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On the consistency of $X^\top AX = B$ when B is either symmetric or skew

Fernando De Terán

Joint work with **A. Borobia** and **R. Canogar**

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The problem

Provide necessary and sufficient conditions for the equation

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being consistent, when B is either symmetric or skew-symmetric.

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☞ A is not necessarily symmetric (or skew). When A is symmetric (skew) the result is **well-known**: $\operatorname{rank} B \leq \operatorname{rank} A$ is a necessary and sufficient condition (even when $m \neq n$).

A remark

$$\boxed{X^\top AX = B,\ A \in \mathbb{C}^{n\times n}, B \in \mathbb{C}^{m\times m}}$$

☞ If X is **invertible**, then A must be symmetric. ✓

Then...

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☞ Then, $n > m$

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$X^\top AX = B$ and bilinear forms

The problem is equivalent to

Given a bilinear form $\mathbb{A} : \mathbb{C}^n \to \mathbb{C}^n$, find the largest dimension of a subspace $V \subseteq \mathbb{C}^n$, such that $\mathbb{A}_{|V} : V \to V$ is symmetric (skew) and non-degenerate.

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(If A is a matrix of $\mathbb{A}$ in some basis, and the columns of X are a basis of V, then $X^\top AX$ is a matrix for $\mathbb{A}_{|V}$.)

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Some references on this problem

- A, B with entries over finite fields (or fields with characteristic 2):

J. H. M. Wedderburn.
The automorphic transformation of a bilinear form.
Ann. of Math. 2, 23 (1921) 122–134.

L. Carlitz.
Representations by skew forms in a finite field.
Arch. Math., V (1954) 19–31.

J. H. Hodges.
A skew matrix equation over a finite field.
Math. Nachr., 17 (1966) 49–55.

P. G. Buckhiester.
Rank r solutions to the matrix equation $XAX^t = C$, A alternate, over GF(2^y).
Trans. Amer. Math. Soc., 189 (1974) 201–209.

- Recent references (mainly connected to applications):

P. Benner, D. Palitta.
On the solution of the non-symmetric T-Riccati equation.
Electron. Trans. Numer. Anal., 54 (2021) 66–88.

P. Benner, B. Iannazzo, B. Meini, D. Palitta.
Palindromic linearization and numerical solution of nonsymmetric algebraic T-Riccati equations.
(2021) arXiv:2110.03254

M. Benzi, M. Viviani.
Solving cubic matrix equations arising in conservative dynamics.
(2021) arXiv:2111.12373

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Some examples (B symmetric)

$X^\top AX = B$ with ...

- $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is consistent ($X = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$)

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(B is symmetric in all cases, but A **is not**).

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$X^\top J_n(0)X = I_m \oplus 0_{s\times s}$ is consistent $\Leftrightarrow$ $n \geq 2m-1, \ n > 1$.

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The Canonical form for congruence (CFC)

$$J_k(\lambda) := \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix}, \quad \Gamma_k := \begin{bmatrix} 0 & & & & & (-1)^{k+1} \\ & & & & \iddots & (-1)^k \\ & & & -1 & \iddots & \\ & & 1 & 1 & & \\ & -1 & -1 & & & \\ 1 & 1 & & & & 0 \end{bmatrix}, \quad H_{2k}(\lambda) := \begin{bmatrix} 0 & I_k \\ J_k(\lambda) & 0 \end{bmatrix}.$$

Theorem (CFC) [Horn & Sergeichuk, 2006]

Each square complex matrix is congruent to a direct sum, uniquely determined up to permutation of addends, of matrices of the form:

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$$(\Gamma_1 = [1], \qquad H_2(-1) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.)$$

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Notation: C_M= CFC of M.

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☞ We can restrict ourselves to A and B given in CFC.

Some basic remarks

- B symmetric $\Rightarrow C_B = I_m \oplus 0_{s\times s}$
- B skew $\Rightarrow C_B = H_2(-1)^{\oplus k} \oplus 0_{s\times s} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{\oplus k} \oplus 0_{s\times s}$.
- $X^\top(A \oplus 0_{\ell\times\ell})X = B \oplus 0_{s\times s}$ is consistent $\Leftrightarrow X^\top AX = B$ is consistent.

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- $X^\top (A \oplus 0_{\ell\times\ell}) X = B \oplus 0_{s\times s}$ is consistent $\Leftrightarrow X^\top A X = B$ is consistent.

