Flanders' Theorem for many matrices under commutativity assumptions

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Joint work with
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Outline

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JCF(AB) vs JCF(BA)

Notation:

- JCF(M) $=$ Jordan Canonical Form of M.
- $\mathcal{S}_\lambda(M) = (n_1, n_2, \ldots, 0, 0, \ldots)$ $=$ **Segré characteristic** of M at $\lambda \in \mathbb{C}$ (infinite sequence of ordered sizes $n_1 \geq n_2 \geq \ldots$ of Jordan blocks at λ in JCF(M)).

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Theorem (Flanders, 1951)

Given $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times m}$, set $M = AB, N = BA$.

(i) $\mathcal{S}_\lambda(M) = \mathcal{S}_\lambda(N)$ for all $\lambda \neq 0$.

(ii) $\|\mathcal{S}_0(M) - \mathcal{S}_0(N)\|_\infty \leq 1$.

Conversely, if $M \in \mathbb{C}^{m\times m}$ and $N \in \mathbb{C}^{n\times n}$ satisfy (i)–(ii), then $M = AB$ and $N = BA$, for some A, B.

In plain words: JCF(AB) and JCF(BA) can only differ in the J-blocks at 0, and the corresponding sizes differ, at most, by 1, and this happens **only** for matrices of the form AB and BA.

Some history

Proved in:

H. Flanders
The elementary divisors of *AB* and *BA*.
Proc. Am. Math. Soc. 2 (1951) 871–874.

And later in:

W. V. Parker, B. E. Mitchell.
Elementary divisors of certain matrices.
Duke Math. J. 19 (1952) 483–485.

R. C. Thompson.
On the matrices *AB* and *BA*.
Linear Algebra Appl. 1 (1968) 43–58.

S. Bernau, A. Abian.
Jordan canonical forms of matrices *AB* and *BA*.
Rend. Istit. Mat. Univ. Trieste. 20 (1988) 101–108.

C. R. Johnson, E. S. Schreiner.
The relationship between *AB* and *BA*.
Amer. Math. Monthly 103 (1996) 578–581.

R. A. Lippert, G. Strang.
The Jordan form of *AB* and *BA*.
Electron. J. Linear Algebra 18 (2009) 281–288.

Flanders again: exhaustivity

Moreover:

Theorem (Flanders, 1951)

Let $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots)$, and $\boldsymbol{\mu}' = (\mu_1', \mu_2', \ldots)$ be two lists of integers with $\mu_1 \geq \mu_2 \geq \ldots \geq 0$, and $\mu_1' \geq \mu_2' \geq \ldots \geq 0$, with:

(i) $\|\boldsymbol{\mu} - \boldsymbol{\mu}'\|_\infty \leq 1$, and

(ii) $\|\boldsymbol{\mu}\|_1 = m$, $\|\boldsymbol{\mu}'\|_1 = n$.

Then, there are $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times m}$ with $\mathcal{S}_0(AB) = \boldsymbol{\mu}$ and $\mathcal{S}_0(BA) = \boldsymbol{\mu}'$.

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Three matrices: $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{p\times q}, C \in \mathbb{C}^{r\times s}$ of “appropriate" sizes. What dose this mean?:

- ABC must be defined $(m \times n) \cdot (p \times q) \cdot (r \times s) \Rightarrow n = p,\ q = r$.
- ACB must be defined $(m \times n) \cdot (r \times s) \cdot (p \times q) \Rightarrow n = r,\ s = p$.
- CAB must be defined $(r \times s) \cdot (m \times n) \cdot (p \times q) \Rightarrow s = m,\ n = p$.

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Then: $m = n = p = q = r = s$.

More than two matrices? Cyclic permutations.

Three matrices: $A, B, C \in \mathbb{C}^{n\times n}$. Then

- $\mathcal{S}_1 = \{ABC, CAB, BCA\}$: Any two here satisfy Flanders' Theorem.
- $\mathcal{S}_2 = \{ACB, BAC, CBA\}$: Any two here satisfy Flanders' Theorem.

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Q: What happens with one from $\mathcal{S}_1$ and another one from $\mathcal{S}_2$?

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Theorem

Let

$$\Lambda_1 = \{\lambda_{11}, \dots, \lambda_{n1}\}, \qquad \Lambda_2 = \{\lambda_{12}, \dots, \lambda_{n2}\}$$

be two sets of n nonzero complex numbers (with possible repetitions).

