Chapter 1

Experimental facts

1.1 The principles of relativity and determinacy

1.2 The galilean group and Newton's equations

Exercise 1.1 证明空间$\mathbb{R}\times\mathbb{R}^3$的每个加利略变换都可唯一得表示为一个旋转, 一个平移, 一个匀速运动的组合($g = g_1 \circ g_2 \circ g_3$)(所以, 加利略变换群的维数是$3+4+3=10$)

Proof： 对任意加利略变换g, 先考虑$g(t,\mathbf{0})$由g是线性的, 可知, 有$\boldsymbol{v}$, $\boldsymbol{a}$,s, 使得$g(t,\mathbf{0}) = (t-s,\boldsymbol{a}+t\boldsymbol{v}) \in \mathbb{R}\times\mathbb{R}^3$. 下面考虑$\mathbb{R}^3$中的标准正交基$\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3$. 令$g(t,\boldsymbol{e}_i)-g(t,\mathbf{0}) = (t-s,\boldsymbol{u}_i)$. 由于同时事件空间的差是一个3维线性空间(定义).

所以$g(t,(a,b,c))-g(t,\mathbf{0}) = (t-s,G(a,b,c)^T)$, 其中$G = (\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3)$. 令$g_3(t,\boldsymbol{x}) = (t,G\boldsymbol{x})$, $g_2(t,\boldsymbol{x}) = (t,t\boldsymbol{v}+\boldsymbol{x})$, $g_1(t,\boldsymbol{x}) = (t-s,\boldsymbol{a}+\boldsymbol{x})$. 则$g = g_1\circ g_2\circ g_3$, 即为所求.

Exercise 1.2 证明一切加利略空间均互相同构, 而特别是同构于坐标空间$\mathbb{R}\times\mathbb{R}^3$.

Proof： 由于$t:\mathbb{R}^4\to\mathbb{R}$, 是一个Hilbert 空间上的线性范函, 有如下分解:$\mathbb{R}^4 = \operatorname{Ker} t \oplus \operatorname{Ker} t^{\perp}$, 且$\operatorname{Ker} t$的维数是3. 设$P,Q$为对应的投影映射, 使$\boldsymbol{x} = P\boldsymbol{x}+Q\boldsymbol{x}$, 其中$P\boldsymbol{x}\in\operatorname{Ker} t$, $Q\boldsymbol{x}\in\operatorname{Ker} t^{\perp}$. 又令$R$为$\operatorname{Ker} t$到$\mathbb{R}^3$的一个等距同构.

任取A^4中一点$\boldsymbol{x}_0$. 可作$F: A^4\to\mathbb{R}\times\mathbb{R}^3$, 其中$F(\boldsymbol{x}) = (t(\boldsymbol{x}-\boldsymbol{x}_0), R\,P\boldsymbol{x})$. 即为所求同构.

Exercise 1.3 平面上一个可微运动的轨迹是否有图3形状? 其加速度适量是否有图上所示的值?

Solution： 可能, 如$\boldsymbol{x}(t) = (t^3,t^2)$; 不可能$\ddot{\boldsymbol{x}}$ 指向曲线凹处.

Exercise 1.4 若一力学系只由一点组成, 证明它在惯性参考系中的加速度为零.

Proof： 由例1,例2,加速度不依赖于$\boldsymbol{x},\dot{\boldsymbol{x}}$和$t$, 所以$F(\boldsymbol{x},\dot{\boldsymbol{x}},t) = \boldsymbol{v}$, 又由旋转不变性对任意$G$, $\boldsymbol{v} = F(G\boldsymbol{x},G\dot{\boldsymbol{x}},t) = GF(\boldsymbol{x},\dot{\boldsymbol{x}},t) = G\boldsymbol{v}$ 所以只能有$\boldsymbol{v}=0$. 即加速度为零.

Exercise 1.5 设一个力学系由两点组成, 在初始时刻其速度均为零, 证明他们将停留在初始时刻联结他们的直线上.