(We can get rid of possible null diagonal blocks in the CFC of A and B, namely blocks $J_1(0)$. In particular, B may be assumed to be invertible).

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A necessary condition

Let C_A consist of (exactly):

(i) j_1 Type-0 blocks with size 1;

(ii) j_O Type-0 blocks with odd size at least 3;

(iii) γ_ε Type-I blocks with even size;

(iv) h^-_{2O} Type-II blocks of the form $H_{4k-2}(-1)$, for any $k \geq 1$;

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(vi) an arbitrary number of other blocks.

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Set:

$$\rho_{\mathrm{sym}}(A) := \frac{n - j_1 + j_o + \gamma_\varepsilon + 2h_4^+}{2}, \quad \rho_{\mathrm{skew}}(A) := \frac{n - j_1 + j_o + \gamma_\varepsilon + 2h_{2o}^-}{2}.$$

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Theorem

- $X^\top AX = B$ consistent (B symmetric) $\Rightarrow$ rank $B \leq \rho_{\mathrm{sym}}(A)$.
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☞ It depends on (certain kinds of blocks in) the CFC of A.

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1. $A = H_2(-1) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $B = I_1 = [1]$.

   The condition is satisfied:

   $$n = 2, \quad \operatorname{rank} B = 1, \quad j_1 = j_O = \gamma_\varepsilon = 2h_4^+ = 0,$$

   so it reads

   $$1 \leq \rho_{\mathrm{sym}}(A) = \frac{2}{2}.$$

   However,

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The answer is NO. (But there are just a few exceptions).

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2. $A = H_4(1) = \left[\begin{array}{cc|cc} & & 1 & 0 \\ & & 0 & 1 \\ \hline 1 & 1 & & \\ 0 & 1 & & \end{array}\right]$, $B = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

   The condition is satisfied:

   $$n = 4, \quad \operatorname{rank} B = 3, \quad j_1 = j_O = \gamma_\varepsilon = 0,\ 2h_4^+ = 2,$$

   so it reads

   $$3 \leq \rho_{\mathrm{sym}}(A) = \frac{4+2}{2}.$$

   However,

   $$X^\top \left[\begin{array}{cc|cc} & & 1 & 0 \\ & & 0 & 1 \\ \hline 1 & 1 & & \\ 0 & 1 & & \end{array}\right] X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   is not consistent.

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Is it sufficient (B symmetric)?

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☞ When B is symmetric, the blocks $H_2(-1)$ and $H_4(1)$ in C_A are problematic.

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Is it sufficient (B skew)?

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The answer is, again, NO.

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Is it sufficient (B skew)?

The answer is, again, NO. (But, again, there are just a few exceptions).

1. $A = \Gamma_2^{\oplus 4} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}^{\oplus 4}$, $B = H_2(-1)^{\oplus 3} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{\oplus 3}$.

   The condition is satisfied:

   $$n = 8, \quad \operatorname{rank} B = 6, \quad j_1 = j_O = 2h_{2O}^- = 0, \ \gamma_\varepsilon = 4,$$

   so it reads

   $$6 \leq \rho_{\text{skew}}(A) = \frac{8+4}{2}.$$

   However,

   $$X^\top \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}^{\oplus 4} X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{\oplus 3}$$

   is not consistent.

Is it sufficient (B skew)?

The answer is, again, NO. (But, again, there are just a few exceptions).

1. $A = \Gamma_2^{\oplus 4} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}^{\oplus 4}$, $B = H_2(-1)^{\oplus 3} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{\oplus 3}$.

2. $A = \Gamma_1^{\oplus n} = I_n,\ B = H_2(-1) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

   The condition is satisfied for $n \geq 4$:

$$\operatorname{rank} B = 2, \quad j_1 = j_O = \gamma_\varepsilon = 2h_{2O}^- = 0,$$

   so it reads

$$2 \leq \rho_{\mathrm{sym}}(A) = \frac{n}{2}.$$

   However,

$$X^\top I_n X = X^\top X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

   is not consistent.

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Is it sufficient (B skew)?

The answer is, again, NO. (But, again, there are just a few exceptions).

1. $A = \Gamma_2^{\oplus 4} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}^{\oplus 4}$, $B = H_2(-1)^{\oplus 3} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{\oplus 3}$.

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$$2 \leq \rho_{\mathrm{sym}}(A) = \frac{n}{2}.$$

However,

$$X^\top I_n X = X^\top X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

is not consistent (note that $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ is skew).

Is it sufficient (B skew)?

