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Departamento de Matemáticas

A survey on NLEVPs and multiparameter eigenvalue problems

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Outline

1. Basic notions
2. Applications
3. How to solve them?
   - Small-Moderate size
   - Large scale

Outline

NLEVP: definition

NLEVP Given $\emptyset \neq \Omega \subseteq \mathbb{C}$ (open), and:

$$\begin{array}{rccc} F: & \Omega & \to & \mathbb{C}^{n\times n} \\ & \lambda & \mapsto & [F_{ij}(\lambda)] \end{array}$$

Definition (right and left eigenpair)

(λ_0, v) right eigenpair of F if $F(\lambda_0)v = 0$ $(v \neq 0)$,
(λ_0, v) left eigenpair of F if $w^* F(\lambda_0) = 0$ $(w \neq 0)$.

($\lambda_0 \in \mathbb{C}$: eigenvalue, $v \in \mathbb{C}^n$: right eigenvector, $w \in \mathbb{C}^n$: left eigenvector).

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Notation: $\Lambda(F) = \{\lambda \in \mathbb{C} : \lambda \text{ is an e-val of } F\}$ (spectrum).

MPEP: definition

MPEP In this case,

$$\begin{array}{rccc} W: & \Omega \subseteq \mathbb{C}^m & \rightarrow & \mathbb{C}^{n_1 \times n_1} \times \cdots \times \mathbb{C}^{n_m \times n_m} \\ & \lambda & \mapsto & W(\lambda) := (W_1(\lambda), \ldots, W_m(\lambda)) \end{array}$$

For:

$$x := x_1 \otimes \cdots \otimes x_m \in \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_m}$$

set

$$W(\lambda)x := (W_1(\lambda)x_1, \ldots, W_n(\lambda)x_n), \qquad x^* W(\lambda) := (x_1^* W_1(\lambda), \ldots, x_m^* W_m(\lambda)).$$

Then

(λ_0, x) is a right eigenpair of W if $W(\lambda_0)x = 0$
(λ_0, y) is a left eigenpair of W if $y^* W(\lambda_0) = 0$

($\lambda_0 \in \mathbb{C}^m$ is an eigenvalue, x is a right eigenvector, and y is a left eigenvector).

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Some particular (but relevant) cases

1. Standard e-val problem: $F(\lambda) = A - \lambda I$, $A \in \mathbb{C}^{n\times n}$.
2. Generalized e-val problem: $F(\lambda) = A - \lambda B$, $A, B \in \mathbb{C}^{n\times n}$.
3. Polynomial e-val problem (PEP): $F_{ij}(\lambda) = p_{ij}(\lambda)$, a polynomial in λ.

$$F(\lambda) = A_0 + \lambda A_1 + \cdots + \lambda^d A_d, \qquad A_0, A_1, \ldots, A_d \in \mathbb{C}^{n\times n}.$$

4. Rational e-val problem (REP): $F_{ij}(\lambda) = \frac{p_{ij}(\lambda)}{q_{ij}(\lambda)}$, a rational function in λ.

$$F(\lambda) = P(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda) \quad \text{or} \quad F(\lambda) = P(\lambda) + R_{sp}(\lambda),$$

with P, C, A, B matrix polynomials (A nonsingular), and R_{sp} **strictly proper** ($\deg p_{ij} < \deg q_{ij}$).

5. In general: $F : \Omega \to \mathbb{C}^{n\times n}$ holomorphic.
6. E-vec dependent NLEVPs: $F(V)V = V\Lambda$, with $F \in \mathbb{C}^{n\times n}$, $V \in \mathbb{C}^{n\times k}$ (with orthonormal columns), $\Lambda \in \mathbb{C}^{k\times k}$ (diagonal)

$$V^* F(V) V = \Lambda \Rightarrow \Lambda \text{ contains some e-vals of } F(V).$$

(Bai, Wed. 17:00; Truhar, Tue. 17:30).