If $\lambda_{11} \cdots \lambda_{n1} = \lambda_{12} \cdots \lambda_{n2}$, there are $A, B, C \in \mathbb{C}^{n\times n}$, such that

$$\Lambda(ABC) = \Lambda_1, \qquad \Lambda(CBA) = \Lambda_2.$$

More than two matrices? Anything may happen with the zero e-val.

☞ The sizes of Jordan blocks at 0 in JCF(ABC) and JCF(CBA) can be arbitrarily different !!

$$A = \begin{bmatrix} 1 & & & \\ & 1/2 & & \\ & & \ddots & \\ & & & 1/n \end{bmatrix}, \quad B = \begin{bmatrix} 1 & & & \\ -1 & \ddots & & \\ & \ddots & 1 & \\ & & -1 & 1 \end{bmatrix}, \quad C = (AB)^{-1} \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}.$$

- $ABC = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} (= J_n(0)).$
- The e-vals of CBA are: $0, \lambda_1, \ldots, \lambda_{n-1}$, with $\lambda_1 \cdots \lambda_{n-1} \neq 0$.

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- The e-vals of *CBA* are: $0, \lambda_1, \ldots, \lambda_{n-1}$, with $\lambda_1 \cdots \lambda_{n-1} \neq 0$.

(IDEA: 0 is a simple eigenvalue of *CBA*: $\operatorname{rank}(CBA) = n-1$ and $(CBA)\,[\,1\ 2\ \cdots\ n\,]^\top = 0$. But $(CBA)v_1 = [\,1\ 2\ \cdots\ n\,]^\top$ is impossible, since this would imply $Cw = [\,1\ 2\ \cdots\ n\,]^\top$, but the last two entries of Cw must coincide, since the last two rows of C are the same.)

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Which ones ?

Flanders pairs and bridges

Set $M \in \mathbb{C}^{m\times m}$, $N \in \mathbb{C}^{n\times n}$.

Definition

(M, N) is a Flanders pair if $M = AB, N = BA$, for some $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times m}$.

There is a Flanders bridge between M and N if (M, N) is a Flanders pair.

Note: Not transitive !!!

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Example:

$M = J_3(0),\ Q = \mathrm{diag}(J_2(0), J_1(0)),\ N = \mathrm{diag}(J_1(0), J_1(0), J_1(0)) \equiv 0_{3\times 3}$.

Then (M, Q) and (Q, N) are Flanders pairs, but (M, N) is **not**.

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Corollary (of Flanders' Theorem)

If $(M_1, M_2), (M_2, M_3), \ldots, (M_d, M_{d+1})$ are Flanders pairs, then:

(i) $\mathcal{S}_\lambda(M_1) = \mathcal{S}_\lambda(M_{d+1})$, for all $\lambda \neq 0$.

(ii) $\|\mathcal{S}_0(M_1) - \mathcal{S}_0(M_{d+1})\|_\infty \leq d$.

Sequences of Flanders pairs allow us to **relate the JCF of two matrices**

The problems

Given $A_1, \ldots, A_k \in \mathbb{C}^{n\times n}$, we set:
$\mathcal{P}(A_1, \ldots, A_k) := \{A_{i_1} \cdots A_{i_k} : (i_1, \ldots, i_k) \text{ a permutation of } (1, \ldots, k)\}$
("Permuted products" of $A_1, \ldots, A_k$)

Three questions (after Flanders' Theorem):

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Question 1: Find necessary and sufficient conditions on $A_1, \ldots, A_k$ such that:

(i) $\mathcal{S}_\lambda(M) = \mathcal{S}_\lambda(N)$, for all $\lambda \neq 0$ and all $M, N \in \mathcal{P}(A_1, \ldots, A_k)$, and

(ii) $\|\mathcal{S}_0(M) - \mathcal{S}_0(N)\|_\infty \leq d$, for any $M, N \in \mathcal{P}(A_1, \ldots, A_k)$ and
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Three questions (after Flanders’ Theorem):

Question 2: If M, N satisfy

(i) $\mathcal{S}_\lambda(M) = \mathcal{S}_\lambda(N)$, $\forall \lambda \neq 0$, and

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then $M, N \in \mathcal{P}(A_1, \ldots, A_k)$, for some $A_1, \ldots, A_k$ satisfying the conditions obtained in **Question 1**?

