Multidimensional Digital Signal Processing

Or plain: Image processing for dummies.

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1 MD signals and systems

1.1 Periodic sequences

Let $\mathbf{n}, \mathbf{r} \in \mathbb{N}^k$ and $\mathbf{N}, \mathbf{P} \in \mathbb{N}^{k \times k}$. $x(\mathbf{n})$ is peridioc if

$$x(\mathbf{n} + \mathbf{N}\mathbf{r}) = x(\mathbf{n}). \tag{1}$$

The number of samples in one period is

$$|\det(\mathbf{N})|, \tag{2}$$

and $\mathbf{NP}$ is also a valid periodicity matrix.

1.2 Linear shift-invariant systems

Let $a, b \in \mathbb{C}$ and let $v(\mathbf{n}) = au_1(\mathbf{n}) + bu_2(\mathbf{n})$ and $w(\mathbf{n}) = u(\mathbf{n} - \mathbf{m})$. A system $\mathcal{L}$ is linear and shift-invariant (LSI) if

1. $\mathcal{L}[v](\mathbf{n}) = a\mathcal{L}[u_1](\mathbf{n}) + b\mathcal{L}[u_2](\mathbf{n})$,
2. $\mathcal{L}[w](\mathbf{n}) = \mathcal{L}[u](\mathbf{n} - \mathbf{m})$.

If $\mathcal{L}$ is linear and $y(n_1, n_2) := L[x](n_1, n_2)$ then

$$y(n_1, n_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x(k_1, k_2) \underbrace{\mathcal{L}[\delta](n_1 - k_1, n_2 - k_2)}_{h_{k_1 k_2}(n_1, n_2)}. \tag{3}$$

If $\mathcal{L}$ is also shift-invariant then

$$h := h_{00}(n_1 - k_1, n_2 - k_2) = h_{k_1 k_2}(n_1, n_2). \tag{4}$$

1.3 2D convolution

1. $x ** h = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x(k_1, k_2) h(n_1 - k_1, n_2 - k_2)$
2. $x ** h = h ** x$
3. $(x ** h) ** g = x ** (h ** g)$
4. $x ** (h + g) = (x ** h) + (x ** g)$

1.4 Separable systems

If $h(n_1, n_2) = h_1(n_1) h_2(n_2)$ then

$$y(n_1, n_2) = h_1 * (h_2 * x). \tag{5}$$

That is

1. $g(n_1, n_2) = h_2 * x = \sum_{k_2=-\infty}^{\infty} h_2(k_2) x(n_1, n_2 - k_2)$,
2. $y(n_1, n_2) = h_1 * g = \sum_{k_1=-\infty}^{\infty} h_1(k_1) g(n_1 - k_1, n_2)$.

1.5 BIBO stability

A system is BIBO stable if

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} |h(n_1, n_2)| = S_1 < \infty \tag{6}$$

1.6 Fourier transform

The Fourier transform is defined as

$$X(\omega_1, \omega_2) = \sum_{n_1} \sum_{n_2} x(n_1, n_2) \exp(-j\omega_1 n_1 - j\omega_2 n_2). \tag{7}$$

X is periodic with $\mathbf{N} = \begin{bmatrix} 2\pi & 0 \\ 0 & 2\pi \end{bmatrix}$. Its inverse transform is

$$x(n_1, n_2) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(\omega_1, \omega_2) \exp(j\omega_1 n_1 + j\omega_2 n_2) d\omega_1 d\omega_2. \tag{8}$$

Some nice properties:

1. $a, b \in \mathbb{C} : ax_1 + by_1 \circ\!\!-\!\!\bullet\ aX_1 + bX_2$
2. $x(n_1 - k_1, n_2 - k_2) \circ\!\!-\!\!\bullet\ \exp(-j\omega_1 k_1 - j\omega_2 k_2) X(\omega_1, \omega_2)$
3. $\exp(j\theta_1 n_1 + j\theta_2 n_2) x(n_1, n_2) \circ\!\!-\!\!\bullet\ X(\omega_1 - \theta_1, \omega_2 - \theta_2)$
4. $h ** x \circ\!\!-\!\!\bullet\ HX$

2 Discrete fourier transform

2.1 Discrete fourier series

Let x be periodic with with $\mathbf{N} = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}$. Them its 2D fourier series is

$$x(n_1, n_2) = \frac{1}{N_1 N_2} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} X(k_1, k_2) \exp(j \frac{2\pi}{N_1} n_1 k_1 + j \frac{2\pi}{N_2} n_2 k_2), \tag{9}$$

with

$$X(k_1, k_2) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x(n_1, n_2) \exp(-j \frac{2\pi}{N_1} n_1 k_1 - j \frac{2\pi}{N_2} n_2 k_2). \tag{10}$$

It also holds that

$$X(k_1, k_2) = X(\omega_1, \omega_2)\Big|_{\omega_1 = 2\pi \frac{k_1}{N_1}, \omega_2 = 2\pi \frac{k_2}{N_2}} \tag{11}$$

if $X(\omega_1, \omega_2)$ is the FT of one period of x.