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NLEVP vs Standard e-val problem

- $F(\lambda)$ can be singular: $\det F(\lambda) \equiv 0 \rightsquigarrow$ requires another def'n of e-val/e-vec.
  (In most talks, but not all, $F(\lambda)$ is regular, $\det F(\lambda) \not\equiv 0$).
- E-vecs of different e-vals are not necessarily linearly independent: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an evec of $\begin{bmatrix} \lambda(\lambda-1) & 0 \\ 0 & 1 \end{bmatrix}$ for $\lambda = 0, 1$.
- $\Omega \setminus \Lambda(F)$ is **open**.
- If F is regular, $\Lambda(F) = \{\lambda \in \Omega : \det F(\lambda) = 0\}$. Then any $\lambda_0 \in \Lambda(F)$ is isolated (i.e., there is an open set $\mathscr{U} \subseteq \Omega : \mathscr{U} \cap \Lambda(F) = \{\lambda_0\}$).
- There can be an infinite e-val: When $G(\lambda) := F(1/\lambda)$ has a zero e-val. (For polynomials, $P(\lambda)$, of degree d, we consider $\lambda^d P(1/\lambda)$).
- $F(\lambda)$ may have poles.
- Algebraic and geometric multiplicities, Jordan chains, etc. can also be defined. (Bora, after this talk; Marcaida, Tue. 18:00).

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Regular vs Singular (matrix polynomials)

If $P(\lambda) = \sum_{i=0}^{d} \lambda^i A_i$ is singular, then it has right and left minimal bases and right and left minimal indices:

- Related to the fact that $P(\lambda)$ has non-trivial left and/or right null-spaces over the field $\mathbb{C}(\lambda)$ of rational functions:

$$\begin{aligned} \mathcal{N}_\ell(P) &:= \left\{ y(\lambda)^\top \in \mathbb{C}(\lambda)^{1\times m} : y(\lambda)^\top P(\lambda) \equiv 0^\top \right\}, \\ \mathcal{N}_r(P) &:= \left\{ x(\lambda) \in \mathbb{C}(\lambda)^{n\times 1} : P(\lambda) x(\lambda) \equiv 0 \right\}, \end{aligned}$$

- which have bases consisting entirely of vector polynomials.
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Eigenstructure: $\left\{\begin{array}{l}\text{Eigenvalues (with multiplicities) + Minimal indices}\\ \text{Eigenvectors/Jordan chains + minimal bases}\end{array}\right.$

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Quadratic PEPs

Usually associated with $\boxed{Mx''(t) + Cx'(t) + Kx(t) = f(t)}$ $(M, C, K \in \mathbb{C}^{n\times n}, x(t) \in \mathbb{C}[t]^n)$.

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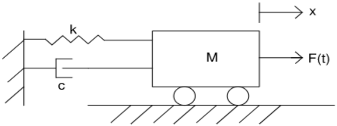

mass-spring system ($n=1$)

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Usually associated with $\boxed{Mx''(t) + Cx'(t) + Kx(t) = f(t)}$ $(M, C, K \in \mathbb{C}^{n\times n}, x(t) \in \mathbb{C}[t]^n)$.

If all e-vals of $Q(\lambda) := \lambda^2 M(\lambda) + \lambda C + K$ are semisimple and finite, set

$$\begin{array}{rl} \Lambda := \mathrm{diag}(\lambda_1, \dots, \lambda_{2n}) & \text{(e-vals)}, \\ X := \left[\begin{array}{c|c|c} x_1 & \dots & x_{2n} \end{array}\right] & \text{(right e-vecs)}, \\ Y := \left[\begin{array}{c|c|c} y_1 & \dots & y_{2n} \end{array}\right] & \text{(left e-vecs)}. \end{array}$$

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$$x(t) = Xe^{\Lambda t}c + \int_{-\infty}^{t} Xe^{\Lambda(t-s)} Y^* f(s)ds, \quad c \in \mathbb{C}^n \text{ arbitrary.}$$

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Tisseur–Meerbergen. The quadratic eigenvalue problem. SIAM Rev. 43 (2001)

