1. 求 $\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\cdots\sin\frac{(n-1)\pi}{n}$ 的值

解：设 $\varepsilon = e^{\frac{\pi}{n}i}$，则 $1, \varepsilon, \varepsilon^2, \cdots \varepsilon^{2(n-1)}$ 为 $x^{2n}-1=0$ 的根.

$$\sin\frac{k\pi}{n} = \frac{\varepsilon^k - \varepsilon^{-k}}{2i} = \frac{\varepsilon^{2k}-1}{2i\varepsilon^k}$$

$$\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\cdots\sin\frac{(n-1)\pi}{n} = \frac{(\varepsilon^2-1)(\varepsilon^4-1)\cdots(\varepsilon^{2(n-1)}-1)}{2^{n-1}\, i^{n-1}\, \varepsilon^{\frac{1}{2}n(n-1)}}$$

$$= \frac{(-1)^{n-1}(1-\varepsilon^2)(1-\varepsilon^4)\cdots(1-\varepsilon^{2(n-1)})}{2^{n-1}(i)^{2(n-1)}} = \frac{(1-\varepsilon^2)(1-\varepsilon^4)\cdots(1-\varepsilon^{2(n-1)})}{2^{n-1}}$$

而 $(x^2-\varepsilon^2)(x^2-\varepsilon^4)\cdots(x^2-\varepsilon^{2(n-1)}) = x^{2(n-1)}+\cdots+x^2+1$.

$\therefore (1-\varepsilon^2)(1-\varepsilon^4)\cdots(1-\varepsilon^{2(n-1)}) = n$

$\therefore \sin\frac{\pi}{n}\sin\frac{2\pi}{n}\cdots\sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}$

2. (a) 设 $n = a_k p^k + a_{k-1}p^{k-1} + \cdots + a_0$. $a_i \in \{0, 1, \cdots, p-1\}$.

则 $\operatorname{ord}_p(n!) = \sum_{i=1}^{\infty}\left[\frac{n}{p^i}\right]$

$$= a_k p^{k-1} + a_{k-1}p^{k-2} + \cdots + a_2 p + a_1$$
$$+ a_k p^{k-2} + a_{k-1}p^{k-3} + \cdots + a_2$$
$$+ \cdots$$
$$+ a_k$$

$$= a_k\frac{p^k-1}{p-1} + a_{k-1}\frac{a^{k-1}-1}{p-1} + \cdots + a_2\frac{p^2-1}{p-1} + a_1\frac{p-1}{p-1}$$

$$= \frac{(a_k p^k + a_{k-1}p^{k-1} + \cdots + a_1 + a_0) - (a_k + a_{k-1} + \cdots + a_0)}{(p-1)}$$

$$= \frac{n - S_p(n)}{p-1}$$

$S_p(mn) = b_k + \cancel{b_{k-1}} + \cdots + b_1 + b_0$

$\because b_0 \geqslant 1$

$\therefore S_p(mn+n) = (a_k + b_k) + \cdots + (a_\alpha + b_{\alpha-1}) - \alpha(p-1) - t(p-1)$

$(a_k + b_k, a_{k-1} + b_{k-1}, \cdots, a_\alpha + b_\alpha$ 共有 t 次进位

显然 $\dfrac{(n(m+1))!}{(mn)!\,(n+1)!} = C_{mn}^{n} / n+1$.

$\therefore \mathrm{Ord}_p(C_{mn+n}^{n}) = \mathrm{Ord}_p((n(m+1))!) - \mathrm{Ord}_p(n!) - \mathrm{Ord}_p((mn)!)$

$$= \frac{S_p(mn) + S_p(n) - S_p(mn+n)}{p-1}$$

$$= \alpha + t \geqslant \alpha$$

$\therefore p^\alpha \mid C_{mn+n}^{n}$ 即 $\dfrac{(n(m+1))!}{(mn)!\,(n+1)!} \in N^*$.

3、(a) 易知：$(x+y)^{n+1} = \sum_{k=0}^{n+1} a_{n+1,k} x^{n+1-k} y^k$

$$= \sum_{k=0}^{n} a_{n,k}\, t x^{n-k} y^k (x + ty)$$

$\therefore a_{n+1,k} = a_{n,k-1} + p a_{n,k}$. 且 $a_{k,0} = a_{k,k} = 1$.

$\therefore a_{n,k}$ 是 p 的多项式.