3 Linear block transforms

3.1 Separable unitary block transforms

The forward transform is

$$\mathbf{Y} = \mathbf{A}\mathbf{X}\mathbf{A}^T \tag{12}$$

and the backward transform is

$$\mathbf{X} = \mathbf{A}^H \mathbf{Y} \mathbf{A}^* \tag{13}$$

Let $\mathbf{A} = (\mathbf{a}_1, \ldots, \mathbf{a}_N)$. Then, $\mathbf{X}$ can be written as a linear combination of basis images.

$$\mathbf{X} = \sum_{k=1}^{N} \sum_{l=1}^{N} y_{kl} \mathbf{a}_k \mathbf{a}_l^T \tag{14}$$

3.2 Separable block transforms

The forward transform is

$$\mathbf{Y} = \mathbf{A}_1 \mathbf{X} \mathbf{A}_2^T. \tag{15}$$

It can be also written as

$$\mathbf{y} = \mathbf{A}_{2D} \mathbf{x} \tag{16}$$

if is $\mathbf{x}$ is extraced row-wise from $\mathbf{X}$ and

$$\mathbf{A}_{2D} = \mathbf{A}_1 \otimes \mathbf{A}_2 = \begin{bmatrix} (\mathbf{A}_1)_{11} \mathbf{A}_2 & \ldots & (\mathbf{A}_1)_{1N} \mathbf{A}_2 \\ \vdots & \ddots & \vdots \\ (\mathbf{A}_1)_{N1} \mathbf{A}_2 & \ldots & (\mathbf{A}_1)_{NN} \mathbf{A}_2 \end{bmatrix}. \tag{17}$$

3.3 Haar transform

$$\mathbf{H}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{18}$$

$$\mathbf{H}_{2^{k+1}} = \begin{bmatrix} \mathbf{H}_{2^k} \otimes \begin{bmatrix} 1 & 1 \end{bmatrix} \\ 2^{\frac{k}{2}} \mathbf{I}_{2^k} \otimes \begin{bmatrix} 1 & -1 \end{bmatrix} \end{bmatrix} \tag{19}$$

$$\mathbf{A}_{2^k}^{Haar} = \frac{1}{\sqrt{2^k}} H_{2^k} \tag{20}$$

3.4 Hadamard transform

$$\mathbf{A}_2^{Had} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{21}$$

$$\mathbf{A}_{2^k}^{Had} = \bigotimes_{l=1}^{k} \mathbf{A}_2^{Had} \tag{22}$$

3.5 Unitary DFT

$$\mathbf{A}_M^{DFT} = \frac{1}{\sqrt{M}} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & e^{-j\frac{2\pi}{M}} & e^{-j\frac{4\pi}{M}} & \dots & e^{-j\frac{2\pi(M-1)}{M}} \\ 1 & e^{-j\frac{4\pi}{M}} & e^{-j\frac{8\pi}{M}} & \dots & e^{-j\frac{4\pi(M-1)}{M}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-j\frac{2\pi(M-1)}{M}} & e^{-j\frac{4\pi(M-1)}{M}} & \dots & e^{-j\frac{2\pi(M-1)^2}{M}} \end{bmatrix} \tag{23}$$

3.6 Unitary DCT

$$(\mathbf{A}_M^{DCT})_{kn} = \begin{cases} \frac{1}{\sqrt{M}} & k = 0 \\ \sqrt{\frac{2}{M}} \cos \frac{\pi(2n+1)k}{2M} & k \neq 0 \end{cases} \tag{24}$$

$$\mathbf{A}_M^{DCT} = \sqrt{\frac{2}{M}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \dots & \frac{1}{\sqrt{2}} \\ \cos \frac{\pi}{2M} & \cos \frac{3\pi}{2M} & \dots & \cos \frac{(2M-1)\pi}{2M} \\ \cos \frac{2\pi}{2M} & \cos \frac{6\pi}{2M} & \dots & \cos \frac{2(2M-1)\pi}{2M} \\ \vdots & \vdots & \ddots & \vdots \\ \cos \frac{(M-1)\pi}{2M} & \cos \frac{(M-1)3\pi}{2M} & \dots & \cos \frac{(M-1)(2M-1)\pi}{2M} \end{bmatrix} \tag{25}$$