Polynomial root-finding using companion matrices

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Outline

Goal

Compute the roots of (scalar) polynomials

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \qquad (a_k \in \mathbb{C})$$

using companion forms.

We can restrict ourselves to **monic** polynomials (after dividing by a_n, if necessary).

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$A \in \mathbb{C}[a_0, a_1, \ldots, a_{n-1}]^{n\times n}$ such that

$$p_A(z) = \det(zI - A) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = p(z).$$

(Only for **monic** polynomials).

Roots of $p(z)$ = Eigenvalues of A (i.e.: $p(z) = 0 \Leftrightarrow \det(zI - A) = 0$).

Theoretically:

Polynomial root-finding $\overset{\text{Companion matrix}}{\Longleftrightarrow}$ Standard eigenvalue problem

But numerically, they are different problems !!!

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Motivation

Frobenius companion matrices:

$$C_1 = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_0 \\ 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & 1 & 0 \end{bmatrix}, \qquad C_2 = C_1^\top$$

MATLAB's command `roots`: QR algorithm on C_2.

Outline

Basic definitions

Problem:	Algorithm:
$f: \underbrace{X}_{data} \longrightarrow \underbrace{Y}_{solution}$	$\widetilde{f}: \underbrace{X}_{data} \longrightarrow \underbrace{Y}_{solution}$

$\widetilde{f}$ is backward stable if $\widetilde{f}(x) = f(x+\delta x), \qquad \|\delta x\| = O(u)\|x\|$

($u =$ unit roundoff)

☞ B'stability for **poly root-finding** using **companion matrices**:

1. B'stability on the companion matrix (e-vals):

   $\widetilde{f} =$ e-val algorithm, $f =$ e-val problem, $x =$ companion matrix

   (the computed roots (e-vals) are the e-vals of a nearby matrix, not necessarily companion!!!).

2. B'stability on the polynomial (roots):

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Given $p(z)$ → Choose A such that $p(z) = \det(zI - A)$ → Compute the e-vals of A = e-vals of $A + E$, $\|E\| = O(u)\|A\|$

(if we use a backward stable algorithm, like QR)

Set $\widetilde{p}(z) = \det(zI - (A + E))$

Question: Is $\widetilde{p}(z)$ close to $p(z)$?

$$\frac{\|\widetilde{p} - p\|}{\|p\|} = O(u) \text{ ?}$$

$\frac{\|\widetilde{p} - p\|}{\|p\|}$: b'err of polynomial root-finding as an eigenvalue problem (using A).

Goal:

Analyze $\frac{\|\widetilde{p} - p\|}{\|p\|}$.

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Perturbation of the characteristic polynomial: first order term

Using Jacobi's formula:

$$\widetilde{p}(z) - p(z) = \det(zI - (A+E)) - \det(zI - A) = -\operatorname{tr}(\operatorname{adj}(zI - A) \cdot E) + O(\|E\|^2)$$

$$\operatorname{adj}(zI - A) = \sum_{k=0}^{n-1} A_k z^k \text{ (matrix polynomial of degree } n-1\text{)}.$$

Hence, if we set: $\det(zI - X) = z^n + \sum_{k=0}^{n-1} a_k(X) z^k$, then, to **first order** in E:

$$a_k(A+E) - a_k(A) = -\operatorname{tr}(A_k E).$$

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Q: Explicit formula for A_k?

Recursive formula for the adjugate

$p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k = \det(zI - A)$

Proposition [Gantmacher, 1959]

Set:

$$\begin{cases} A_{n-1} = I, \quad \text{and} \\ A_k = A \cdot A_{k+1} + a_k I, \quad \text{for } k = n-2, \ldots, 1, 0. \end{cases}$$

Then,

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Note:

$$A_{k-1} = p_{n-k}(A) = A^{n-k} + a_{n-1} A^{n-k-1} + \cdots + a_{k+1} A + a_k I$$

($(n-k)$th Horner shift of $p(z)$ evaluated at A).