PEPs with higher (low) degree and moderate size

$P(\lambda)=\sum_{i=0}^{d}\lambda^i A_i$: Associated with $A_0X(t)+A_1X'(t)+\cdots+A_kX^{(d)}(t)=f(t)$

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Some examples:

- Orr-Sommerfeld equation ($d = 4$):

$$\left[\left(\frac{d^2}{dy^2} - \lambda^2\right)^2 - iR\left\{(\lambda U - \omega)\left(\frac{d^2}{dy^2} - \lambda^2\right) - \lambda U''\right\}\right]\phi = 0.$$

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- Planar waveguide ($\lambda^4 A_4 + \cdots + A_0$):

$$A_1 = \frac{\delta^2}{4} \mathrm{diag}(-1, 0, 0, \ldots, 0, 0, 1), \quad A_3 = \mathrm{diag}(1, 0, 0, \ldots, 0, 0, 1),$$

$$A_0(i,j) = \frac{\delta^4}{16}(\phi_i, \phi_j) \quad A_2(i,j) = (\phi_i', \phi_j') - (q\phi_i, \phi_j) \quad A_4(i,j) = (\phi_i, \phi_j).$$

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Betcke-Higham-Mehrmann-Schröder-Tisseur. NLEVP: A collection of nonlinear eigenvalue problems. ACM TOMS, 39 (2010)

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☞ Differential Algebraic Equations (Systems)

REPs

Loaded elastic string: Finite element discretization of a boundary problem describing the eigenvibration of a string with a load of mass m attached by an elastic spring of stiffness k.

$$R(\lambda)x = \left(A - \lambda B + \frac{\lambda}{\lambda - \sigma} C\right) x = 0, \qquad \sigma = k/m$$

with

$$A = n \begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & 2 & -1 \\ & & -1 & 1 \end{bmatrix}, \quad B = \frac{1}{6n} \begin{bmatrix} 4 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & 4 & 1 \\ & & 1 & 2 \end{bmatrix}, \quad C = k \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix}$$

(n up to 10^3).

Betcke-Higham-Mehrmann-Schröder-Tisseur. NLEVP: A collection of nonlinear eigenvalue problems. ACM TOMS, 39 (2010)

REPs

Damped vibration on a viscoelastic structure: A FEM takes the form:

$$R(\lambda) = \left(\lambda^2 M + K - \sum_{j=1}^{d} \frac{1}{1 + b_j\lambda} \Delta K_j\right) x = 0,$$

with

$d =$ number of regions,
$b_j =$relaxation parameters,
$\Delta K_j =$stiffness matrices over each region.

$(M, K > 0.)$

Mehrmann-Voss. Nonlinear eigenvalue problems: a challenge for modern eigenvalue methods. GAMM, 27 (2004)

Other NLEVPs

Typical example: Look for solutions $x(t) = e^{\lambda t} v$ in a system of 1st order **delayed** differential equations:

$$B_0 x'(t) = A_0 x(t) + A_1 x(t-\tau) \Longrightarrow (\lambda B_0 - A_0 - A_1 e^{-\lambda \tau}) v = 0$$

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- **The radio-frequency gun cavity problem**:

$$\left[(K - \lambda M) + i\sqrt{\lambda - \sigma_1^2}\, W_1 + i\sqrt{\lambda - \sigma_2^2}\, W_2 \right] v = 0,$$

  where M, K, W_1, W_2 are real sparse symmetric 9956×9956.

- **Bound states in semiconductor devices problems**:

$$\left[(H - \lambda I) + \sum_{j=0}^{80} e^{i\sqrt{\lambda - \alpha_j}}\, S_j \right] v = 0,$$

  where $H, S_j \in \mathbb{R}^{16281 \times 16281}$, H symmetric and S_j have low rank.

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MPEPs

- **Multiparameter Sturm-Liouville eigenvalue problems**:

$$-\frac{d}{d\xi_i}\left(p_i(\xi_i)\frac{d}{d\xi_i}y_i(\xi_i)\right)+q_i(\xi_i)y_i(\xi_i)=\sum_{j=1}^{k}\lambda_j a_{ij}(\xi_i)y_i(\xi_i)$$

Dai. Numerical methods for solving multiparameter eigenvalue problems. Int. J. Comp. Math. 72 (1999).