Proof： 只要证明$\ddot{\boldsymbol{x}}_i$平行于$\boldsymbol{x}_1-\boldsymbol{x}_2$. 显然$\boldsymbol{x}_1-\boldsymbol{x}_2$非零, 可令$\boldsymbol{u} = (\boldsymbol{x}_1-\boldsymbol{x}_2)/\|\boldsymbol{x}_1-\boldsymbol{x}_2\|$, 对任意与$\boldsymbol{u}$垂直的单位向量$\boldsymbol{v}$,令$\boldsymbol{w} = \boldsymbol{u}\times\boldsymbol{v}$ 令$G = (\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})^T$, 为一个旋转变换. 由例3的相同方法,可知$F = \ddot{\boldsymbol{x}} = (\ddot{\boldsymbol{x}}_1,\ddot{\boldsymbol{x}}_2)$ 只依赖于$\boldsymbol{x}_1-\boldsymbol{x}_2$和$\dot{\boldsymbol{x}}_1-\dot{\boldsymbol{x}}_2$. 则$G\ddot{\boldsymbol{x}} = G\,F(\boldsymbol{x}_1-\boldsymbol{x}_2,\dot{\boldsymbol{x}}_1-\dot{\boldsymbol{x}}_2,t) = F(G\boldsymbol{u},G\mathbf{0},t) \equiv F((\|\boldsymbol{u}\|,0,0)^T,\mathbf{0},t)$. 由$\boldsymbol{v}$的任意性, $\ddot{\boldsymbol{x}}_i$垂直于$\boldsymbol{v}$. 又考虑到$\boldsymbol{x}_1 = \boldsymbol{x}_2 = 0$ 所以$\boldsymbol{x}_i(t)$始终在他们的连线上.

Exercise 1.6 设一个力学系由三点组成, 在初始时刻其速度均为零, 证明它们总位于初始时刻包含它们的平面上.

Proof： 类似前一题, 构造正交变换G. 令$\boldsymbol{u}$是垂直于$\boldsymbol{x}_1-\boldsymbol{x}_2,\boldsymbol{x}_1-\boldsymbol{x}_3$的向量. $\boldsymbol{v}$平行于$\boldsymbol{x}_1-\boldsymbol{x}_2$, $\boldsymbol{w} = \boldsymbol{u}\times\boldsymbol{v}$, $G = (\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})$,$G' = (-\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})$,$\boldsymbol{x} = (\ddot{\boldsymbol{x}}_1,\ddot{\boldsymbol{x}}_2,\ddot{\boldsymbol{x}}_3)$. 由书中例, F只与$\boldsymbol{x}_1-\boldsymbol{x}_2,\boldsymbol{x}_1-\boldsymbol{x}_3$有关. 则($\star$中部分不必考虑具体形式)

$$G\ddot{\boldsymbol{x}} = F(G(\boldsymbol{x}_1-\boldsymbol{x}_2,\boldsymbol{x}_1-\boldsymbol{x}_3),G(\dot{\boldsymbol{x}}_1,\dot{\boldsymbol{x}}_2,\dot{\boldsymbol{x}}_3),t) = F((0,0,0;\star),\mathbf{0},t) = G'\ddot{\boldsymbol{x}}$$

, 得$\boldsymbol{u}\ddot{\boldsymbol{x}} = \mathbf{0}$, 同上题得证.

Exercise 1.7 设一个力学系由两点组成, 证明, 对任意初始条件都存在一个惯性参考系, 使得其中这两个点总位于一个固定平面上.