☞ $p_{n-k-1}(A)$ encodes the information on the variation $a_k(A+E) - a_k(A)$:

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Fiedler matrices: definition

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$$M_0 = \begin{bmatrix} I_{n-1} & \\ & -a_0 \end{bmatrix}, \quad M_k = \begin{bmatrix} I_{n-k-1} & & \\ & \boxed{\begin{matrix} -a_k & 1 \\ 1 & 0 \end{matrix}} & \\ & & I_{k-1} \end{bmatrix}, \quad k = 1, \ldots, n-1.$$

Let $\sigma : \{0, 1, \ldots, n-1\} \to \{1, \ldots, n\}$ be a bijection. Then:

$$M_\sigma := M_{\sigma^{-1}(1)} \cdots M_{\sigma^{-1}(n)}$$

Fiedler matrix of p associated with the bijection σ

- Introduced by **Fiedler** in 2003.

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Fiedler matrices: some examples

- Frobenius companion matrices:

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$$C_2 = M_0 M_1 \cdots M_{n-1} = C_1^\top$$

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- $$M_6 (M_4 M_5)(M_2 M_3)(M_0 M_1) = \begin{bmatrix} -a_5 & 1 & 0 & 0 & 0 & 0 \\ -a_4 & 0 & -a_3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & -a_1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_0 & 0 \end{bmatrix} \quad (n = 6)$$

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Fiedler matrices: Basic properties

- All M_σ contain the same entries (located in different positions):

$$-a_0, \ldots, -a_{n-1} \quad \& \quad \overbrace{1, \ldots, 1}^{n-1} \quad \& \quad 0's$$

- M_σ is a (sparse) companion matrix ($\det(zI - M_\sigma) = p(z)$).

- There are 2^{n-1} different Fiedler matrices.

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Formula for the adjugate: main features

To first order in E:

$$a_k(M_\sigma + E) - a_k(M_\sigma) = -\sum_{i,j=1}^{n} p_{ij}^{(\sigma,k)}(a_0, a_1, \ldots, a_{n-1})\, E_{ij}, \qquad k = 0, 1, \ldots, n-1,$$

where:

- $p_{ij}^{(\sigma,k)}(a_0, a_1, \ldots, a_{n-1})$ is a polynomial in a_i with **degree at most** 2.
- If $M_\sigma = C_1, C_2$, then all $p_{ij}^{(\sigma,k)}(a_0, a_1, \ldots, a_{n-1})$ have **degree** 1.
- If $M_\sigma \neq C_1, C_2$, then there is at least one k and some (i,j) such that $p_{ij}^{(\sigma,k)}(a_0, a_1, \ldots, a_{n-1})$ has **degree** 2.

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- If $M_\sigma = C_1, C_2$, then all $p_{ij}^{(\sigma,k)}(a_0, a_1, \ldots, a_{n-1})$ have **degree** 1.
- If $M_\sigma \neq C_1, C_2$, then there is at least one k and some (i, j) such that $p_{ij}^{(\sigma,k)}(a_0, a_1, \ldots, a_{n-1})$ has **degree** 2.

Formula for the adjugate: main features

To first order in E:

$$a_k(M_\sigma + E) - a_k(M_\sigma) = -\sum_{i,j=1}^{n} p_{ij}^{(\sigma,k)}(a_0, a_1, \ldots, a_{n-1})\, E_{ij}, \qquad k = 0, 1, \ldots, n-1,$$

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Some particular examples

Frobenius companion matrices:

$$p_{n-k-1}(C_1^\top) = p_{n-k-1}(C_2) = \left[\begin{array}{cccc|cccc} 0 & \dots & 0 & & 1 & & & 0 \\ -a_k & & & & a_{n-1} & 1 & & \\ \vdots & \ddots & & & \vdots & a_{n-1} & \ddots & \\ -a_1 & \ddots & -a_k & & a_{k+1} & \vdots & \ddots & 1 \\ -a_0 & \ddots & \vdots & & & a_{k+1} & \ddots & a_{n-1} \\ & \ddots & -a_1 & & & & \ddots & \vdots \\ 0 & & -a_0 & & 0 & & & a_{k+1} \end{array}\right].$$

These are the **only** Fiedler matrices M_σ for which **all** $p_k(M_\sigma)$ have entries of **degree 1** !!!!