- **Stability of delay-differential equations**: From

$$\lambda B_0 x = (A_0 + e^{-\lambda\tau} A_1)x$$

setting $\mu = e^{-\lambda\tau}$ and, assuming $\lambda = \mathrm{i}\omega \Rightarrow \overline{\lambda} = -\lambda, \overline{\mu} = \mu^{-1}$ we get

$$\begin{cases} A_0 x &= \lambda B_0 x - \mu A_1 x \\ \overline{A}_1 x &= -\lambda\mu \overline{B}_0 y - \mu \overline{A}_0 y. \end{cases}$$

Jarlebring-Hochstenbach. Polynomial two-parameter eigenvalue problems and matrix pencil methods for stability of delay-differential equations. Linear Algebra Appl. 431 (2009).

PEPs with large degree and/or large size

- PEPs used to approximate other NLEVPs:
  - Galerkin-type discretization of a $3D$ Laplace e-val problem with boundary conditions + interpolating Chebyshev polynomials: $3 \leq d \leq 30$.
  - Finite element/boundary element discretization of a 3D fluid-structure interaction problem: $d = 11$, $n = 6223$.
  - Loaded-string from NLEVP collection + Chebyshev interpolation: $d = 20$, $n = 10^4$.

Kressner–Román. Memory-efficient Arnoldi algorithms for linearizations of matrix polynomials in Chebyshev basis. Numer. Lin. Algebra Appl. 21 (2014).

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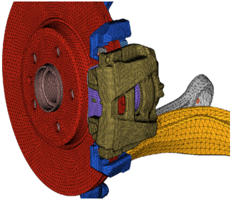
- Large-scale problems:

Brake squeal simulation

$\mathbf{n \approx 10^6}$

(V. Mehrmann, Tue. 11:00h)



Gräbner–Mehrmann–Quraishi–Schröder–von Wagner. Numerical methods for parametric model reduction in the simulation of disk brake squeal. ZAMM, 96 (2016).

In this MS

Other applications:

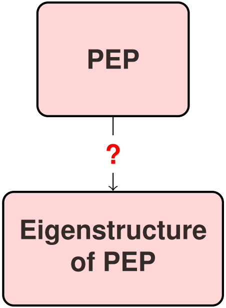
- Dynamics of systems with radiation and delay (Bindel, Tue. 12:30).
- Electronic structure calculations (Bai, Tue. 17:00).
- Quantum mechanics and machine learning (Upadhyaya, Wed. 11:00).
- Optimization problems (Lu, Wed. 15:30).
- Computer-aided geometric design (González-Vega, Wed. 16:00).
- Computational nanoelectronics (Miedlar, Wed. 17:30).

Outline

1. Basic notions
2. Applications
3. How to solve them?
   - Small-Moderate size
   - Large scale

PEPs

For small/medium size: **LINEARIZATION**:



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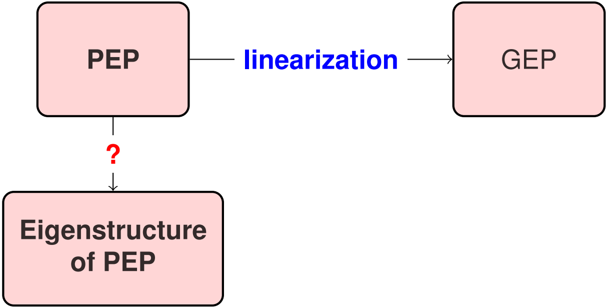