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Given $A_1, \ldots, A_k \in \mathbb{C}^{n \times n}$, we set:
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Three questions (after Flanders' Theorem):

Question 3 (exhaustivity):

Given: two non-increasing sequences of nonnegative integers $\boldsymbol{\mu}, \boldsymbol{\mu}'$ such that $\|\boldsymbol{\mu} - \boldsymbol{\mu}'\|_\infty = d$,

are there: $A_1, \ldots, A_k$ satisfying the conditions obtained in **Question 1** and $\mathcal{S}_0(\Pi_1) = \boldsymbol{\mu},\ \mathcal{S}_0(\Pi_2) = \boldsymbol{\mu}'$, for some $\Pi_1, \Pi_2 \in \mathcal{P}(A_1, \ldots, A_k)$?

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(Only for $k = 3$).

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Given M and N with

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(ii) $\|\mathcal{S}_0(M) - \mathcal{S}_0(N)\|_\infty \leq d$,

then

$$M \sim \left[\begin{array}{c|c} J_{\neq 0}(M) & 0 \\ \hline 0 & J_0(M) \end{array}\right] \sim \left[\begin{array}{c|c} J_{\neq 0}(M) & 0 \\ \hline 0 & \Pi_1 \end{array}\right] = \widetilde{\Pi}_1,$$
$$N \sim \left[\begin{array}{c|c} J_{\neq 0}(M) & 0 \\ \hline 0 & J_0(N) \end{array}\right] \sim \left[\begin{array}{c|c} J_{\neq 0}(M) & 0 \\ \hline 0 & \Pi_2 \end{array}\right] = \widetilde{\Pi}_2.$$

($\sim$: similar).

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($\sim$: similar).

So $M \sim \widetilde{\Pi}_1$ and $N \sim \widetilde{\Pi}_2$, with $\widetilde{\Pi}_1, \widetilde{\Pi}_2 \in \mathcal{P}(\widetilde{A}_1, \ldots, \widetilde{A}_k)$.

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Permuted products of $A, B, C \in \mathbb{C}^{n\times n}$

$$\mathcal{P}(A,B,C) = \{ABC, ACB, BCA, BAC, CBA, CAB\}$$

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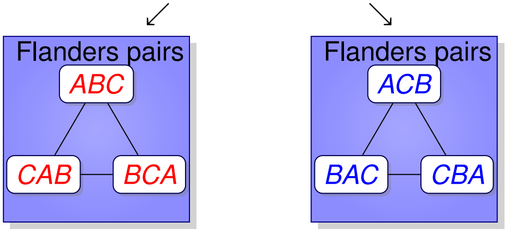

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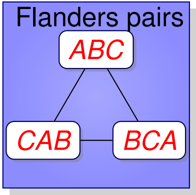
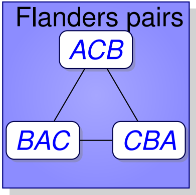
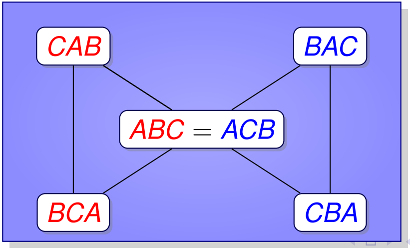




If $A(BC) = A(CB)$:



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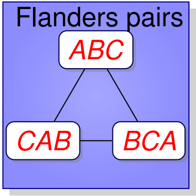
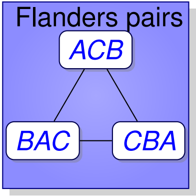
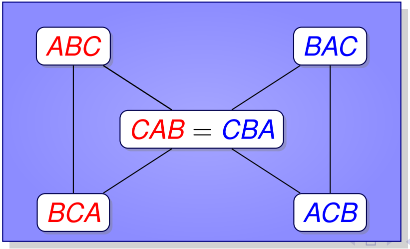
$\swarrow$ $\searrow$





If $C(AB) = C(BA)$:



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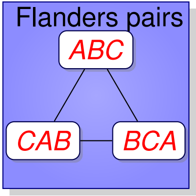
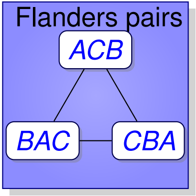
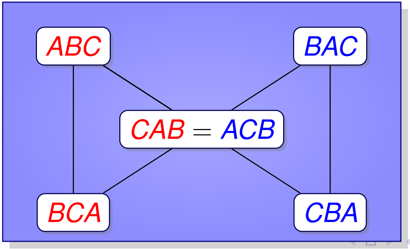




If $(CA)B = (AC)B$:



Commutativity relations

If at least **two of** A, B, C **commute** then, for any $\Pi_1, \Pi_2 \in \mathcal{P}(A, B, C)$:

(i) $\mathcal{S}_\lambda(\Pi_1) = \mathcal{S}_\lambda(\Pi_2)$, for all $\lambda \neq 0$.

