \\ \\ \\ \\ \\ \\ \end{bmatrix} x \qquad (x \text{ e-vec of } P(\lambda))$$

$\iota_k = 1, (\mathrm{rev}\,\widehat{\sigma}_1, \tau(\sigma_0,3)) = (7,4,5,6,1,2,3,0) \Rightarrow$ *number of inversions of* $(\mathrm{rev}\,\widehat{\sigma}_1, \tau(\sigma_0,3))$ *from 0 to* $k - \iota_k - 0 - 2 = 3$.

Example

Let $k=9, h=6$ and $\sigma_0=(3,4,5,6,1,2,0), \sigma_1=(7,9,8), S=(3)$. Then, a right e-vec of the **type 1 FPR**: $F_{(\sigma_0,\sigma_1,S)}$ is an e-vec of $F_{(\tau(\sigma_0,S),\sigma_1)} = \lambda M_7^{-1} M_9 M_8^{-1} - M_4 M_5 M_6 M_1 M_2 M_3 M_0$. This is:

$$v = \begin{bmatrix} \lambda^4 A_9 \\ \hline \lambda^3 I \\ \\ \\ \\ \\ \end{bmatrix} x \qquad (x \text{ e-vec of } P(\lambda))$$

$\iota_k = 1, (\mathrm{rev}\,\widehat{\sigma}_1, \tau(\sigma_0,3)) = (7,4,5,6,1,2,3,0) \Rightarrow$ *number of inversions of* $(\mathrm{rev}\,\widehat{\sigma}_1, \tau(\sigma_0,3))$ *from 0 to* $k-\iota_k-1-2=2$ + $(\mathrm{rev}\,\widehat{\sigma}_1, \tau(\sigma_0,3))$ has inversion at $6 = k-\iota_k-1-1$.

Example

Let $k=9, h=6$ and $\sigma_0=(3,4,5,6,1,2,0), \sigma_1=(7,9,8), S=(3)$. Then, a right e-vec of the **type 1 FPR**: $F_{(\sigma_0,\sigma_1,S)}$ is an e-vec of $F_{(\tau(\sigma_0,S),\sigma_1)} = \lambda M_7^{-1} M_9 M_8^{-1} - M_4 M_5 M_6 M_1 M_2 M_3 M_0$. This is:

$$v = \begin{bmatrix} \lambda^4 A_9 \\ \hline \lambda^3 I \\ \lambda^2 I \\ \\ \\ \\ \\ \\ \end{bmatrix} x \qquad (x \text{ e-vec of } P(\lambda))$$

$\iota_k = 1, (\operatorname{rev}\widehat{\sigma}_1, \tau(\sigma_0,3)) = (7,4,5,6,1,2,3,0) \Rightarrow$ *number of inversions of* $(\operatorname{rev}\widehat{\sigma}_1, \tau(\sigma_0,3))$ *from 0 to* $k-\iota_k-1-2=2$ + $(\operatorname{rev}\widehat{\sigma}_1, \tau(\sigma_0,3))$ has inversion at $6=k-\iota_k-1-1$.

Example

Let $k=9, h=6$ and $\sigma_0=(3,4,5,6,1,2,0), \sigma_1=(7,9,8), S=(3)$. Then, a right e-vec of the **type 1 FPR**: $F_{(\sigma_0,\sigma_1,S)}$ is an e-vec of $F_{(\tau(\sigma_0,S),\sigma_1)} = \lambda M_7^{-1} M_9 M_8^{-1} - M_4 M_5 M_6 M_1 M_2 M_3 M_0$. This is:

$$v = \begin{bmatrix} \lambda^4 A_9 \\ \hline \lambda^3 I \\ \lambda^2 I \\ \\ \\ \\ \\ \\ \end{bmatrix} x \qquad (x \text{ e-vec of } P(\lambda))$$

$\imath_k = 1, (\mathrm{rev}\,\widehat{\sigma}_1, \tau(\sigma_0,3)) = (7,4,5,6,1,2,3,0) \Rightarrow$ *number of inversions of* $(\mathrm{rev}\,\widehat{\sigma}_1, \tau(\sigma_0,3))$ *from 0 to* $k-\imath_k-2-2=2$ + $(\mathrm{rev}\,\widehat{\sigma}_1, \tau(\sigma_0,3))$ has consecution at $5=k-\imath_k-1-1$.