Proof : 设两点初始时刻位置和速度分别为$\boldsymbol{x}_1, \boldsymbol{x}_2$和$\dot{\boldsymbol{x}}_1, \dot{\boldsymbol{x}}_2$, 考虑一个以$\dot{\boldsymbol{x}}_1$运动的参考系, 在其中$\dot{\boldsymbol{x}}_1 = \mathbf{0}$. 考虑单位向量$\boldsymbol{u}$同时垂直于$\boldsymbol{x}_1 - \boldsymbol{x}_2$和$\dot{\boldsymbol{x}}_2$. 另作单位向量$\boldsymbol{v}, \boldsymbol{w}$, 使它们成为两两正向量. 令$G = (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$, $G' = (-\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$. $\ddot{\boldsymbol{x}} = (\ddot{\boldsymbol{x}}_1, \ddot{\boldsymbol{x}}_2)$. 同时, F 只与$\boldsymbol{x}_1 - \boldsymbol{x}_2$, $\dot{\boldsymbol{x}}_1$,$\dot{\boldsymbol{x}}_2$有关. 于是

$$G\ddot{\boldsymbol{x}} = F(G(\boldsymbol{x}_1 - \boldsymbol{x}_2), G\dot{\boldsymbol{x}}_1, G\dot{\boldsymbol{x}}_2) = F((0, \star)^T, \mathbf{0}, (0, \star)) = F(G'(\boldsymbol{x}_1 - \boldsymbol{x}_2), G'\dot{\boldsymbol{x}}_1, G'\dot{\boldsymbol{x}}_2) = G'\ddot{\boldsymbol{x}}$$

于是$\ddot{\boldsymbol{x}}$和$\boldsymbol{u}$垂直, 结合$\dot{\boldsymbol{x}}_1, \dot{\boldsymbol{x}}_2$和$\boldsymbol{u}$垂直即得, 运动始终在和$\boldsymbol{u}$垂直的平面内.

Exercise 1.8 证明镜子里的力学和我们的力学是一样的

Proof : 令

$$G = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

即得.

Exercise 1.9 惯性参考系类是否唯一

Proof : 将惯性参考系类理解为, 若两个参考系为同一类, 则它们之间有一个加利略变换. 那么改变时间间隔或者时间方向的两个惯性参考系之间是没有加利略变换的. 若有, 则这个变换不保持时间间隔数t, 所以不是加利略变换.

1.3 Examples of mechanical systems

Chapter 2

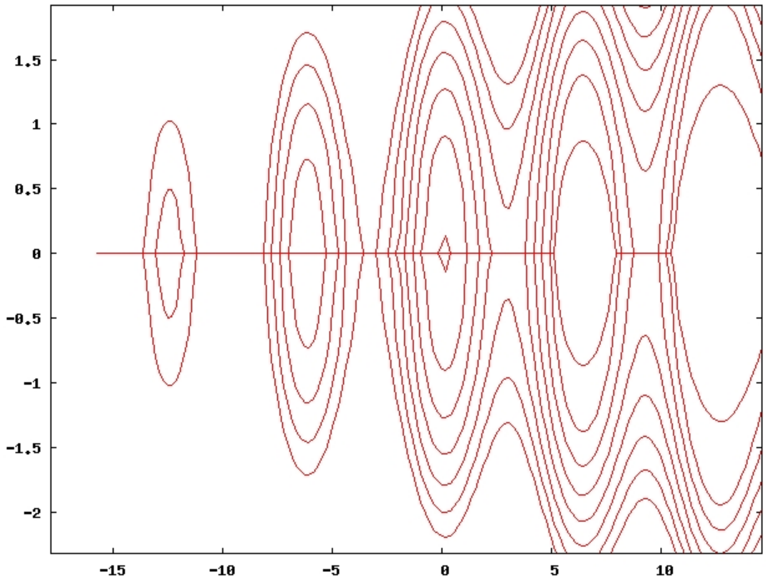
Investigation of the equations of motion

2.1 Systems with one degree of freedom

Exercise 2.1 Draw the phase curves for the “equation of a pendulum one a rotating axis”: $\ddot{\boldsymbol{x}} = -\sin \boldsymbol{x} + M$

Exercise 2.2 Find the tangent lines to the branches of the critical level corresponding to maximal potential energy $E = U(\xi)$

Solution : 由于$dx/dt = y$, $dy/dt = -\sin \boldsymbol{x} + M$. $dy/dx = (-\sin \boldsymbol{x} + M)/y$. 解常微分方程得. $y^2/2 = E + \cos \boldsymbol{x} + M\boldsymbol{x}$. 所以相图如下(M = 0.15).