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Some particular examples (II)

$$F = M_{n-1} \cdots M_2 M_0 M_1$$

$$p_{n-k-1}(F) = \left[\begin{array}{cccc|cccc}
0 & & & & 1 & & & 0 \\
-a_k & & & & a_{n-1} & \ddots & & \vdots \\
\vdots & \ddots & & & \vdots & \ddots & 1 & 0 \\
-a_1 & & -a_k & & a_{k+2} & & a_{n-1} & -a_0 \\
-a_0 & \ddots & \vdots & -a_k & a_{k+1} & \ddots & \vdots & -a_0 a_{n-1} \\
& \ddots & -a_1 & \vdots & & \ddots & a_{k+2} & \vdots \\
& & -a_0 & -a_1 & & & a_{k+1} & -a_0 a_{k+2} \\
& & & 1 & & & & a_{k+1}
\end{array}\right], \quad \text{for } k = 0 : n-3,$$

$$p_1(F) = \left[\begin{array}{cccccc}
0 & & & & & 0 \\
-a_{n-2} & 1 & & & & \\
-a_{n-3} & a_{n-1} & 1 & & & \\
\vdots & & a_{n-1} & \ddots & & \\
\vdots & & & \ddots & 1 & \\
-a_1 & & & & a_{n-1} & -a_0 \\
1 & & & & 0 & a_{n-1}
\end{array}\right], \quad \text{and} \quad p_0(F) = I.$$

Backward error

Theorem [D., Dopico, Pérez, 2013]

If the roots of $p(z)$ are computed as the e-vals of M_σ with a backward stable algorithm, the computed roots are the exact roots of a polynomial $\widetilde{p}(z)$ with:

(a) If $M_\sigma = C_1, C_2$:

$$\frac{\|\widetilde{p}-p\|_\infty}{\|p\|_\infty} = O(u)\|p\|_\infty,$$ [Edelman-Murakami'95]

(b) if $M_\sigma \neq C_1, C_2$:

$$\frac{\|\widetilde{p}-p\|_\infty}{\|p\|_\infty} = O(u)\|p\|^{2}_\infty.$$

(u is the machine precision)

$$\left(\| \sum_{i=0}^{n} a_i z^i \|_\infty = \max_{i=0,\dots,n} |a_i| \right)$$

Some remarks

(Recall: $\|p\|_\infty \geq 1$, since p is monic).

- For $\|p\|_\infty$ moderate, backward stability of polynomial root-finding is guaranteed using **any Fiedler matrix**.
- Then, particular features of some Fiedler matrices (like low bandwidth) can make them preferable than C_1 and C_2.
- When $\|p\|_\infty$ is large, C_1 and C_2 are expected to give smaller b'err than any other Fiedler.

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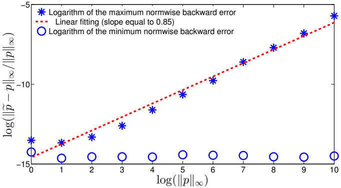
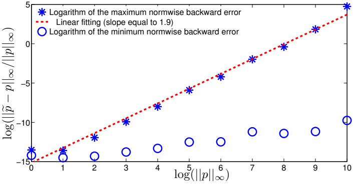
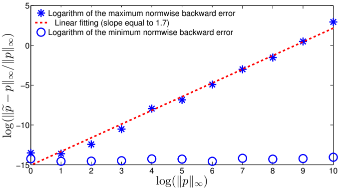
Random polynomials, $n = 20$



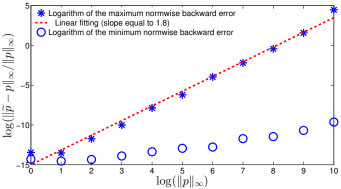
(a) C_2



(b) Pentadiagonal



(c) F



(d) M_σ

Figure: 11 samples, 500 random polys, $\|p\|_\infty = 10^k$ $(k = 0 : 10)$, $a_i = a \cdot 10^c$, $a \in [-1, 1]$, $c \in [-k, k]$, $a_0 = 10^k$.

Outline

Other companion matrices?

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YES: Infinitely many!

