

Multidimensional Digital Signal Processing

Or plain: Image processing for dummies.
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1 MD signals and systems

1.1 Periodic sequences

Let $\mathbf{n}, \mathbf{r} \in \mathbb{N}^k$ and $\mathbf{N}, \mathbf{P} \in \mathbb{N}^{k \times k}$. $x(\mathbf{n})$ is periodic if

$$x(\mathbf{n} + \mathbf{N}\mathbf{r}) = x(\mathbf{n}). \quad (1)$$

The number of samples in one period is

$$|\det(\mathbf{N})|, \quad (2)$$

and \mathbf{NP} is also a valid periodicity matrix.

1.2 Linear shift-invariant systems

Let $a, b \in \mathbb{C}$ and let $v(\mathbf{n}) = au_1(\mathbf{n}) + bu_2(\mathbf{n})$ and $w(\mathbf{n}) = u(\mathbf{n} - \mathbf{m})$. A system \mathcal{L} is linear and shift-invariant (LSI) if

1. $\mathcal{L}[v](\mathbf{n}) = a\mathcal{L}[u_1](\mathbf{n}) + b\mathcal{L}[u_2](\mathbf{n})$,
2. $\mathcal{L}[w](\mathbf{n}) = \mathcal{L}[u](\mathbf{n} - \mathbf{m})$.

If \mathcal{L} is linear and $y(n_1, n_2) := \mathcal{L}[x](n_1, n_2)$ then

$$y(n_1, n_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x(k_1, k_2) \underbrace{\mathcal{L}[\delta](n_1 - k_1, n_2 - k_2)}_{h_{k_1 k_2}(n_1, n_2)}. \quad (3)$$

If \mathcal{L} is also shift-invariant then

$$h := h_{00}(n_1 - k_1, n_2 - k_2) = h_{k_1 k_2}(n_1, n_2). \quad (4)$$

1.3 2D convolution

1. $x ** h = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x(k_1, k_2)h(n_1 - k_1, n_2 - k_2)$
2. $x ** h = h ** x$
3. $(x ** h) ** g = x ** (h ** g)$
4. $x ** (h + g) = (x ** h) + (x ** g)$

1.4 Separable systems

If $h(n_1, n_2) = h_1(n_1)h_2(n_2)$ then

$$y(n_1, n_2) = h_1 * (h_2 * x). \quad (5)$$

That is

1. $g(n_1, n_2) = h_2 * x = \sum_{k_2=-\infty}^{\infty} h_2(k_2)x(n_1, n_2 - k_2)$,
2. $y(n_1, n_2) = h_1 * g = \sum_{k_1=-\infty}^{\infty} h_1(k_1)g(n_1 - k_1, n_2)$.

1.5 BIBO stability

A system is BIBO stable if

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} |h(n_1, n_2)| = S_1 < \infty \quad (6)$$

1.6 Fourier transform

The Fourier transform is defined as

$$X(\omega_1, \omega_2) = \sum_{n_1} \sum_{n_2} x(n_1, n_2) \exp(-j\omega_1 n_1 - j\omega_2 n_2). \quad (7)$$

X is periodic with $\mathbf{N} = \begin{bmatrix} 2\pi & 0 \\ 0 & 2\pi \end{bmatrix}$. Its inverse transform is

$$x(n_1, n_2) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(\omega_1, \omega_2) \exp(j\omega_1 n_1 + j\omega_2 n_2) d\omega_1 d\omega_2. \quad (8)$$

Some nice properties:

1. $a, b \in \mathbb{C} : ax_1 + by_1 \circ \bullet aX_1 + bX_2$
2. $x(n_1 - k_1, n_2 - k_2) \circ \bullet \exp(-j\omega_1 k_1 - j\omega_2 k_2)X(\omega_1, \omega_2)$
3. $\exp(j\theta_1 n_1 + j\theta_2 n_2)x(n_1, n_2) \circ \bullet X(\omega_1 - \theta_1, \omega_2 - \theta_2)$
4. $h ** x \circ \bullet HX$

2 Discrete fourier transform

2.1 Discrete fourier series

Let x be periodic with $\mathbf{N} = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}$. Then its 2D fourier series is

$$x(n_1, n_2) = \frac{1}{N_1 N_2} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} X(k_1, k_2) \exp(j\frac{2\pi}{N_1} n_1 k_1 + j\frac{2\pi}{N_2} n_2 k_2), \quad (9)$$

with

$$X(k_1, k_2) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x(n_1, n_2) \exp(-j\frac{2\pi}{N_1} n_1 k_1 - j\frac{2\pi}{N_2} n_2 k_2). \quad (10)$$

It also holds that

$$X(k_1, k_2) = X(\omega_1, \omega_2)|_{\omega_1=2\pi\frac{k_1}{N_1}, \omega_2=2\pi\frac{k_2}{N_2}} \quad (11)$$

if $X(\omega_1, \omega_2)$ is the FT of one period of x .