Example

Let $k=9, h=6$ and $\sigma_0=(3,4,5,6,1,2,0), \sigma_1=(7,9,8), S=(3)$. Then, a right e-vec of the **type 1 FPR**: $F_{(\sigma_0,\sigma_1,S)}$ is an e-vec of $F_{(\tau(\sigma_0,S),\sigma_1)} = \lambda M_7^{-1} M_9 M_8^{-1} - M_4 M_5 M_6 M_1 M_2 M_3 M_0$. This is:

$$v = \begin{bmatrix} \lambda^4 A_9 \\ \hline \lambda^3 I \\ \lambda^2 I \\ \lambda^2 P_3 \\ \\ \\ \\ \\ \end{bmatrix} x \qquad (x \text{ e-vec of } P(\lambda))$$

$\imath_k = 1, (\mathrm{rev}\,\widehat{\sigma}_1, \tau(\sigma_0,3)) = (7,4,5,6,1,2,3,0) \Rightarrow$ *number of inversions of* $(\mathrm{rev}\,\widehat{\sigma}_1, \tau(\sigma_0,3))$ *from 0 to* $k - \imath_k - 2 - 2 = 2$ + $(\mathrm{rev}\,\widehat{\sigma}_1, \tau(\sigma_0,3))$ has consecution at $5 = k - \imath_k - 2 - 1$.

Example

Let $k=9, h=6$ and $\sigma_0=(3,4,5,6,1,2,0), \sigma_1=(7,9,8), S=(3)$. Then, a right e-vec of the **type 1 FPR**: $F_{(\sigma_0,\sigma_1,S)}$ is an e-vec of $F_{(\tau(\sigma_0,S),\sigma_1)} = \lambda M_7^{-1} M_9 M_8^{-1} - M_4 M_5 M_6 M_1 M_2 M_3 M_0$. This is:

$$v = \left[\begin{array}{c} \lambda^4 A_9 \\ \hline \lambda^3 I \\ \lambda^2 I \\ \lambda^2 P_3 \\ \\ \\ \\ \\ \\ \end{array}\right] x \qquad (x \text{ e-vec of } P(\lambda))$$

$\iota_k = 1, (\operatorname{rev}\widehat{\sigma}_1, \tau(\sigma_0,3)) = (7,4,5,6,1,2,3,0) \Rightarrow$ *number of inversions of* $(\operatorname{rev}\widehat{\sigma}_1, \tau(\sigma_0,3))$ *from 0 to* $k-\iota_k-3-2=2$ + $(\operatorname{rev}\widehat{\sigma}_1, \tau(\sigma_0,3))$ has consecution at $4=k-\iota_k-3-1$.

Example

Let $k=9, h=6$ and $\sigma_0=(3,4,5,6,1,2,0), \sigma_1=(7,9,8), S=(3)$. Then, a right e-vec of the **type 1 FPR**: $F_{(\sigma_0,\sigma_1,S)}$ is an e-vec of $F_{(\tau(\sigma_0,S),\sigma_1)} = \lambda M_7^{-1} M_9 M_8^{-1} - M_4 M_5 M_6 M_1 M_2 M_3 M_0$. This is:

$$v = \begin{bmatrix} \lambda^4 A_9 \\ \hline \lambda^3 I \\ \lambda^2 I \\ \lambda^2 P_3 \\ \lambda^2 P_4 \\ \\ \\ \\ \\ \end{bmatrix} x \qquad (x \text{ e-vec of } P(\lambda))$$

$\imath_k = 1, (\operatorname{rev}\widehat{\sigma}_1, \tau(\sigma_0,3)) = (7,4,5,6,1,2,3,0) \Rightarrow$ *number of inversions of* $(\operatorname{rev}\widehat{\sigma}_1, \tau(\sigma_0,3))$ *from 0 to* $k-\imath_k-3-2=2$ + $(\operatorname{rev}\widehat{\sigma}_1, \tau(\sigma_0,3))$ has consecution at $4=k-\imath_k-3-1$.

Example

Let $k=9, h=6$ and $\sigma_0=(3,4,5,6,1,2,0), \sigma_1=(7,9,8), S=(3)$. Then, a right e-vec of the **type 1 FPR**: $F_{(\sigma_0,\sigma_1,S)}$ is an e-vec of $F_{(\tau(\sigma_0,S),\sigma_1)} = \lambda M_7^{-1} M_9 M_8^{-1} - M_4 M_5 M_6 M_1 M_2 M_3 M_0$. This is:

$$v = \begin{bmatrix} \lambda^4 A_9 \\ \hline \lambda^3 I \\ \lambda^2 I \\ \lambda^2 P_3 \\ \lambda^2 P_4 \\ \\ \\ \\ \\ \end{bmatrix} x \qquad (x \text{ e-vec of } P(\lambda))$$

$\imath_k = 1, (\mathrm{rev}\,\widehat{\sigma}_1, \tau(\sigma_0,3)) = (7,4,5,6,1,2,3,0) \Rightarrow$ *number of inversions of* $(\mathrm{rev}\,\widehat{\sigma}_1, \tau(\sigma_0,3))$ *from 0 to* $k-\imath_k-4-2=1$ + $(\mathrm{rev}\,\widehat{\sigma}_1, \tau(\sigma_0,3))$ has inversion at $3=k-\imath_k-4-1$.

