Sylvester and $\star$-Sylvester equations: analogies and differences

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ILAS 2014, August 6–9, 2014

Collaborators

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Outline

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The equations

Consider the equations:

- $AX + X^\star A = 0$ $\quad A \in \mathbb{C}^{n\times n}$
- $AX + X^\star B = 0$ $\quad A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{n\times m}$
- $AX + BX^\star = 0$ $\quad A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{m\times n}$

where $\star$ can be either $*$ (conjugate transpose) or T (transpose).

GOALS:

- Find necessary and sufficient conditions for consistency.
- Find the dimension of the solution space.
- Find an expression for the solution.
- Find necessary and sufficient conditions for uniqueness of the solution.

Motivation

- $AX + X^\star A = 0$: Naturally arises when studying $\star$-congruence orbits of $A \in \mathbb{C}^{n\times n}$ [De Terán & Dopico, 2011].
- $AX + X^\star B = C$: Related to block-antidiagonalization of block anti-triangular matrices via $\star$-congruence $\rightsquigarrow$ $\star$-palindromic eigenvalue problems [Byers & Kressner, 2006], [Kressner, Schröder & Watkins, 2009].

Some history

$AX + X^{\star}B = C$ $(\star = T \text{ or } *)$ $\star$-Sylvester equation
(or “Sylvester equation for congruence")

(a) Sylvester equation: $AX - XB = C$ (A, B must be **square**!!)

- Solution know since (at least) the 1950's (Gantmacher).
- Characterization of consistency and uniqueness of solution known for long (Roth, 1952, Gantmacher, 1959).
- Efficient algorithm for the unique solution already known (Bartels-Stewart, 1971).

Some history (II)

$AX + X^\star B = C \qquad (\star = T \text{ or } *) \qquad \star$-Sylvester equation

(b) $AX \pm X^\star A^\star = C, \qquad A \in \mathbb{F}^{m\times n}, C \in \mathbb{F}^{m\times m}$:

- Hodges (1957): Solution over finite fields.
- Taussky-Wielandt (1962): Eigenvalues of $g(X) = A^T X + X^T A$.
- Lancaster-Rozsa (1983), Braden (1999): Necessary and sufficient conditions for consistency. Closed-form formula for the solution (using projectors and generalized inverses) and dimension of the solution space.
- Djordjević (2007): Extends Lancaster-Rozsa to A, C, X bounded linear operators on Hilbert spaces (with closed rank).

(c) $AX + X^\star A = C, \qquad A, C \in \mathbb{C}^{n\times n}$:

- Ballantine (1969): $H = PA + AP^*$, with H hermitian and A, P with certain structure.
- DT-Dopico (2011): Complete solution for $C = 0$ (explicit except for two particular sub-cases). Related to the theory of (congruence) orbits.
- García-Shoemaker (2013), and Chan-García-García-Shoemaker (2013): Explicit solution for the remaining sub-cases.

Some history (III)

(d) The $\star$-Sylvester equation: $AX + X^\star B = C$:

- Necessary and sufficient conditions for **consistency**: **Wimmer (1994)**, Piao-Zhang-Wang (2007, involved), DT-Dopico (2011, another proof of Wimmer's, valid for arbitrary fields with char$\neq 2$).
- Necessary and sufficient conditions for **unique solution**: Byers-Kressner (2006, $\star = T$), Kressner-Schröder-Watkins (2009, $\star = *$).
- **Closed-form formula** for the solution: Piao-Zhang-Wang (2007, involved), Cvetković-Ilić (2008, operators with certain restrictions). In terms of generalized inverses.
- **Complete solution** ($C = 0$): DT-Dopico-Guillery-Montealegre-Reyes (2013).
- **Algorithm** for the (unique) solution: DT-Dopico (2011, $O(n^3)$), Vorontsov-Ivanov (2011), Chiang-Chu-Lin (2013).

(e) $AX + BX^\star = 0$:

- **Complete solution**: DT (2013).

Related work

- Dmytryshyn-Kågström-Sergeichuk (2014): Solution of the system $\begin{cases} X^TA + AX = 0 \\ X^TB + BX = 0 \end{cases}$, with A, B skew-symmetric.
- Dmytryshyn-Kågström-Sergeichuk (2014): Solution of the system $\begin{cases} X^TA + AX = 0 \\ X^TB + BX = 0 \end{cases}$, with A, B symmetric.
- Zhou-Lam-Duan (2011): Closed-form formula for the solution of $X = AX^\star B + C$

Outline

Orbit theory

$XA + AX^\star = 0, \qquad A \in \mathbb{C}^{n\times n}$

Set:

$\mathcal{O}_c(A) = \{PAP^\star \ : \ P \text{ nonsingular}\}$ $\quad$ **$\star$-congruence** orbit of A

(complex manifold if $\star = T$, real manifold if $\star = *$).