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☞ commutativity of (A, B) **or** (A, C), **or** (B, C) is the answer to **Question 1** for three matrices.

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(ii) $\|\mathcal{S}_0(\Pi_1) - \mathcal{S}_0(\Pi_2)\|_\infty \leq 2$.

☞ commutativity of (A, B) **or** (A, C), **or** (B, C) is the answer to **Question 1** for three matrices.

☞ Moreover, it is the answer to **Question 3**:

Theorem

Let $\boldsymbol{\mu}, \boldsymbol{\mu}'$ be two non-increasing sequences of nonnegative integers such that

(i) $\|\boldsymbol{\mu} - \boldsymbol{\mu}'\|_\infty \leq 2$, and

(ii) $\|\boldsymbol{\mu}\|_1 = \|\boldsymbol{\mu}'\|_1 = n$.

Then, there are three matrices $A, B, C \in \mathbb{C}^{n\times n}$, such that $AC = CA$ and

$$\mathcal{S}_0(ABC) = \boldsymbol{\mu}, \qquad \text{and} \qquad \mathcal{S}_0(CBA) = \boldsymbol{\mu}'.$$

Answer to Question 2?

As for Question 2, we have:

Corollary

Let $M, N \in \mathbb{C}^{n\times n}$. Then the following are equivalent:

(a) There is $Q \in \mathbb{C}^{n\times n}$ such that (M, Q) and (Q, N) are Flanders pairs.

(b) $\mathcal{S}_\lambda(M) = \mathcal{S}_\lambda(N)$, for all $\lambda \neq 0$, and $\|\mathcal{S}_0(M) - \mathcal{S}_0(N)\|_\infty \leq 2$.

(c) There are $A, B, C \in \mathbb{C}^{n\times n}$ such that $AC = CA$, M is similar to ABC, and N is similar to CBA.

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Let $M, N \in \mathbb{C}^{n \times n}$. Then the following are equivalent:

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Not necessarily: $M = ABC$ and $N = CBA$!!!

Answer to Question 2? (proof)

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Proof: (a) $\Rightarrow$ (b): Corollary of Flanders' Th. (already seen).

Answer to Question 2? (proof)

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(c) There are $A, B, C \in \mathbb{C}^{n\times n}$ such that $AC = CA$, M is similar to ABC, and N is similar to CBA.

Proof:

(b) $\Rightarrow$ (c): Taking M, N to JCF:

$$\begin{aligned} M &\sim \text{JCF}(M) = \text{diag}(M_r, M_s) \\ N &\sim \text{JCF}(N) = \text{diag}(N_r, N_s) \\ & \text{(nonzero e-vals, zero e-val)} \end{aligned}$$

Answer to Question 2? (proof)

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Let $M, N \in \mathbb{C}^{n\times n}$. Then the following are equivalent:

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By hypothesis: $M_r = N_r$ and $\|\mathcal{S}_0(M) - \mathcal{S}_0(N)\|_\infty \leq 2$. Therefore (last Thm.) there are A_s, B_s, C_s with $A_sC_s = C_sA_s$ and $A_sB_sC_s = M_s,\ C_sB_sA_s = N_s$.

Answer to Question 2? (proof)

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$\Rightarrow$ $A = \text{diag}(I, A_s)$, $B = \text{diag}(M_r, B_s)$, $C = \text{diag}(I, C_s)$ fulfill the conditions in (c).

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Proof:

(c) $\Rightarrow$ (a): Let $M = P(ABC)P^{-1}, N = R(CBA)R^{-1}$, and set $Q := BCA$.

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Proof:

(c) $\Rightarrow$ (a): Let $M = P(ABC)P^{-1}, N = R(CBA)R^{-1}$, and set $Q := BCA$.