Example

Let $k=9, h=6$ and $\sigma_0=(3,4,5,6,1,2,0), \sigma_1=(7,9,8), S=(3)$. Then, a right e-vec of the **type 1 FPR**: $F_{(\sigma_0,\sigma_1,S)}$ is an e-vec of $F_{(\tau(\sigma_0,S),\sigma_1)} = \lambda M_7^{-1} M_9 M_8^{-1} - M_4 M_5 M_6 M_1 M_2 M_3 M_0$. This is:

$$v = \begin{bmatrix} \lambda^4 A_9 \\ \hline \lambda^3 I \\ \lambda^2 I \\ \lambda^2 P_3 \\ \lambda^2 P_4 \\ \lambda^1 I \\ \\ \\ \\ \end{bmatrix} x \qquad (x \text{ e-vec of } P(\lambda))$$

$\imath_k = 1, (\operatorname{rev}\widehat{\sigma}_1, \tau(\sigma_0,3)) = (7,4,5,6,1,2,3,0) \Rightarrow$ *number of inversions of* $(\operatorname{rev}\widehat{\sigma}_1, \tau(\sigma_0,3))$ *from 0 to* $k-\imath_k-4-2=1$ + $(\operatorname{rev}\widehat{\sigma}_1, \tau(\sigma_0,3))$ has inversion at $3=k-\imath_k-4-1$.

Example

Let $k=9, h=6$ and $\sigma_0=(3,4,5,6,1,2,0), \sigma_1=(7,9,8), S=(3)$. Then, a right e-vec of the **type 1 FPR**: $F_{(\sigma_0,\sigma_1,S)}$ is an e-vec of $F_{(\tau(\sigma_0,S),\sigma_1)} = \lambda M_7^{-1} M_9 M_8^{-1} - M_4 M_5 M_6 M_1 M_2 M_3 M_0$. This is:

$$v = \begin{bmatrix} \lambda^4 A_9 \\ \hline \lambda^3 I \\ \lambda^2 I \\ \lambda^2 P_3 \\ \lambda^2 P_4 \\ \lambda I \\ \\ \\ \\ \end{bmatrix} x \qquad (x \text{ e-vec of } P(\lambda))$$

$\imath_k=1, (\mathrm{rev}\,\widehat{\sigma}_1, \tau(\sigma_0,3)) = (7,4,5,6,1,2,3,0) \Rightarrow$ *number of inversions of* $(\mathrm{rev}\,\widehat{\sigma}_1, \tau(\sigma_0,3))$ *from 0 to* $k-\imath_k-5-2=1$ + $(\mathrm{rev}\,\widehat{\sigma}_1, \tau(\sigma_0,3))$ has consecution at $3=k-\imath_k-5-1$.

Example

Let $k=9, h=6$ and $\sigma_0=(3,4,5,6,1,2,0), \sigma_1=(7,9,8), S=(3)$. Then, a right e-vec of the **type 1 FPR**: $F_{(\sigma_0,\sigma_1,S)}$ is an e-vec of $F_{(\tau(\sigma_0,S),\sigma_1)} = \lambda M_7^{-1}M_9M_8^{-1} - M_4M_5M_6M_1M_2M_3M_0$. This is:

$$v = \begin{bmatrix} \lambda^4 A_9 \\ \hline \lambda^3 I \\ \lambda^2 I \\ \lambda^2 P_3 \\ \lambda^2 P_4 \\ \lambda I \\ \lambda^1 P_6 \\ \\ \\ \end{bmatrix} x \qquad (x \text{ e-vec of } P(\lambda))$$

$\imath_k = 1, (\operatorname{rev}\widehat{\sigma}_1, \tau(\sigma_0,3)) = (7,4,5,6,1,2,3,0) \Rightarrow$ *number of inversions of* $(\operatorname{rev}\widehat{\sigma}_1, \tau(\sigma_0,3))$ *from 0 to* $k-\imath_k-5-2=1$ + $(\operatorname{rev}\widehat{\sigma}_1, \tau(\sigma_0,3))$ has consecution at $2=k-\imath_k-5-1$.