$\mathcal{O}_s(A) = \left\{PAP^{-1} \ : \ P \text{ nonsingular}\right\}$ $\quad$ Similarity orbit of A

Then:

$T_{\mathcal{O}_c(A)}(A) = \{XA + AX^\star \ : \ X \in \mathbb{C}^{n\times n}\}$ Tangent space of $\mathcal{O}_c(A)$ at A

$T_{\mathcal{O}_s(A)}(A) = \{XA - AX \ : \ X \in \mathbb{C}^{n\times n}\}$ Tangent space of $\mathcal{O}_s(A)$ at A

(a) $\operatorname{codim} \mathcal{O}_c(A) = \operatorname{codim} T_{\mathcal{O}_c(A)}(A) = \dim(\text{solution space of } XA + AX^\star = 0)$

(complex dimension if $\star = T$, real dimension if $\star = *$).

(b) $\operatorname{codim} \mathcal{O}_s(A) = \operatorname{codim} T_{\mathcal{O}_s(A)}(A) = \dim(\text{solution space of } XA - AX = 0)$

Reduction by congruence to anti-triangular form

$$\overbrace{\begin{bmatrix} I & 0 \\ X^\star & I \end{bmatrix}}^{P} \begin{bmatrix} 0 & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \overbrace{\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}}^{P^\star} = \begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix}$$

$$\Leftrightarrow A_{21}X + X^\star A_{12} = -A_{22}.$$

Application: Anti-triangular form of palindromic pencils $A + \lambda A^\star$.

(Analogous to:

$$\overbrace{\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}}^{P} \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \overbrace{\begin{bmatrix} I & -X \\ 0 & I \end{bmatrix}}^{P^{-1}} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$$

$$\Leftrightarrow A_{11}X - XA_{22} = A_{12} \rightsquigarrow$$ **Sylvester equation**)

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Outline

$AX - XB = 0$: Resolution procedure

$$AX - XB = 0$$

$$P^{-1} \cdot \downarrow \cdot Q$$

$$(P^{-1}AP)(P^{-1}XQ) - (P^{-1}XQ)(Q^{-1}BQ) = 0$$

$$\Updownarrow \; Y = P^{-1}XQ$$

$$J_A Y - Y J_B = 0$$

J_M = **Jordan canonical form of** M

1. Solve: $J_A Y - Y J_B = 0$ (Explicit solution available).
2. Undo: $X = P Y Q^{-1}$

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2. Undo: $X = PYQ^{-1}$

$AX - XB = 0$: Summary

- The solution depends on the Jordan canonical form (JCF) of A, B: J_A, J_B.
- Explicit solution is available (up to the knowledge of the change matrices P, Q such that $P^{-1}AP = J_A$, $Q^{-1}BQ = J_B$).
- Unique solution if and only if $\sigma(A) \cap \sigma(B) = \emptyset$.

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$AX - XB = C$: consistency

Theorem (Roth, 1952)

$AX - XB = C$ is consistent if and only if

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \text{ and } \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \text{ are similar.}$$

Outline

$XA + AX^\star = 0$: Resolution procedure

$$XA + AX^\star = 0$$

$$P \cdot \downarrow \cdot P^\star$$

$$(PXP^{-1})PAP^\star + PAP^\star(P^{-\star}X^\star P^\star) = 0$$

$$\Updownarrow \; Y = PXP^{-1}$$

$$YC_A + C_AY^\star = 0$$

$C_A = PAP^\star$: **canonical form for $\star$-congruence of A**

1. Solve: $YC_A + C_AY^\star = 0$
2. Undo: $X = P^{-1}YP$

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$C_A = PAP^\star$: **canonical form for $\star$-congruence of A**

1. Solve: $YC_A + C_A Y^\star = 0$
2. Undo: $X = P^{-1} Y P$

$XA + AX^\star = 0$: Resolution procedure

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Solution through canonical form for congruence

Canonical form for $\star$-congruence of A [Horn & Sergeichuk, 2006]: $PAP^\star = C_A$

$C_A = D_1 \oplus \cdots \oplus D_s$ $\rightsquigarrow$ direct sum of three types of blocks (different for $\star = T$ and $\star = *$).