Solution : By $U(x) + y^2/2 = E$, $dy/dx = -U'(x)/y$ (Since $E = U(\xi) > U(x)$,$y \neq 0$). By L'Hospital's Rule

$$\lim_{x \to \xi} dy/dx = -U''(x)/y'$$

. So $y'(\xi) = \pm\sqrt{-U''(\xi)}$. Then tangent lines are $y = \pm\sqrt{-U''(\xi)}(x - \xi)$. ps 我的做法是对$U(x) + y^2/2 = E$求两次导数。

Exercise 2.3 Let $S(E)$ be the area enclosed by the closed phase curve coresponding to the energy level E. Show that the period of mothion along this curve is equal to

$$T = \frac{dS}{dE}$$

.

Proof : 由唯一性定理, 首先知道这样的相曲线与x轴只能有两个交点设为$x_1 < x_2$ (否则如果有一个中间点, 那么相曲线必然在那边断开了,建议画草图看一下). 另外可知对每个半平面(上

下), x到y是一一对应的(否则, 有一个转折点,于是就有$\dot{\boldsymbol{x}} = 0$,这于$y = \dot{\boldsymbol{x}} \neq 0$即在上, 下半平面内矛盾). 它们与能级$E$有关. 考虑相曲线围成的面积

$$S = \int_{x_1}^{x_2} y dx + \int_{x_2}^{x_1} y dx.$$

(注意后面一项的积分方向). 结合$y = \pm\sqrt{2(E-U)}$. 所以

$$dS/dE = \int_{x_1}^{x_2} \frac{dx}{\sqrt{2(E-U)}} + \int_{x_1}^{x_2} \frac{dx}{\sqrt{2(E-U)}} + y dx/dE \Big|_{x_1}^{x_2} + y dx/dE \Big|_{x_2}^{x_1}$$

注意到y在x_1, x_2处取零, 上式第一项就是运动从x_1到x_2的时间, 后一项是x_2到x_1的时间, 和即为一个周期的运动时间. 所以$T = dS/dE$.

ps 由公式$U(x)+y^2/2 = E$可以看到相曲线是关于x轴对称的, 所以$T/2 = \int_{x_1}^{x_2} \frac{dx}{\sqrt{2(E-U)}}$由Green公式，$S/2 = \int_{x_1}^{x_2} y dx$求导即得结论。

Exercise 2.4 Let E_0 be the value of the potential function at a minimum point ξ. Find the period $T_0 = \lim_{E \to E_0} T(E)$ of small oscillations in a neighborhood of the point ξ.

Exercise 2.5 Consider a periodic motion along the closed phase curve corresponding to the energy level E. Is it stable in the sense of Liapunov?

Solution : The Definition of Liapunove stability: (By Nonlinear Ordinary Differential Equations: nonlin Ordinary Diff Equat 3e P286)

Let $\boldsymbol{x}^*(t)$ be a given real or complex solution vector of the n-dimentional system $\dot{\boldsymbol{x}} = X(\boldsymbol{x}, t)$. Then $\boldsymbol{x}^*$ is Liapunov stable for $t \geq t_0$ if and only if, to each value of $\varepsilon > 0$, however small, there corresponds a value of $\delta > 0$ (where δ may depend only on ε and t_0) such that