The answer is, again, NO. (But, again, there are just a few exceptions).

1. $A = \Gamma_2^{\oplus 4} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}^{\oplus 4}$, $B = H_2(-1)^{\oplus 3} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{\oplus 3}$.
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☞ When B is skew, the blocks Γ_1 and Γ_2 in C_A are problematic.

It is sufficient “in general"

Theorem (consistency of $X^\top AX = B$, with B symmetric).

If C_A does not contain blocks of the form $H_2(-1)$ and $H_4(1)$, then

$$X^\top AX = B \qquad (B \text{ symmetric})$$

is consistent if and only if $\operatorname{rank} B \le \rho_{\mathrm{sym}}(A)$.

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Theorem (consistency of $X^\top AX = B$, with B skew).

If C_A does not contain blocks of the form Γ_1 and Γ_2, then

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is consistent if and only if $\operatorname{rank} B \leq \rho_{\mathrm{skew}}(A)$.

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When CFC(A) contains blocks $H_2(-1)$: B symmetric

Set:

$$\sigma_{\rm sym}(A) := n - j_1 - j_o - \gamma_\varepsilon - 2h_{2o}^-$$

(recall: $\rho_{\rm sym}(A) := \frac{n - j_1 + j_O + \gamma_O + 2h_{2\varepsilon}^+}{2}$)

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If CFC(A) has no blocks of type $H_2(-1)$ then $\rho_{\mathrm{sym}}(A) \leq \sigma_{\mathrm{sym}}(A)$.

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If CFC(A) has no blocks of type $H_2(-1)$ then $\rho_{\rm sym}(A) \leq \sigma_{\rm sym}(A)$.

Theorem (consistency of $X^\top AX = B$, with B symmetric, improved).

If C_A does not contain blocks of the form $H_4(1)$, then

$$X^\top AX = B \qquad (B \text{ symmetric})$$

is consistent if and only if $\operatorname{rank} B \leq \min\left\{\rho_{\rm sym}(A), \sigma_{\rm sym}(A)\right\}$.

Generic case in terms of bilinear forms

Theorem

$\mathbb{A} : \mathbb{C}^n \to \mathcal{R}$ a **bilinear form.**

$A \in \mathbb{C}^{n\times n}$ a matrix representation of $\mathbb{A}$.

If CFC(A) does not contain blocks $H_4(1)$, the largest dimension of a subspace, V of $\mathbb{C}^n$ such that $\mathbb{A}|_V$ is a symmetric (non-degenerate) is $\min\{\rho_{\rm sym}(A), \sigma_{\rm sym}(A)\}$.

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Theorem

$\mathbb{A} : \mathbb{C}^n \to \mathcal{R}$ a **bilinear form.**

$A \in \mathbb{C}^{n\times n}$ a matrix representation of $\mathbb{A}$.

If CFC(A) does not contain blocks Γ_1 and Γ_2, the largest dimension of a subspace, V, of $\mathbb{C}^n$ such that $\mathbb{A}|_V$ is skew-symmetric (non-degenerate) is $\rho_{\mathrm{skew}}(A)$.

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The generic case

The “generic" CFC in $\mathbb{C}^{n\times n}$ is:

$$\mathrm{CFC}_g(n) := \begin{cases} H_2(\mu_1)\oplus\cdots\oplus H_2(\mu_k), & \text{if } n=2k,\\ H_2(\mu_1)\oplus\cdots\oplus H_2(\mu_k)\oplus\Gamma_1, & \text{if } n=2k+1 \end{cases}$$

($\mu_1,\ldots,\mu_k$ different to each other and to $\mu_1^{-1},\ldots,\mu_k^{-1}$, ± 1).

FDT, F. M. Dopico.
The solution of the equation $XA + AX^T = 0$ and its application to the theory of orbits.
Linear Algebra Appl., 434 (2011) 44–67

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Theorem

If $C_A=\mathrm{CFC}_g(n)$, then

$$X^\top AX=B \qquad (B \text{ symmetric or skew})$$

is consistent if and only if $\operatorname{rank} B\leq n/2$.

Open questions

Analyze the consistency of:

- $X^\top AX = B$, when C_A contains blocks $H_4(1)$ (B symmetric) or Γ_1, Γ_2 (B skew).
- $X^* AX = B$, with B Hermitian or skew-Hermitian.
- $X^\top AX = B$ with B symmetric/skew but A, B, X having **real entries**.
- (Hard) $X^\top AX = B$, with B arbitrary.

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