Then (M, Q) and (Q, N) are Flanders pairs:

$$M = P(ABC)P^{-1} = (PA)(BCP^{-1}) \sim (BCP^{-1})(PA) = BCA = Q.$$
$$N = R(CBA)R^{-1} = (RC)(BAR^{-1}) \sim (BAR^{-1})(RC) = BAC = BCA = Q. \qquad \square$$

Outline

Basic definitions

Path of a graph: Sequence of adjacent edges containing no cycles. Its length is the number of edges.

Forest: A graph containing no cycles.

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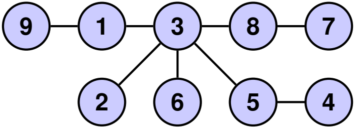
Example:



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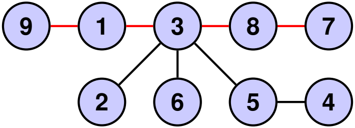
Example: – – – –– Path (of length 4)



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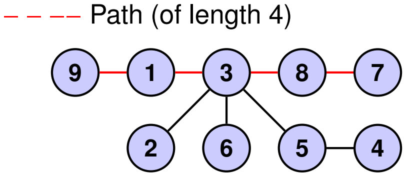

Definition

The graph of non-commutativity relations of $A_1, \ldots, A_k$ is the graph $\mathcal{G} = (V, E)$ with $V = \{1, 2, \ldots, k\}$, such that $\{i, j\} \in E$ if and only if $A_i A_j \neq A_j A_i$, for $1 \leq i, j \leq k$ with $i \neq j$.

Sequences of Flanders bridges

Definition

$M_1, M_{d+1} \in \mathbb{C}^{n\times n}$ are connected by a sequence of Flanders bridges if $(M_1, M_2), (M_2, M_3), \ldots, (M_d, M_{d+1})$ are Flanders pairs, for some $M_2, \ldots, M_d$.

$\mathcal{G}(A_1, \ldots, A_k)$: the graph of non-commutativity relations of $A_1, \ldots, A_k$.

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Then, if products in $\mathcal{P}(A_1, \ldots, A_k)$ are considered as **formal products**:

Theorem

Any two products in $\mathcal{P}(A_1, \ldots, A_k)$ are related by a sequence of Flanders bridges $\Leftrightarrow \mathcal{G}(A_1, \ldots, A_k)$ is a forest.

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The main result

Theorem

1. $\mathcal{G}(A_1,\ldots,A_k)$ a forest. Set $d=$ length of the longest path in $\mathcal{G}(A_1,\ldots,A_k)$. Given $\Pi_1,\Pi_2 \in \mathcal{P}(A_1,\ldots,A_k)$:

$$\|\mathcal{S}_0(\Pi_1)-\mathcal{S}_0(\Pi_2)\|_\infty \leq d.$$

2. This bound is **attainable**: Let $\mathcal{G}$ be any forest with k vertices, and let $d \leq k$ be the length of the longest path in $\mathcal{G}$. Then there are $A_1,\ldots,A_k \in \mathbb{C}^{n\times n}$ whose graph of non-commutativity relations is $\mathcal{G}$, and $\Pi_1,\Pi_2 \in \mathcal{P}(A_1,\ldots,A_k)$ with

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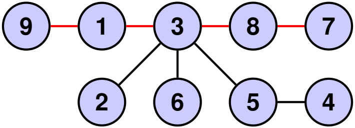
$$\|\mathcal{S}_0(\Pi_1) - \mathcal{S}_0(\Pi_2)\|_\infty = d.$$

Comment on the **Proof:**

- For 1 Uses tools from theory of permutations and graph theory.
- For 2: Constructive, just matrix manipulations.

Example



Set:

$$A_1 = \operatorname{diag}(\widetilde{A}_1, I_8), \quad A_2 = \operatorname{diag}(I_7, D_2^{(2)}, I_4), \quad A_3 = \operatorname{diag}(\widetilde{A}_3, D_3^{(1)}, D_3^{(2)}, D_3^{(3)}, I_2),$$
$$A_4 = \operatorname{diag}(I_{11}, D_4^{(4)}), \quad A_5 = \operatorname{diag}(I_9, D_5^{(3)}, D_5^{(4)}), \quad A_6 = \operatorname{diag}(I_5, D_6^{(1)}, I_6),$$
$$A_7 = \operatorname{diag}(\widetilde{A}_7, D_2^{(2)}, I_4), \quad A_8 = \operatorname{diag}(\widetilde{A}_8, I_8), \quad A_9 = (\widetilde{A}_9, I_8),$$