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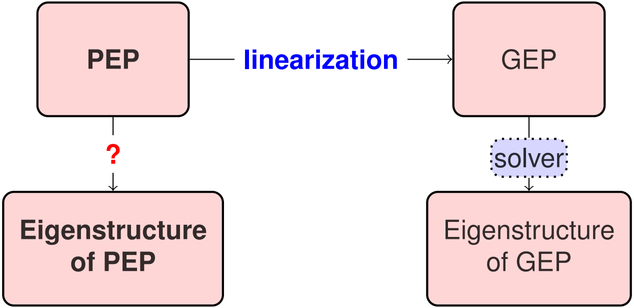

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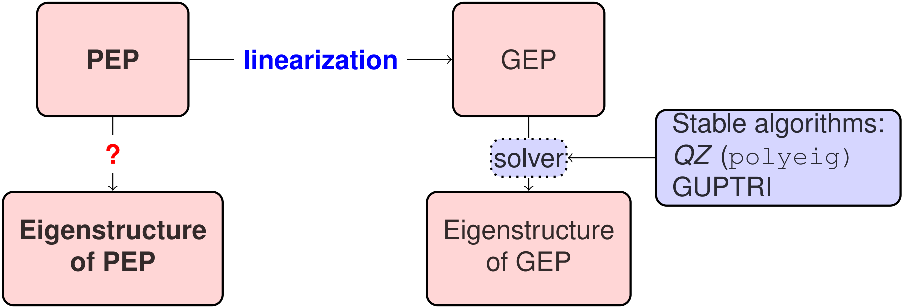

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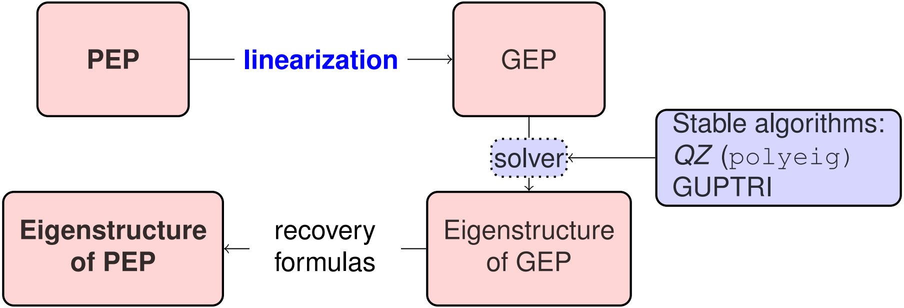

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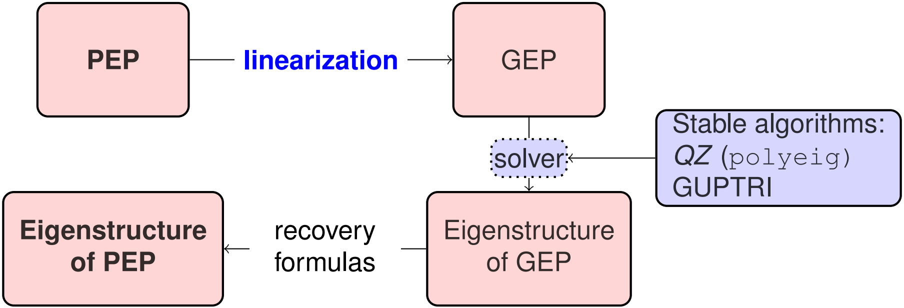

☞ Linz's preserve e-vals (and multiplicities), but **NOT** e-vecs, minimal bases and minimal indices.

Linearization: example

$$F(\lambda) = A_0 + \lambda A_1 + \cdots + \lambda^d A_d, \qquad A_0, A_1, \ldots, A_d \in \mathbb{C}^{m\times n}$$

Frobenius companion lin'z of $P(\lambda)$:

$$F(\lambda) := \begin{bmatrix} \lambda A_d + A_{d-1} & A_{d-2} & \cdots & A_1 & A_0 \\ -I_n & \lambda I_n & & & \\ & \ddots & \ddots & & \\ & & \ddots & \lambda I_n & \\ & & & -I_n & \lambda I_n \end{bmatrix} \in \mathbb{C}[\lambda]^{(m+n(d-1))\times nd}$$

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☞ There are **many other** linearizations !!!!