3 Linear block transforms

3.1 Separable unitary block transforms

The forward transform is

$$\mathbf{Y} = \mathbf{A}\mathbf{X}\mathbf{A}^T \quad (12)$$

and the backward transform is

$$\mathbf{X} = \mathbf{A}^H \mathbf{Y} \mathbf{A}^* \quad (13)$$

Let $\mathbf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_N)$. Then, \mathbf{X} can be written as a linear combination of basis images.

$$\mathbf{X} = \sum_{k=1}^N \sum_{l=1}^N y_{kl} \mathbf{a}_k \mathbf{a}_l^T \quad (14)$$

3.2 Separable block transforms

The forward transform is

$$\mathbf{Y} = \mathbf{A}_1 \mathbf{X} \mathbf{A}_2^T. \quad (15)$$

It can be also written as

$$\mathbf{y} = \mathbf{A}_{2D} \mathbf{x} \quad (16)$$

if \mathbf{x} is extracted row-wise from \mathbf{X} and

$$\mathbf{A}_{2D} = \mathbf{A}_1 \otimes \mathbf{A}_2 = \begin{bmatrix} (\mathbf{A}_1)_{11} \mathbf{A}_2 & \dots & (\mathbf{A}_1)_{1N} \mathbf{A}_2 \\ \vdots & \ddots & \vdots \\ (\mathbf{A}_1)_{N1} \mathbf{A}_2 & \dots & (\mathbf{A}_1)_{NN} \mathbf{A}_2 \end{bmatrix}. \quad (17)$$

3.3 Haar transform

$$\mathbf{H}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (18)$$

$$\mathbf{H}_{2^{k+1}} = \begin{bmatrix} \mathbf{H}_{2^k} \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ \frac{1}{\sqrt{2}} \mathbf{I}_{2^k} \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \end{bmatrix} \quad (19)$$

$$\mathbf{A}_{2^k}^{Haar} = \frac{1}{\sqrt{2^k}} \mathbf{H}_{2^k} \quad (20)$$

3.4 Hadamard transform

$$\mathbf{A}_2^{Had} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (21)$$

$$\mathbf{A}_{2^k}^{Had} = \bigotimes_{l=1}^k \mathbf{A}_2^{Had} \quad (22)$$

3.5 Unitary DFT

$$\mathbf{A}_M^{DFT} = \frac{1}{\sqrt{M}} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & e^{-j\frac{2\pi}{M}} & e^{-j\frac{4\pi}{M}} & \dots & e^{-j\frac{2\pi(M-1)}{M}} \\ 1 & e^{-j\frac{4\pi}{M}} & e^{-j\frac{8\pi}{M}} & \dots & e^{-j\frac{4\pi(M-1)}{M}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-j\frac{2\pi(M-1)}{M}} & e^{-j\frac{4\pi(M-1)}{M}} & \dots & e^{-j\frac{2\pi(M-1)^2}{M}} \end{bmatrix} \quad (23)$$

3.6 Unitary DCT

$$(\mathbf{A}_M^{DCT})_{kn} = \begin{cases} \frac{1}{\sqrt{M}} & k=0 \\ \sqrt{\frac{2}{M}} \cos \frac{\pi(2n+1)k}{2M} & k \neq 0 \end{cases} \quad (24)$$

$$\mathbf{A}_M^{DCT} = \sqrt{\frac{2}{M}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \dots & \frac{1}{\sqrt{2}} \\ \cos \frac{\pi}{2M} & \cos \frac{3\pi}{2M} & \dots & \cos \frac{(2M-1)\pi}{2M} \\ \cos \frac{2\pi}{2M} & \cos \frac{6\pi}{2M} & \dots & \cos \frac{2(2M-1)\pi}{2M} \\ \vdots & \vdots & \ddots & \vdots \\ \cos \frac{(M-1)\pi}{2M} & \cos \frac{(M-1)3\pi}{2M} & \dots & \cos \frac{(M-1)(2M-1)\pi}{2M} \end{bmatrix} \quad (25)$$