with:

$$\widetilde{A}_9 = \operatorname{diag}(I_3, J_2(0)) \quad \widetilde{A}_1 = \operatorname{diag}(I_2, J_2(0), 1), \quad \widetilde{A}_3 = \operatorname{diag}(1, J_2(0), I_2),$$
$$\widetilde{A}_8 = \operatorname{diag}(J_2(0), I_3), \quad \widetilde{A}_7 = \operatorname{diag}(0, I_4), \quad \widetilde{A}_i = I_5, \text{ for } i \neq 1, 3, 7, 8, 9,$$

and $D_j^{(i)} \in \mathbb{C}^{2\times 2}$ nonsingular such that $D_3^{(1)} D_6^{(1)} \neq D_6^{(1)} D_3^{(1)}$, $D_3^{(2)} D_2^{(2)} \neq D_2^{(2)} D_3^{(2)}$, $D_3^{(3)} D_5^{(3)} \neq D_5^{(3)} D_3^{(3)}$, and $D_4^{(4)} D_5^{(4)} \neq D_5^{(4)} D_4^{(4)}$. Then:

$$\Pi_1 = (A_9 A_1 A_3 A_8 A_7) A_6 A_2 A_5 A_4 = \operatorname{diag}(J_5(0), J), \Pi_2 = (A_7 A_8 A_3 A_1 A_9) A_6 A_2 A_5 A_4 = \operatorname{diag}(0_5, J),$$

with $J = \operatorname{diag}\left(D_3^{(1)} D_6^{(1)}, D_3^{(2)} D_2^{(2)}, D_3^{(3)} D_5^{(3)}, D_5^{(4)} D_4^{(4)}\right)$, nonsingular.

Hence: $\mathcal{S}_0(\Pi_1) = (5)$ and $\mathcal{S}_0(\Pi_2) = (1, 1, 1, 1, 1)$, so $\|\mathcal{S}_0(\Pi_1) - \mathcal{S}_0(\Pi_2)\|_\infty = 4$.

Open Problems

1. Given $d \geq 4$ and two non-increasing sequences $\boldsymbol{\mu}, \boldsymbol{\mu}'$ of nonnegative integers such that $\|\boldsymbol{\mu} - \boldsymbol{\mu}'\|_\infty \leq d - 1$, is it always possible to find d matrices, $A_1, \ldots, A_d$, such that $\mathcal{G}(A_1, \ldots, A_k)$ is a path, and $\mathcal{S}_0(A_1 \cdots A_d) = \boldsymbol{\mu}$, $\mathcal{S}_0(A_d \cdots A_1) = \boldsymbol{\mu}'$?

2. If $M, Q \in \mathbb{C}^{n \times n}$ are such that $\mathcal{S}_\lambda(M) = \mathcal{S}_\lambda(Q)$, for all $\lambda \neq 0$, and $\|\mathcal{S}_0(M) - \mathcal{S}_0(Q)\|_\infty \leq 2$, are there three matrices $A, B, C \in \mathbb{C}^{n \times n}$ with $AC = CA$, such that $M = ABC$ and $Q = CBA$?

Simple cases for Open Problem 2 (I)

The simplest case is

$$M = J_3(0) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad N = J_1(0) \oplus J_1(0) \oplus J_1(0) \equiv 0_{3\times 3}.$$

$$\mathcal{S}_0(M) = (3,0,0), \mathcal{S}_0(N) = (1,1,1) \Rightarrow \|\mathcal{S}_0(M) - \mathcal{S}_0(N)\|_\infty = 2.$$

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In this case, the answer is **affirmative**:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

satisfy:

- $ABC = M$
- $CBA = N$
- $AC = CA$

Simple cases Open Problem 2 (II)

The second simplest case is

$$M = J_4(0) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad N = J_2(0) \oplus J_2(0) = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\mathcal{S}_0(M) = (4,0), \mathcal{S}_0(N) = (2,2) \Rightarrow \|\mathcal{S}_0(M) - \mathcal{S}_0(N)\|_\infty = 2.$