Just multiply: $PM_\sigma P^{-1}$ (P invertible)$\rightsquigarrow$ In general, not sparse (**exception**: P is a **permutation** matrix).

☞ We look for sparse companion matrices.

Sparse companion matrices (I)

Sparse: It has the smallest number of nonzero entries

☞ For companion matrices, this number is $2n-1$ [Ma-Zhan'13]

(we focus on: $\underbrace{1,\ldots,1}_{n-1}, -a_0,\ldots,-a_{n-1}$).

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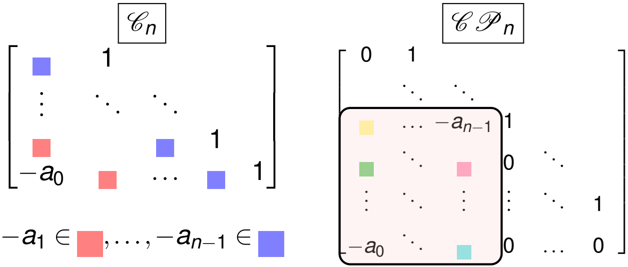
We define the following (lower Hessenberg) classes of matrices:



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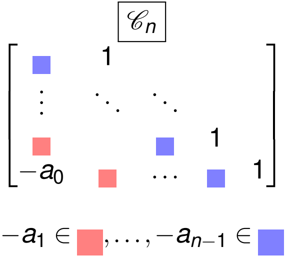

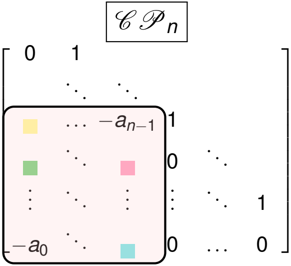

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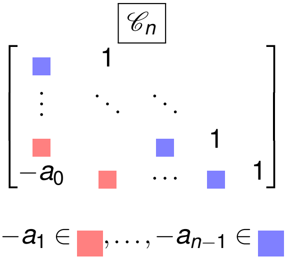

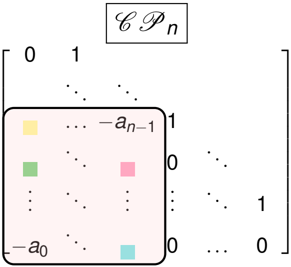

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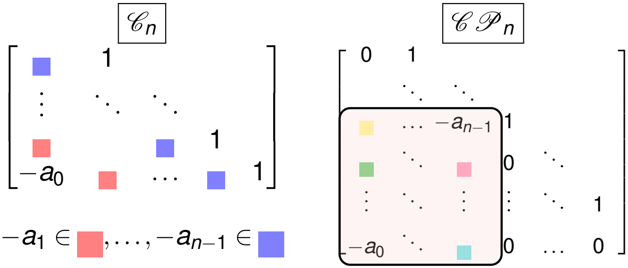

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Theorem [Eastman-etal'14]

$A(a_0, \ldots, a_{n-1}) \in \mathscr{C}_n$ is a (sparse) companion matrix $\Leftrightarrow A(a_0, \ldots, a_{n-1}) \in \mathscr{C}\mathscr{P}_n$.

Outline

Why monic polynomials?

If $q(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ (not necessarily monic) $(a_n \neq 0)$.

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Then (setting $\widetilde{q} := a_n \widetilde{p}$):

$$\frac{\|q - \widetilde{q}\|}{\|q\|} = \frac{\|\frac{q}{a_n} - \frac{\widetilde{q}}{a_n}\|}{\|\frac{q}{a_n}\|} = \frac{\|p - \widetilde{p}\|}{\|p\|} = O(u).$$

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$\Rightarrow$ It is enough to prove b'stability for monic polys.

However...

- B'stability (in the poly sense) is only guaranteed when $\|p\|$ is moderate.
- The QZ algorithm on the Frobenius companion form (non-monic) gives b'stability if $\|p\|_\infty \approx 1$ ([Van Dooren-Dewilde'83]).
- If we divide by $\|p\|_\infty \rightsquigarrow$ the polynomial may become non-monic!

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Companion forms

Companion form: Valid for non-monic polynomials.