(b) $a_{n+1,k} = a_{n,k-1} + p a_{n,k}$

$= C_1^0 (a_{n-1,k-2} + p a_{n-1,k-1}) + C_1^1 (a_{n-1,k-1} + p a_{n-1,k}) \cdot p$

$= C_2^0 a_{n-1,k-2},\ C_2^1 p a_{n-1,k-1} + C_2^2 p^2 a_{n-1,k}$

$= \cdots\cdots$

$= C_k^0 a_{n-k+1,0} + C_k^1 p a_{n-k+1,2} + C_k^2 p^2 a_{n-k+1,3} + \cdots + C_k^{k-1} p^{k-1} a_{n-k+1,k}$

(b). 设 $p^\alpha \| n+1$. (p为$n+1$的任一素因子)

即: $n+1 = a_k p^k + a_{k-1}p^{k-1} + \cdots + a_\alpha p^\alpha$, $0\le a_i \le p-1$ 且 $1\le a_\alpha \le p-1$

则 $n = a_k p^k + \cdots + (a_\alpha - 1)p^\alpha + (p-1)p^{\alpha-1} + (p-1)p^{\alpha-2} + \cdots + (p-1)$

$2n = 2a_k p^k + \cdots + 2(a_\alpha-1)p^\alpha + 2(p-1)p^{\alpha-1}+\cdots+2(p-1)$.

显然 $\frac{(2n)!}{n!(n+1)!} = C_{2n}^n/_{n+1}$. $C_{2n}^n \in N^*$.

$S_p(n) = a_k + a_{k-1} + \cdots + (a_\alpha-1) + \alpha(p-1)$

$S_p(2n) = 2(a_k+a_{k-1}+\cdots+(a_\alpha-1)) - \alpha(p-1) - t(p-1)$ ($t\ge 0$)

($2a_k, 2a_{k-1}, \ldots, 2a_{\alpha+1}$中有$t$次进位)

$$Ord_p(C_{2n}^n) = Ord_p((2n)!) - 2Ord_p(n!)$$

$$= \frac{2S_p(n) - S_p(2n)}{p-1} = \frac{\alpha(p-1)+t(p-1)}{p-1}$$

$$= \alpha + t \ge \alpha. \quad \therefore p^\alpha | C_{2n}^n.$$

由 $p^\alpha \| n+1$

即 $\frac{(2n)!}{n!(n+1)!} \in N^*$

(c). 由题知 若 p为$n+1$的素因子, 则 $(p,m)=1, (p,n)=1$.

设 $n+1 = a_k p^k + a_{k-1}p^{k-1} + \cdots + a_\alpha p^\alpha$ (其中 $a_\alpha \ge 1$).

$mn = b_k p^k + b_{k-1}p^{k-1} + \cdots + b_0$ ($b_0\ge 1$).

则 $n = a_k p^k + a_{k-1}p^{k-1} + \cdots + (a_\alpha-1)p^\alpha + (p-1)p^{\alpha-1}+\cdots+(p-1)$

$mn+n = (a_k+b_k)p^\alpha + (a_{k-1}+b_{k-1})p^{k-1} + \cdots + (a_\alpha+b_\alpha-1)p^\alpha$
$+ (b_{\alpha-1}+p-1)p^{\alpha-1} + \cdots + (b_0+p-1)$.

$S_p(n) = a_k + a_{k-1} + \cdots + (a_\alpha-1) + \alpha(p-1)$

$$= C_k^0 + (C_k^0 + C_k^1) p\, a_{n-k,1} + (C_k^1 + C_k^2) p^2 a_{n-k,2} + \cdots + C_k^k p^{k+1} a_{n-k,k}$$

$$= C_k^0 + C_{k+1}^1 p\, a_{n-k,1} + C_{k+1}^2 p^2 a_{n-k,2} + \cdots + C_{k+1}^{k+1} p^{k+1} a_{n-k,k}$$

$$= \cdots$$

$$= C_k^0 + C_{k+1}^1 p + C_{k+2}^2 p^2 + \cdots + C_n^{n-k+1} p^{n-k+1}$$

$$\therefore a_{n,k} = 1 + C_k^1 p + C_{k+1}^2 p^2 + \cdots + C_{n-1}^{n-k} p^{n-k}$$

4、设 $\varepsilon_n = e^{\frac{2\pi}{n}}$，$\therefore 1, \varepsilon_n, \varepsilon_n^2, \cdots, \varepsilon_n^{n-1}$ 为 $x^n=1$ 的 n 个根、由根与系数的关系知：

$$\sum_{i=1}^{n} \prod \varepsilon_n^{k_1} \cdots \varepsilon_n^{k_i} = 0 \quad (1 \le i < n).$$

$$x^n - 1 = (x-1)(x-\varepsilon_n)\cdots(x-\varepsilon_n^{n-1})$$

$$\therefore \Rightarrow 1 - x^n = (1-x)(1-\varepsilon_n x)\cdots(1-\varepsilon_n^{n-1} x)$$

$$\therefore \sum_{k=0}^{n-1} \frac{1}{1-\varepsilon_n^k t} = \frac{\sum_{i=0}^{n-1} \prod_{k \ne i} (1-\varepsilon_n^k t)}{\prod_{k=0}^{n-1} (1-\varepsilon_n^k t)} = \frac{n}{1-t^n}.$$