Linearization: example

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Frobenius companion lin'z of $P(\lambda)$:

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(Pérez-Álvaro, Today, 18:00; Dmytryshyn, Wed. 11:30; Hernando Wed. 12:00; Saltenberger, Wed. 12:30).

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The size of the problem increases very much !!!!

REPs

- **Linearization** is also the standard tool for small/medium size problems.
- However, the situation is more complicated, due to the presence of denominators.
- No explicit symbolic constructions exist whose e-vals coincide with those of $F(\lambda)$ **for any rational function** F.
- Theoretical background goes back to the 1970's in Control Theory (Rosenbrock, Van Dooren, Verghese, ...) and it is being revisited ([Su-Bai'11], [Amparan-Marcaida-Dopico-Zaballa'18], [Dopico-Quintana-VanDooren]).

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(Van Dooren, Tue. 14:30; Quintana, Tue. 15:00; Hollister Tue. 15:30).

Other NLEVPs

$$\begin{array}{llll} F: & \Omega & \to & \mathbb{C}^{n\times n} \\ & \lambda & \mapsto & F(\lambda) \end{array}$$

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(Telen, Today 18:30; Embree, Wed. 14:30; Guguercin Wed. 15:00; Miedlar, Wed. 17:30)

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Algorithms for large-scale problems

1. PEPs:
   - **Model reduction**: Project the problem into a subspace of much smaller dimension.
   - Large-scale methods (Krylov-type) for **GEP**s over a linearization that take advantage of the structure of the lin'z:
     - TOAR [Su-Bai-Lu'08, Kressner-Román'14].
     - CORK [Van Beeumen-Michiels-Meerbergen'15].
2. REPs: RCORK [Dopico-González Pizarro'19].

(Meerbergen, Tue. 16:00).

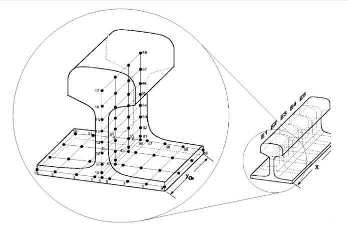
Try to preserve the structure!

In many cases, $F(\lambda)$ coming from applications has some **symmetry structure**. E. g.:

Palindromic matrix polynomials

(arising in the vibration analysis of rails excited by high speed trains):

$$P(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0,$$
$$A_0^\top = A_2, A_1^\top = A_1.$$



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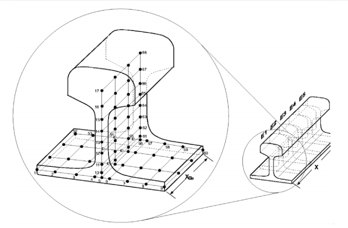

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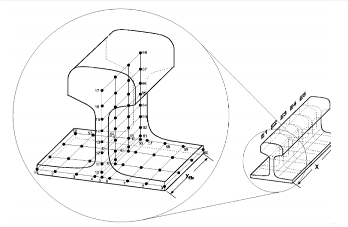

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☞ Rounding errors can destroy this symmetry, if we don't use tools that preserve the symmetry structure. E. g.: the linearization:

$$\begin{bmatrix} \lambda A_2 + A_1 & A_0 \\ -I & \lambda I \end{bmatrix}.$$

Some available software

1. For small-moderate size PEPs: `polyeig` (MATLAB), `quadeig` (QEPs) [Hammarling-Munro-Tisseur'13].

   Large size NLEPs:

2. NLEIGS [Güttel-Van Beeumen-Meerbergen-Michiels'14].
3. Automatic Rational Approximation and Linearization of NEPs [Lietaert-Pérez-Vandereycken-Meerbergen'18].
4. Parallel implementations of TOAR for any degree in SLEPc [Roman-etal'16].

(Tisseur, Wed. 17:00; Román, Wed. 18:00; Jarlebring, Wed. 18:30).

Some surveys on NLEVPs

S. Güttel, F. Tisseur.
The nonlinear eigenvalue problem.
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THANK YOU!!! – ¡GRACIAS!