Companion form:

$A = A_0 + zA_1$ s.t.:

- $A_0, A_1 \in \mathbb{C}[a_0, a_1, \ldots, a_{n-1}, a_n]^{n\times n}$,
- $\det A = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$.

May have entries $a + bz$.

Fiedler companion forms

Frobenius companion forms

$$F_i(z) = z\,\mathrm{diag}(a_n, 1, \ldots, 1) - C_i \qquad\qquad i = 1, 2$$

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Examples: $F_1 = \begin{bmatrix} a_n z + a_{n-1} & a_{n-2} & \cdots & a_0 \\ -1 & z & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & -1 & z \end{bmatrix} \qquad F_2 = F_1^\top$

$$F = \begin{bmatrix} a_6 z + a_5 & -1 & 0 & 0 & 0 & 0 \\ a_4 & z & a_3 & -1 & 0 & 0 \\ -1 & 0 & z & 0 & 0 & 0 \\ 0 & 0 & a_2 & z & a_1 & -1 \\ 0 & 0 & -1 & 0 & z & 0 \\ 0 & 0 & 0 & 0 & a_0 & z \end{bmatrix} \qquad (n = 6)$$

Other companion forms

Companion form

A matrix $A(a_0, a_1, \ldots, a_{n-1}, a_n; z)$ such that:

- The entries are linear polynomials in z.
- $\det A(a_0, a_1, \ldots, a_{n-1}, a_n; z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$.

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Fiedler-like: $$\begin{bmatrix} 0 & 0 & 0 & z & a_0 + z a_1 \\ 0 & 0 & 1 & 0 & -z \\ 0 & z & a_2 + z a_3 & -1 & 0 \\ 1 & 0 & -z & 0 & 0 \\ a_4 + z a_5 & -1 & 0 & 0 & 0 \end{bmatrix} \qquad (n = 5)$$

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Fiedler-like: $$\begin{bmatrix} 0 & 0 & 0 & z & a_0 + za_1 \\ 0 & 0 & 1 & 0 & -z \\ 0 & z & a_2 + za_3 & -1 & 0 \\ 1 & 0 & -z & 0 & 0 \\ a_4 + za_5 & -1 & 0 & 0 & 0 \end{bmatrix} \qquad (n = 5)$$

☞ There are many others [Dopico-Lawrence-Pérez-VanDooren]:

- Permutationally equivalent to companion forms in some "extended $\mathscr{C}\mathscr{P}_n$".
- Most of them are not sparse.

Open questions for companion forms

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- Is there any companion form which provides a smaller b’err than Frobenius ones?

Other issues (not considered in this talk)

- Complexity:
  - Desideratum: $O(n^2)$ flops + $O(n)$ storage.
  - However: `roots` $\rightsquigarrow$ $O(n^3)$ computations + $O(n^2)$ storage.

  ☞ A **fast** ($O(n^2)$ flops + $O(n)$ storage) and b'stable (in the matrix sense) algorithm recently proposed [Aurentz etal'15].

- Coefficient-wise b'err ($|a_i - \widetilde{a}_i| / |a_i|$).

  ☞ No algorithm can provide coefficent-wise b'stability (in the **polynomial** sense) [VanDooren-Mastronardi'15].

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Conclusions

- B'stability on the e-val problem $\not\Rightarrow$ B'stability on the poly root-finding problem.
- When $\|p\|_\infty$ is moderate, a b'stable e-val algorithm implies poly b'stability for any **Fiedler** matrix.
- When $\|p\|_\infty$ is large, Frobenius companion matrices are expected to give **less b'err** than any other Fiedlers.
- Though `roots` is b'stable in practice, it could give non-satisfactory results.
- B'err of the poly root-finding problem can be analyzed, using the adjugate of the characteristic matrix, **for many companion matrices**.
- Characterization of all sparse companion **matrices** is known (only for monic polynomials!).
- Looking at monic polynomials is **not enough** to guarantee b'stability.
- Still more room to look for other companion **forms** and to describe all sparse ones.
- Still open: Fast and b'stable algorithm (in the polynomial sense) for all polynomials?

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