Example

Let $k=9, h=6$ and $\sigma_0=(3,4,5,6,1,2,0), \sigma_1=(7,9,8), S=(3)$. Then, a right e-vec of the **type 1 FPR**: $F_{(\sigma_0,\sigma_1,S)}$ is an e-vec of $F_{(\tau(\sigma_0,S),\sigma_1)} = \lambda M_7^{-1} M_9 M_8^{-1} - M_4 M_5 M_6 M_1 M_2 M_3 M_0$. This is:

$$v = \begin{bmatrix} \lambda^4 A_9 \\ \hline \lambda^3 I \\ \lambda^2 I \\ \lambda^2 P_3 \\ \lambda^2 P_4 \\ \lambda I \\ \lambda P_6 \\ \\ \\ \end{bmatrix} x \qquad (x \text{ e-vec of } P(\lambda))$$

$\imath_k = 1, (\operatorname{rev}\widehat{\sigma}_1, \tau(\sigma_0,3)) = (7,4,5,6,1,2,3,0) \Rightarrow$ *number of inversions of* $(\operatorname{rev}\widehat{\sigma}_1, \tau(\sigma_0,3))$ *from 0 to* $k-\imath_k-6-2=1$ + $(\operatorname{rev}\widehat{\sigma}_1, \tau(\sigma_0,3))$ has consecution at $1 = k-\imath_k-6-1$.

Example

Let $k = 9, h = 6$ and $\sigma_0 = (3,4,5,6,1,2,0), \sigma_1 = (7,9,8), S = (3)$. Then, a right e-vec of the **type 1 FPR**: $F_{(\sigma_0,\sigma_1,S)}$ is an e-vec of $F_{(\tau(\sigma_0,S),\sigma_1)} = \lambda M_7^{-1} M_9 M_8^{-1} - M_4 M_5 M_6 M_1 M_2 M_3 M_0$. This is:

$$v = \begin{bmatrix} \lambda^4 A_9 \\ \hline \lambda^3 I \\ \lambda^2 I \\ \lambda^2 P_3 \\ \lambda^2 P_4 \\ \lambda I \\ \lambda P_6 \\ \lambda^1 P_7 \\ \\ \end{bmatrix} x \qquad (x \text{ e-vec of } P(\lambda))$$

$\imath_k = 1, (\operatorname{rev} \widehat{\sigma}_1, \tau(\sigma_0,3)) = (7,4,5,6,1,2,3,0) \Rightarrow$ *number of inversions of* $(\operatorname{rev} \widehat{\sigma}_1, \tau(\sigma_0,3))$ *from 0 to* $k - \imath_k - 6 - 2 = 1$ + $(\operatorname{rev} \widehat{\sigma}_1, \tau(\sigma_0,3))$ has consecution at $1 = k - \imath_k - 6 - 1$.

Example

Let $k=9, h=6$ and $\sigma_0=(3,4,5,6,1,2,0), \sigma_1=(7,9,8), S=(3)$. Then, a right e-vec of the **type 1 FPR**: $F_{(\sigma_0,\sigma_1,S)}$ is an e-vec of $F_{(\tau(\sigma_0,S),\sigma_1)} = \lambda M_7^{-1} M_9 M_8^{-1} - M_4 M_5 M_6 M_1 M_2 M_3 M_0$. This is:

$$v = \begin{bmatrix} \lambda^4 A_9 \\ \hline \lambda^3 I \\ \lambda^2 I \\ \lambda^2 P_3 \\ \lambda^2 P_4 \\ \lambda I \\ \lambda P_6 \\ \lambda P_7 \\ \\ \end{bmatrix} x \qquad (x \text{ e-vec of } P(\lambda))$$

$\imath_k = 1, (\mathrm{rev}\,\widehat{\sigma}_1, \tau(\sigma_0,3)) = (7,4,5,6,1,2,3,0) \Rightarrow$ *number of inversions of* $(\mathrm{rev}\,\widehat{\sigma}_1, \tau(\sigma_0,3))$ *from 0 to* $k-\imath_k-7-2=0$ + $(\mathrm{rev}\,\widehat{\sigma}_1, \tau(\sigma_0,3))$ has consecution at $0 = k-\imath_k-7-1$.

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Outline

Conditioning for linearizations

Theorem

- $P(\lambda)$ regular,
- $F(\lambda)$, a Fiedler-like lin'z of $P(\lambda)$,
- x/y, right/left e-vecs of $P(\lambda)$ assoc. to λ_0,
- v/w, the associated left and right e-vecs of $F(\lambda)$.

$\Rightarrow$ $|w^* F'(\lambda_0) v| = |y^* P'(\lambda_0) x|$.

Given two linz's $L_1(\lambda) = X_1 + \lambda Y_1$ and $L_2(\lambda) = X_2 + \lambda Y_2$, we have

$$\kappa_{L_1}(\lambda_0)/\kappa_{L_2}(\lambda_0) = \frac{(|\lambda| \cdot \|Y_1\| + \|X_1\|)\|w_1\| \cdot \|v_1\|}{(|\lambda| \cdot \|Y_2\| + \|X_2\|)\|w_2\| \cdot \|v_2\|},$$

and even the conditioning of $L(\lambda) = X + \lambda Y$ with the one the polynomial:

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Work in progress (with F. Tisseur)

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