$XC_A + C_AX^\star = 0$ **decouples** into two kind of equations:

(i) $XD_i + D_iX^T = 0$

(ii) $XD_j + D_iY^T = 0$, $YD_i + D_jX^T = 0$

☞ We have solved all them, for D_i, D_j being any possible combination of canonical blocks (**9 different equations**).

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Solution of $XA + AX^\star = 0$: summary

- Depends on the canonical form for $\star$-congruence of A: C_A.
- Explicit solution available: up to the knowledge of P such that $PAP^\star = C_A$.

Outline

$AX + X^\star B = 0$: Resolution procedure

$$AX + X^\star B = 0$$

$$P\cdot \downarrow \cdot P^\star$$

$$(PAQ)(Q^{-1}XP^\star) + (PX^\star Q^{-\star})(Q^\star BP^\star) = 0$$

$$\Updownarrow \; Y = Q^{-1}XP^\star$$

$$K_A Y + Y^\star K_B = 0$$

$(K_A, K_B^\star) = P(A, B^\star)Q$: **Kronecker canonical form (KCF) of** $(A, B^\star)$

1. Solve: $K_A Y + Y^\star K_B = 0$
2. Undo: $X = QYP^{-\star}$

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Partition into blocks

Lemma

Let $E = \text{diag}(E_1, \ldots, E_d)$ and $F^\star = \text{diag}(F_1^\star, \ldots, F_d^\star)$, and partition $X = [X_{ij}]_{i,j=1:d}$. Then

$$EX + X^\star F = 0$$

is equivalent to the set of equations

$$\begin{aligned} E_i X_{ij} + X_{ji}^\star F_j &= 0 \\ E_j X_{ji} + X_{ij}^\star F_i &= 0, \end{aligned}$$

for $i, j = 1, \ldots, d$.

Note that we have:

$$i = j \to E_i X_{ii} + X_{ii}^\star F_i = 0 \qquad \text{(1 equation)}$$

$$i \neq j \to \begin{cases} E_i X_{ij} + X_{ji}^\star F_j = 0 \\ E_j X_{ji} + X_{ij}^\star F_i = 0 \end{cases} \qquad \text{(system of 2 equations)}$$

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Using the KCF

By particularizing to $(E, F^\star)$ as the KCF of $(A, B^\star)$, i.e.: direct sum of blocks:

- **Type 1**: “finite blocks": $(J_k(\lambda_i), I_k)$
- **Type 2**: “infinite blocks": $(I_m, J_m(0))$
- **Type 3**: “right singular blocks":

$$L_\varepsilon = \left(\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix}_{\varepsilon\times(\varepsilon+1)}, \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}_{\varepsilon\times(\varepsilon+1)} \right)$$

- **Type 4**: “left singular blocks": L_η^T

we have to solve:

(a) $EX + X^\star F = 0$, with $(E, F^\star)$ of **type 1–4** $\rightsquigarrow$ **4** equations

(b) $\begin{array}{l} E_iX + Y^\star F_j = 0 \\ E_jY + X^\star F_i = 0 \end{array}$, with $(E_i, F_i^\star)$, $(E_j, F_j^\star)$ of **type 1–4** $\rightsquigarrow$ **10** systems

☞ We get explicit solutions for all these equations/systems.

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Type 4: “left singular blocks": L_η^T

we have to solve:

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☞ We get explicit solutions for all these equations/systems.

Solution of $AX + X^{\star}B = 0$: summary

- Depends on the Kronecker canonical form of $(A, B^{\star})$: $(K_A, K_B^{\star})$.
- Explicit solution available: up to the knowledge of P, Q such that $P(A, B^{\star})Q = (K_A, K_B^{\star})$.

Uniqueness of solution

Theorem (Byers-Kressner 2006, Kressner-Schröder-Watkins 2009)

$A, B \in \mathbb{C}^{n\times n}$. Then

$$AX + X^{\star}B = C \qquad \text{has a unique solution}$$

if and only if

(1) $(A, B^{\star})$ is regular, and

(2) $\star = T$: If $\mu \in \sigma(A, B^T) \setminus \{-1\}$, then $1/\mu \notin \sigma(A, B^T) \setminus \{-1\}$ and, if $-1 \in \sigma(A, B^T)$, then it has algebraic multiplicity one.
$\star = *$: If $\mu \in \sigma(A, B^*)$, then $1/\overline{\mu} \notin \sigma(A, B^*)$.