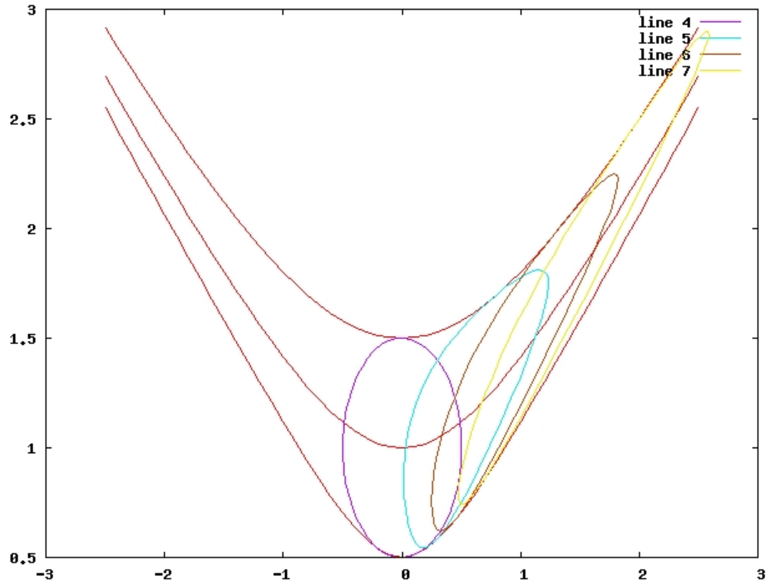
$$\|\boldsymbol{x}(t_0) - \boldsymbol{x}^*(t_0)\| < \delta \Rightarrow \|\boldsymbol{x}(t) - \boldsymbol{x}^*(t)\| < \varepsilon$$

for all $t > t_0$, where $\boldsymbol{x}(t)$ represents any other neighbouring solution.

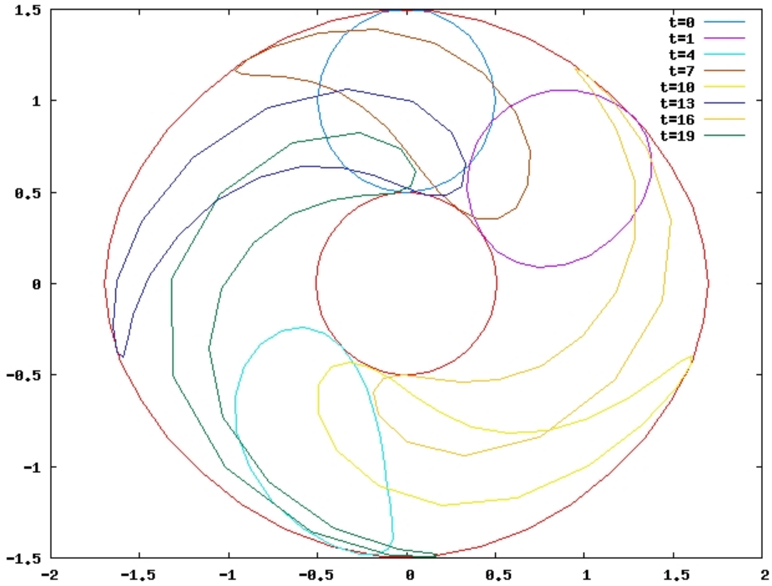
Example:

Exercise 2.6 Draw the imaage of the circle $x^2+(y-1)^2 < \frac{1}{4}$ under the action of a transformation of the phase flow for the equations (a) of the "inverse pendulum", $\ddot{x} = x$ and (b) of the "nonlinear pendulum", $\ddot{x} = -\sin x$.

Solution : 做首次积分, 由$\ddot{x} = x$ 得$dy/dx = x/y$, 所以$y^2 - x^2 = E$. 考虑一支$\dot{x} = y = \sqrt{E + x^2}$. 解得$x = \sqrt{E}\sinh(t+C)$, 于是$y = \dot{x} = \sqrt{E}\cosh(t+C)$. 其中$E = y^2 - x^2$, $C = \operatorname{arc\,sinh}(x/\sqrt{E})$. 于是得到相流$g^t(x, y) = (\sqrt{E}\sinh(t+C), \sqrt{E}\cosh(t+C))$. 如图:



(b) 同样先首次积分, 用数值解法, 算出了图如下:



2.2 Systems with two degrees of freedom

Chapter 3

Variational principles

3.1 Legendre transformations

Exercise 3.1 Legendre transformation takes convex functions to convex functions.

