S.-T. Yau College Student Mathematics Contests 2010

Analysis and Differential Equations

Individual

(Please select 5 problems to solve)

1. (*a*) Let $x_k, k = 1, 2. \cdots, n$ be real numbers from the interval $(0, \pi)$ and define $x = \frac{1}{n}\sum_{i=1}^{n} x_i$. Show that

$$\prod_{k=1}^{n} \frac{\sin x_k}{x_k} \leqslant \left(\frac{\sin x}{x}\right)^n.$$

(*b*) From

$$\int_0^\infty e^{-x^2} \mathrm{d}x = \frac{\sqrt{\pi}}{2},$$

calculate the integral $\int_0^\infty \sin x^2 \mathrm{d}x$.

2. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be any function. Prove that the set of points x in $\mathbb{R}$ where f is continuous is a countable intersection of open sets.

3. Consider the equation $\dot{x} = -x + f(t, x)$, where $|f(t, x)| \leqslant \phi(t)|x|$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}$, $\int^\infty \phi(t)\mathrm{d}t < \infty$. Prove that every solution approaches zero as $t \to \infty$.

4. Find a harmonic function f on the right half-plane such that when approaching any point in the positive half of the y-axis, the function has limit 1, while when approaching any point in the negative half of the y-axis, the function has limit -1.

5. Let $K(x, y) \in C([0, 1] \times [0, 1])$. For all $f \in C[0, 1]$, the space of continuous functions on $[0, 1]$, define a function

$$Tf(x) = \int_0^1 K(x, y)f(y)\mathrm{d}y$$

Prove that $Tf \in C[0, 1]$. Moreover $\Omega = \{Tf| \, ||f||_{\sup} \leqslant 1\}$ is precompact in $C[0, 1]$, i.e every sequence in Ω has a converging subsequence, here $||f||_{\sup} = \sup\{|f(x)||x \in [0, 1]\}$.

6. Prove the Poisson summation formula:

$$\sum_{n=-\infty}^{\infty} f(x+2\pi n) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \hat{f}(k)\mathrm{e}^{\mathrm{i}kx}$$

for all f of functions over $\mathcal{R}$ in the Schwartz space:

$$\mathcal{S} = \{f : (1+x^2)^m |f^{(n)}(x)| \leqslant C_{m,n}, m, n \geqslant 0\}$$

where $\hat{f}(\xi) = \int_{\mathcal{R}} f(x)\mathrm{e}^{-\mathrm{i}x\xi}\mathrm{d}x$.

Geometry and Topology

Individual

(Please select 5 problems to solve)

1. Let $D^* = \{(x,y) \in \mathbb{R}^2 | 0 < x^2+y^2 < 1\}$ be the punctured unit disc in the Euclidean plane. Let g be the complete Riemannian metric on D^* with constant curvature -1. Find the distance under the metric between the points $(\mathrm{e}^{-2\pi}, 0)$ and $(-\mathrm{e}^{-\pi}, 0)$.

2. Show that every closed hypersurface in $\mathbb{R}^n$ has a point at which the second fundamental form is positive definite.

3. Prove that the real projective space $\mathbb{R}P^n$ is orientable if and only if n is odd.

4. Suppose $\pi : M_1 \longrightarrow M_2$ is a C^∞ map of one connected differentiable manifold to another. And suppose for each $p \in M_1$, the differential $\pi_* : T_pM_1 \longrightarrow T_{\pi(p)}M_2$ is a vector space isomorphism.
(*a*) Show that if M_1 is compact, then π is a covering space projection.
(*b*) Given an example where M_2 is compact but $\pi : M_1 \longrightarrow M_2$ is not a covering space (but has the π_* isomorphism property).

5. Let $\sum_g$ be the closed orientable surface of genus g. Show that if $g > 1$, then $\sum_g$ is a covering space of $\sum_2$.

6. Let M be a smooth 4-dimensional manifold. A symplectic form is a closed 2-form ω on M such that $\omega \wedge \omega$ is a nowhere vanishing 4-form.
(*a*) Construct a symplectic form on $\mathbb{R}^4$.
(*b*) Show that there are no symplectic forms on S^4.

Algebra, Number Theory and Combinatorics

Individual

(Please select 5 problems to solve)

1. Let V be a finite dimensional complex vector space. Let A, B be two linear endomorphism of V satisfying $AB - BA = B$. Prove that there is a common eigenvector for A and B.

2. Let $M_2(\mathbb{R})$ be the ring of 2×2 matrices with real entries. Its group of multiplicative units is $GL_2(\mathbb{R})$, consisting of invertible matrices in $M_2(\mathbb{R})$.
(*a*) Find an injective homomorphism from the field $\mathbb{C}$ of complex numbers into the ring $M_2(\mathbb{R})$.
(*b*) Show that if ϕ_1 and ϕ_2 are two such homomorphisms, then there exists a $g \in GL_2(\mathbb{R})$ such that $\phi_2(x) = g\phi_1(x)g^{-1}$ for all $x \in \mathbb{C}$.
(*c*) Let h be an element in $GL_2(\mathbb{R})$ whose characteristic polynomial $f(x)$ is irreducible over $\mathbb{R}$. Let $F \subset M_2(\mathbb{R})$ be the subring generated by h and $a \cdot I$ for all $a \in \mathbb{R}$, where I is the idenity matrix. Show that F is isomorphic to $\mathbb{C}$.
(*d*) Let h' be any element in $GL_2(\mathbb{R})$ with the same characteristic polynomial $f(x)$ as h in (*c*). Show that h and h' are conjugate in $GL_2(\mathbb{R})$.
(*e*) If $f(x)$ in (*c*) and (*d*) is reducible over $\mathbb{R}$, will the same conclusion on h and h' hold? Give reasons.

3. Let G be a non-abelian finite group. Let $c(G)$ be the number of conjugacy classes in G. Define $\bar{c}(G) := c(G)/|G|,\ (|G| = \text{Card}(G))$.
(*a*) Prove that $\bar{c}(G) \leqslant \dfrac{5}{8}$.
(*b*) Is there a finite group H with $\bar{c}(H) = \dfrac{5}{8}$.
(*c*) (open ended question) Suppose that there exists a prime number p and an element $x \in G$ such that the cardinality of the conjugacy class of x is divisible by p. Find a good/sharp upper bound for $\bar{c}(G)$.

4. Let F be a splitting field over $\mathbb{Q}$ the polynomial $x^8 - 5 \in \mathbb{Q}[x]$. Recall that F is the subfield of $\mathbb{C}$ generated by all roots of this polynomial.
(*a*) Find the degree $[F : \mathbb{Q}]$ of the number field F.
(*b*) Determine the Galois group Gal($F/\mathbb{Q}$).

5. Let $T \subset N_{>0}$ be an finite set of positive integers. For each integer $n > 0$, define a_n to be the number of all finite sequences $(t_1, \cdots, t_m)$ with $m \leqslant n, t_i \in T$ for all $i = 1, \cdots, m$ and $t_1 + \cdots + t_m = n$. Prove that the infinite series

$$1 + \sum_{n \geqslant 1} a_n z^n \in \mathbb{C}[[z]]$$

is a *rational* function in z, and find this rational function.

6. Describe all the irreducible complex representations of the group S_4 (the symmetric group on four letters).