$$\text{而} \sum_{k=1}^{n-1} \frac{1}{1-\varepsilon_n^k t} = \frac{n}{1-t^n} - \frac{1}{1-t}$$

$$= \frac{n(1-t) - (1-t^n)}{(1-t)(1-t^n)}$$

$$= \frac{n - (1 + t + \cdots t^{n-1})}{1-t^n}$$

$$= \frac{(1-t) + (1-t^2) + \cdots + (1-t^{n-1})}{1-t}$$

$$=\frac{1+(1+t)+(1+t+t^2)+\cdots+(1+t+\cdots+t^{n-2})}{1+t+t^2+\cdots+t^{n-1}}$$

当$t=1$时

$$\sum_{k=1}^{n-1}\frac{1}{1-\varepsilon_n^k}=\frac{1+2+\cdots+(n-1)}{n}=\frac{\frac{1}{2}n(n-1)}{n}=\frac{n-1}{2}$$

$$\sum_{k=1}^{n-1}\frac{1}{(1-\varepsilon_n^k)(1-\varepsilon_n^{-k})}=\sum_{k=1}^{n-1}\frac{1}{(1-\cos\frac{2k\pi}{n}-i\sin\frac{2k\pi}{n})(1-\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n})}$$

$$=\sum_{k=1}^{n-1}\frac{1}{(1-\cos\frac{2k\pi}{n})^2+\sin^2\frac{2k\pi}{n}}$$

$$=\sum_{k=1}^{n-1}\frac{1}{2-2\cos\frac{2k\pi}{n}}$$

$$=\sum_{k=1}^{n-1}\frac{1}{4\sin^2\frac{k\pi}{n}}$$

$$=\frac{n-1}{4}+\frac{1}{4}\sum_{k=1}^{n-1}\cot^2\frac{k\pi}{n}$$

又 $(\cos\frac{k\pi}{2n}+i\sin\frac{k\pi}{2n})^{2n}=1=\sum_{t=0}^{2n}C_{2n}^{t}(\cos\frac{k\pi}{2n})^{2n-t}(i\sin\frac{k\pi}{2n})^{t}$

其中 $\sum_{t=1}^{n}C_{2n}^{2t-1}(\cos\frac{k\pi}{2n})^{2(n-t)+1}(i\sin\frac{k\pi}{2n})^{2t-1}=0$ 0

而 $\sum_{t=1}^{n+1}C_{2n}^{2t-1}\cos^{2(n-t)}\frac{k\pi}{2n}(-1)^{t}\sin^{2t}\frac{k}{2n}=0$

又∵ $\left(\cos\frac{k\pi}{n}+i\sin\frac{k\pi}{n}\right)^{k}=(-1)^{k}=\sum_{t=0}^{n}C_n^t\cos^{n-t}\frac{k\pi}{n}\left(i\sin\frac{k\pi}{n}\right)^{t}$

当 $t=2m$ 时

$$\sum_{t=1}^{m}C_{2m}^{2t-1}\cos^{2m-2t+1}\frac{2k\pi}{n}\left(i\sin\frac{2k\pi}{n}\right)^{2t-1}=0$$

$$\sum_{t=1}^{m}C_{2m}^{2t-1}\left(\cot\frac{2k\pi}{n}\right)^{2(m-t)}(-1)^{t}=0$$

∴ $\cot^2\frac{2\pi}{n}$，$\cot^2\frac{2\cdot 2\pi}{n}$，$\cdots$，$\cot^2\frac{(m-1)\pi}{n}$ 为上式 $m-1$ 个根

$$\cot^2\frac{\pi}{n}+\cot^2\frac{2\pi}{n}+\cdots+\cot^2\frac{m-1}{n}\pi=\frac{C_{2m}^3}{C_{2m}^1}=\frac{1}{3}(m-1)(2m-1)$$

$$\cot^2\frac{\pi}{n}+\cot^2\frac{2\pi}{n}+\cdots+\cot^2\frac{m-1}{n}\pi=\frac{1}{3}(2m-2)(2m-1)$$

$$=\frac{1}{3}(n-2)(n-1)$$

当 $t=2m+1$ 时

类似可得，$\cot^2\frac{\pi}{n}+\cot^2\frac{2\pi}{n}+\cdots+\cot^2\frac{m}{n}\pi$

$$=2\,\frac{C_{2m+1}^3}{C_{2m+1}^1}=\frac{2m(2m-1)}{3}=\frac{(n-2)(n-1)}{3}$$

原式 $=\frac{n-1}{4}+\frac{1}{4}\left(\frac{(n-2)(n-1)}{3}+\frac{1+(-1)^{n+1}}{2}\right)$

$$=\frac{1}{12}(n^2-1)+\frac{1+(-1)^{n+1}}{8}$$