Proof : 由$d(px-f(x))/dx=0$得, $p=f'(x)$, 所以$dg/dp=x+pdx/dp-f'(x)dx=x$. 于是$d^2g/dp^2=dx/dp$. 另一方面由$dp/dx=f''(x)$, $dx/dp=1/f''(x)>0$ 所以g也是凸的.

Exercise 3.2 The Legendre transformation is involutive.

Proof : 考虑$g(p)$的Legendre变换. 作函数$G(x,p)=px-g(p)$. 那么只要证明$\max_p G(x,p)=f(x)$. 首先, 若取$x=x(p)$(原Legendre变换,即$f\to g$,中的最大值点), 则利用原变换中的等式, 可知$\max_p G(x,p)\geq px(p)-g(p)=f(x)$. 另一方面, 原Legendre变换中有对任意$p$, $px-f(x)\leq g(p)$, 于是有, $\max_p G(x,p)=\max_p px-g(p)\leq f(x)$. 结合以上, 得结论.

3.2 Liouville's theorem

Exercise 3.3 Prove Liouville's formula $W=W_0e^{\int \operatorname{tr} A dt}$ for the Wronskian determinant of the linear system $\dot{\boldsymbol{x}}=A(t)\boldsymbol{x}$.

Proof :

Exercise 3.4 Show that in a hamiltonian system it is impossible to have asymptotically stable equilibrum positions and asymptotically stable limit cycles in the phase space.

Proof : 若有, 对于某区域D其体积将趋于0, 这与Liouville定理矛盾.

Exercise 3.5 Consider the first digits of the numbers 2^n: $1,2,4,8,1,3,6,1,2,5,1,2,4,\ldots$

Dose the digit 7 appears in this sequence? Which digit appears more often, 7 or 8? How many times more often?

Chapter 4

Lagrangian mechanics on manifolds

4.1 Holonomic constraints

4.2 Differentiable manifolds

Exercise 4.1 Show that SO(3) is homeomorphic to three-dimensional real projective space.

4.3 E. Noether's theorem

Exercise 4.2 Suppose that a particle moves in the field of the uniform helical line $x = \cos\phi$, $y = \sin\phi$, $z = c\phi$. Find the law of conservation corresponding to this helical symetry.

Solution : 任一质点若容许在螺线上运动, 且L在其上不变于是有容许映射h, 设$\boldsymbol{x} = (\cos\phi, \sin\phi, c\phi)$. 则$h^s\boldsymbol{x} = (\cos(\phi+s), \sin(\phi+s), c(\phi+s))$. 于是

$$dh^s\boldsymbol{x}/ds\Big|_{s=0} = (-\sin\phi, \cos\phi, c) = (-y, x, c)$$

. 又$M_3 = ([\boldsymbol{x}, m\dot{\boldsymbol{x}}], \boldsymbol{e}_3) = (m\boldsymbol{x}, [\boldsymbol{x}, \boldsymbol{e}_3]) = (m(\dot{x}, \dot{y}, \dot{z}), [(x, y, z), (0, 0, 1)]) = -m\dot{x}y + m\dot{y}x$ 于是$I = m\dot{z}c + m\dot{x}(-y) + m\dot{y}x = cP_3 + M_3$守恒.

Exercise 4.3 Suppose that a rigid body is moving under its own inertia. Show that its center of mass moves linearly and uniformly. If the center of mass is at rest, then the angular momentum with repect to its is conserved

Proof :

Exercise 4.4 What quantity is conserved under the motion of a heavy rigid body if it is fixed at some point O? What if, in addition, the body is symmetric with respect to an axis passing through O?

Exercise 4.5 Extend Noether's theorem to non-autonomous lagrangian systems.