$AX + X^\star B = C$: Consistency

Theorem (Wimmer 1994, DT-Dopico 2011)

Let $\mathbb{F}$ be a field with $\operatorname{char}\mathbb{F} \neq 2$, $A \in \mathbb{F}^{m\times n}$, $B \in \mathbb{F}^{n\times m}$, $C \in \mathbb{F}^{m\times m}$. Then

$$AX + X^\star B = C \qquad \text{is consistent}$$

if and only if

$$\begin{bmatrix} C & A \\ B & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \quad \text{are } \textbf{congruent}$$

(Compare with Roth's criterion:

"$AX - XB = C$ is consistent if and only if

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \qquad \text{are } \textbf{similar}\text{")}$$

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Outline

$AX + BX^\star = 0$: Resolution procedure

$A, B \in \mathbb{C}^{m\times n}$, $X \in \mathbb{C}^{n\times n}$ (**square** !!)

$$AX + BX^\star = 0$$

$$P \cdot \downarrow \cdot Q^\star$$

$$(PAQ^{-1})(QXQ^\star) + (PBQ^{-1})(QX^\star Q^\star) = 0$$

$$\Updownarrow \; Y = QXQ^\star$$

$$K_A Y + K_B Y^\star = 0$$

$(K_A, K_B) = P(A, B)Q^{-1} \rightsquigarrow$ **KCF**(A, B)

1. Solve: $K_A Y + K_B Y^\star = 0$
2. Undo: $X = Q^{-1} Y Q^{-\star}$

$AX + BX^{\star} = 0$: Resolution procedure

$A, B \in \mathbb{C}^{m\times n}$, $X \in \mathbb{C}^{n\times n}$ (**square** !!)

$$AX + BX^{\star} = 0$$

$$P\cdot \downarrow \cdot Q^{\star}$$

$$(PAQ^{-1})(QXQ^{\star}) + (PBQ^{-1})(QX^{\star}Q^{\star}) = 0$$

$$\Updownarrow \; Y = QXQ^{\star}$$

$$K_A Y + K_B Y^{\star} = 0$$

$(K_A, K_B) = P(A, B)Q^{-1} \rightsquigarrow$ **KCF**(A, B)

1. Solve: $K_A Y + K_B Y^{\star} = 0$
2. Undo: $X = Q^{-1} Y Q^{-\star}$

$AX + BX^\star = 0$: Resolution procedure

$A, B \in \mathbb{C}^{m\times n}, \quad X \in \mathbb{C}^{n\times n}$ (**square** !!)

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Uniqueness of solution

Theorem (DT, 2013)

$A, B \in \mathbb{C}^{n\times n}$. Then

$$AX + BX^\star = C \qquad \text{has a unique solution}$$

if and only if

(1) KCF(A, B) has no right singular blocks

(2) If $\mu \in \sigma(A, B)$, then $1/\mu \notin \sigma(A, B) \setminus \{-1\}$.

Solution of $AX + BX^\star = 0$: summary

With similar developments as for $AX + X^\star B = 0$...

- Depends on the Kronecker canonical form of (A, B): (K_A, K_B).
- Explicit solution available: up to the knowledge of P, Q such that $P(A, B)Q^{-1} = (K_A, K_B)$.

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☞ Characterization of **consistency**??????

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☞ Characterization of **consistency**: **Open problem**

Conclusions

- To solve $\mathbf{AX - XB = 0, AX + X^\star B = 0, AX + BX^\star = 0}$: Reduction to canonical form and explicit solution after decoupling through canonical blocks:
    - $AX - XB = 0$: JCF of A and B.
    - $AX + X^\star A = 0$: Canonical form for $\star$-congruence of A.
    - $AX + X^\star B = 0$: KCF of $(A, B^\star)$.
    - $AX + BX^\star = 0$: KCF of (A, B).

- Consistency of $AX + X^\star B = C$ vs $AX - XB = C$: the role of similarity is now played by congruence.

- Uniqueness of solution: depends on spectral properties of:
    - $AX - XB$: A and B.
    - $AX + X^\star B = 0$: The pencil $(A, B^\star)$
    - $AX + BX^\star = 0$: The pencil (A, B).

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