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In this case, the answer is, again, **affirmative** (but no so simple):

$$A = \begin{bmatrix} 1 & 0 & 0 & -\sqrt{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & \sqrt{2}/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & 2\sqrt{2} \\ 0 & 1 & \sqrt{2} & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2}/2 \\ 0 & 0 & \sqrt{2}/2 & 0 \\ 0 & 0 & 0 & \sqrt{2}/2 \end{bmatrix},$$

satisfy:

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Outline

Fiedler matrices: definition

Given $(a_0, a_1, \dots, a_{n-1}) \in \mathbb{C}^n$:

$$M_0 = \begin{bmatrix} I_{n-1} & \\ & -a_0 \end{bmatrix}, \quad M_k = \begin{bmatrix} I_{n-k-1} & & \\ & \boxed{\begin{matrix} -a_k & 1 \\ 1 & 0 \end{matrix}} & \\ & & I_{k-1} \end{bmatrix}, \quad k = 1, \dots, n-1.$$

Let $\sigma : \{0, 1, \dots, n-1\} \to \{1, \dots, n\}$ be a bijection. Then:

$$M_\sigma := M_{\sigma^{-1}(1)} \cdots M_{\sigma^{-1}(n)}$$

Fiedler matrix associated with the bijection σ

- Introduced by **Fiedler** in 2003.

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Fiedler matrices: some examples

- Frobenius companion matrices:

$$C_1 = M_{n-1} \cdots M_1 M_0 = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_0 \\ 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & 1 & 0 \end{bmatrix} C_2 = M_0 M_1 \cdots M_{n-1} = C_1^\top$$

- $M_{n-1} \cdots M_2 M_0 M_1 = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & -a_0 & 0 \end{bmatrix}$

- $M_6(M_4 M_5)(M_2 M_3)(M_0 M_1) = \begin{bmatrix} -a_5 & 1 & 0 & 0 & 0 & 0 \\ -a_4 & 0 & -a_3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & -a_1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_0 & 0 \end{bmatrix} \quad (n = 6)$

Fiedler matrices: some examples

- Frobenius companion matrices:

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- $$M_{n-1} \cdots M_2 \color{red}{M_0 M_1} = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & \color{red}{1} \\ 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & \color{red}{-a_0} & 0 \end{bmatrix}$$

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Fiedler's Theorem

$$M_0 = \begin{bmatrix} I_{n-1} & \\ & -a_0 \end{bmatrix}, \quad M_k = \begin{bmatrix} I_{n-k-1} & & \\ & \boxed{\begin{matrix} -a_k & 1 \\ 1 & 0 \end{matrix}} & \\ & & I_{k-1} \end{bmatrix}, \quad k = 1, \ldots, n-1.$$

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Theorem [Fiedler, 2003]

All Fiedler matrices M_σ are similar to each other.

☞ All Fiedler matrices have the same eigenvalues (zero or nonzero) with the same multiplicities ⇝ they have the same JCF.

Why commutativity relations?

Fiedler "blocks" satisfy the following commutativity relations:

$$M_i M_j = M_j M_i, \qquad |i - j| \leq 1.$$

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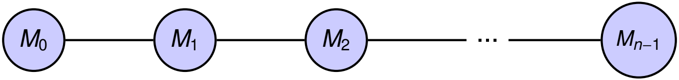
☞ Therefore, the graph of non-commutativity relations of Fiedler blocks, $\mathcal{G}(M_0, \ldots, M_{n-1})$, is a path:



Proof of Fiedler's Theorem

$$M_0 = \begin{bmatrix} I_{n-1} & \\ & -a_0 \end{bmatrix}, \quad M_k = \begin{bmatrix} I_{n-k-1} & & \\ & \boxed{\begin{matrix} -a_k & 1 \\ 1 & 0 \end{matrix}} & \\ & & I_{k-1} \end{bmatrix}, \quad k = 1, \ldots, n-1.$$

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Fiedler's Theorem It is an immediate consequence of:

1. $\mathcal{G}(M_0, \ldots, M_{n-1})$ is a forest (actually, a path).
2. $M_1, \ldots, M_{n-1}$ are invertible.
3. rank $M_0 \geq n-1$.

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1. $\mathcal{G}(M_0, \dots, M_{n-1})$ is a forest (actually, a path).
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because:

1 $\Rightarrow$ all M_σ have the same JCF at nonzero e-vals, and

2+3 $\Rightarrow$ all M_σ have the same JCF at the zero e-val (actually, at most 1 block).

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GRAZIE (THANK YOU)

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