Solution : 令$M_1 = M \times \mathbb{R}$为拓广的构形空间. 定义函数$L_1 : TM_1 \to \mathbb{R}$为$L\frac{dt}{d\tau}$. 局部坐标下:

$$L_1\left(\boldsymbol{q}, t, \frac{d\boldsymbol{q}}{d\tau}, \frac{dt}{d\tau}\right) = L\left(\boldsymbol{q}, \frac{d\boldsymbol{q}/d\tau}{dt/d\tau}, t)dt\right)$$

,

Chapter 5

Differential forms

5.1 Exterior multiplication

Exercise 5.1 Find the exterior k-th power of ω^2.

Solution : $(\omega^2)^k = (\sum_{i=1}^n p_i \wedge q_i)^k = \sum_{i_1 \cdots i_k} p_{i_1} \wedge q_{i_i} \wedge \cdots \wedge p_{i_k} \wedge q_{i_k}$.

考虑其中的项$p_{i_1} \wedge q_{i_i} \wedge \cdots \wedge p_{i_k} \wedge q_{i_k}$, 若$i_1 \cdots i_k$ 中有重复, 则为零. 若不充分, 将其变为形式: $p_{i_1} \wedge \cdots \wedge p_{i_k} \wedge q_{i_1} \wedge \cdots \wedge q_{i_k}$. 这需要经过$1+2+\cdots+(k-1)$次对换, 所以其符号为$(-1)^{k(k-1)/2}$ 再将$\{i_k\}$按大小顺序排列, 此时由于p, q都做同样的置换, 所以这种置换必是偶的. 固定$\{i_k\}$考虑共有多少这样的项?这是从k个ω^2中取出i_k组成排列的个数, 一共有$k!$个. 综合以上, 得

$$(\omega^2)^k = (-1)^{k(k-1)/2} \sum_{i_1 < \cdots < i_k} p_{i_1} \wedge \cdots \wedge p_{i_k} \wedge q_{i_1} \wedge \cdots \wedge q_{i_k}$$

.

Exercise 5.2 Show that every differential 1-form on the line is the differential of some function.

Proof : $\mathbb{R}$上的1-微分形式$\omega : T\mathbb{R} \to \mathbb{R}$, 在每一点$x$处, 可定义函数$f(x) = df(x, 1)$. 显然$f$是连续的, 令$F(x) = \int_0^x f(t)dt$. 则$F$的1-微分形式为$df = f\, dx$, 与$\omega$相同.

Exercise 5.3 Find differential 1-forms on the circle and the plane which are not the differential of any function.

Solution : 定义1微分形式$\omega : TS \to \mathbb{R}$, 其中$\omega(\boldsymbol{x}, t\boldsymbol{w}) = t$ 其中$\boldsymbol{w}$为切空间中与圆周向切, 成逆时针方向的单位切矢量.

此不可能为某函数的微分, 因为圆周上连续函数必有最大值点, 在此点函数导数为零, 其1-形式也必须为零.

Exercise 5.4 Show that every closed form on a vector space in an exterior derivative.

Proof : 证明按提示, 需要证明$(k-1)$-形式$p\omega^k$存在唯一, 其值为

$$(p\omega)_x(\xi_1, \cdots, \xi_{k-1}) = \int_0^1 \omega_{t_x}(x, t\xi_1, \cdots, t\xi_{k-1})dt$$

只对单形证明即可, 设单形$c_{k-1} = \left\{\sum_{i=1}^{k-1} s_i\xi_i : 0 \le s_i \le 1, \forall i\right\}$, 则

$$pc_{k-1,\boldsymbol{x}} = \left\{t(\boldsymbol{x} + \sum_{i=1}^{k-1} s_i\xi_i) : 0 \le s_i \le 1, \forall i, 0 \le t \le 1\right\}$$

由于

$$\int_{c_{k-1}} p\omega^k = \int_{c_{k-1}} p\omega^k(\xi_1, \cdots, \xi_{k-1})ds_1 \cdots, ds_{k-1},$$

$$\int_{pc_{k-1}} \omega^k = \int_{pc_{k-1}} \omega^k(\boldsymbol{x} + \sum_{i=1}^{k-1} s_i\xi_i, t\xi_1, \cdots, t\xi_{k-1})dt\, ds_1 \cdots ds_{k-1}.$$

在$s_i=0$处求导, 于是由定义即得上式.

事实上可直接定义上式为所求$k-1$-形式.

还需证明$d\circ p+p\circ d=1$. 在$\mathbb{R}^m$上考虑k形式.

只需考虑$\omega^k=\phi(x)dx_1\wedge\cdots\wedge dx_n$. 计算$p\omega^k$中项$dx_{i_1}\wedge dx_{i_{k-1}}$ 的系数.

下面$\hat{i}$表示去掉指标i.

将$\boldsymbol{e}_i$的各组合代入得,

$$p\omega^k=\sum_i\left((-1)^{i+1}\int_0^1 t^{k-1}\phi(t\boldsymbol{x})x_i dt\right)dx_1\wedge\cdots\wedge\hat{dx_i}\wedge dx_k$$

下面计算$dp\omega$. 按运算规则. 项$dx_s\wedge dx_1\wedge\cdots\wedge\hat{dx_i}\cdots\wedge d_k$ 前的系数为. 若$s\neq 1,\cdots,k$.

$$a_{(s,1,\cdot,\hat{i},\cdots,k)}=(-1)^{i+1}\int_0^1 t^k\frac{\partial\phi}{\partial x_s}(t\boldsymbol{x})x_i dt$$

若s在$1,\cdots,k$中,设为i, 则会产生项

$$\left((-1)^{i+1}\int_0^1 t^k\frac{\partial\phi}{\partial x_i}(t\boldsymbol{x})x_i dt+(-1)^{i+1}\int_0^1 t^{k-1}\phi(t\boldsymbol{x})dt\right)dx_i\wedge dx_1\wedge\cdots\wedge\hat{dx_i}\wedge\cdots\wedge dx_k$$

再计算$d\omega^k$. 显然

$$d\omega^k=\sum_{s\neq 1,\cdots,k}\frac{\partial\phi}{\partial x_s}(\boldsymbol{x})dx_s\wedge dx_1\wedge\cdots\wedge d_k$$

.

考虑$pd\omega$中项$dx_s\wedge dx_1\wedge\cdots\wedge\hat{dx_i}\wedge\cdots\wedge d_k)$ 的系数若$s\neq 1,\cdots,k$.

则系数为

$$(-1)^{i+2}\int_0^1 t^k\frac{\partial\phi}{\partial x_s}(t\boldsymbol{x})x_i dt$$

而$dx_1\wedge\cdots\wedge d_k$ 的系数为

$$\sum_{s\neq 1,\cdots,k}\int_0^1 t^k\frac{\partial\phi}{\partial x_s}(t\boldsymbol{x})x_s dt$$

合之得

$$\begin{aligned}
dp\omega^k+pd\omega^k&=\left(\int_0^1 kt^{k-1}\phi(t\boldsymbol{x})dt+\sum_{s=1,\cdots,m}\int_0^1 t^k\phi(t\boldsymbol{x})x_s dt\right)dx_1\wedge\cdots\wedge dx_k\\
&=\left(\int_0^1 kt^{k-1}\phi(t\boldsymbol{x})dt+\int_0^1 t^k d\phi(t\boldsymbol{x})\right)dx_1\wedge\cdots\wedge dx_k\\
&=\left(\int_0^1 kt^{k-1}\phi(t\boldsymbol{x})dt+t^k\phi(t\boldsymbol{x})\Big|_0^1-\int_0^1 kt^{k-1}\phi(t\boldsymbol{x})dt\right)dx_1\wedge\cdots\wedge dx_k\\
&=\phi(\boldsymbol{x})dx_1\wedge\cdots\wedge dx_k
\end{aligned}\tag{5.1}$$

得证.

Exercise 5.5 Prove the homotopy formula $i_Xd+di_X=L_x$.

Proof :