3.7 KLT

Let $\mathbf{U}\mathbf{\Lambda}\mathbf{U}^H$ be the eigendecomposition of the covariance matrix $\mathbf{E}[\mathbf{X}\mathbf{X}^T] - \boldsymbol{\mu}\boldsymbol{\mu}^T$ or the autocorrelation matrix $\mathbf{E}[\mathbf{X}\mathbf{X}^T]$. Then the KLT is defined as

$$\mathbf{A}^{KLT} = \mathbf{U}^H \mathbf{x}. \quad (26)$$

The KLT minimises the energy packing coefficient

$$\eta_e(m) = \frac{\sum_{k=0}^{m-1} \mathbf{E}[y_k^2]}{\mathbf{E}[\mathbf{y}^T \mathbf{y}]}. \quad (27)$$

The decorrelation efficiency η_c is 1 for the KLT, with

$$\eta_c = 1 - \frac{\sum_{k=0}^{M-1} \sum_{l=0, k \neq l}^{M-1} |r_{yy}(k, l)|}{\sum_{k=0}^{M-1} \sum_{l=0, k \neq l}^{M-1} |r_{xx}(k, l)|}. \quad (28)$$

4 Multiresolution theory

4.1 Resolution pyramids

The total number of pixels of a $M \times N$ signal in a $P+1$ level pyramid is

$$MN \left(1 + \frac{1}{4} + \frac{1}{4^2} + \dots + \frac{1}{4^P} \right) \leq \frac{4}{3} MN. \quad (29)$$

4.2 Critically sampled filterbank

A filterbank is called critically samples if the amount of samples stays the same before analysis and after synthesis. That is, suppose you have M subsamplers with a subsampling ratio of $1 : k_i$. Then it should hold that

$$\frac{1}{k_0} + \frac{1}{k_1} + \dots + \frac{1}{k_{M-1}} = 1 \quad (30)$$

4.3 Effect of up-/downsampling

4.3.1 Continuous fourier transform

If we decimate $X(\omega)$ by the the factor m , then

$$Y(\omega) = \frac{1}{m} \sum_{l=0}^{m-1} X\left(\frac{\omega}{m} + 2\pi \frac{l}{m}\right). \quad (31)$$

4.3.2 Z-transform

4.3.2.1 Upsampling

Upsampling $x(n)$ by the factor L yields

$$y(n) = \begin{cases} x\left(\frac{n}{L}\right) & : n \bmod L = 0 \\ 0 & : \text{else} \end{cases}. \quad (32)$$

The z-transform of $y(n)$ is

$$Y(z) = \sum_{k=-\infty}^{\infty} y(k) z^{-k} \quad (33)$$

As $y(n)$ is only nonzero for $k = Ln$ it holds that

$$Y(z) = \sum_{n=-\infty}^{\infty} y(Ln) z^{-Ln} = \sum_{n=-\infty}^{\infty} x(n) (z^L)^{-n} = X(z^L). \quad (34)$$

4.3.2.2 Downsampling

Let

$$q_L(k) = \begin{cases} 1 & : k \bmod L = 0 \\ 0 & : \text{else} \end{cases}. \quad (35)$$

Let $y(k) = x(Lk)$ and let $l = \frac{n}{L}$. Then

$$Y(z) = \sum_{l=-\infty}^{\infty} x(Ll) z^{-l} = \sum_{n=-\infty}^{\infty} x(n) q_L(n) z^{-\frac{n}{L}}. \quad (36)$$

For $L = 2$ one can write $q_2(k)$ as

$$q_2(k) = \frac{1}{2}((-1)^{-k} + 1). \quad (37)$$

Thus

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} x(n) q_2(n) z^{-\frac{n}{2}} \\ &= \sum_{n=-\infty}^{\infty} x(n) \frac{1}{2}((-1)^{-n} + 1) z^{-\frac{n}{2}} \\ &= \frac{1}{2} \left(\sum_{n=-\infty}^{\infty} (-1)^{-n} x(n) z^{-\frac{n}{2}} + \sum_{n=-\infty}^{\infty} x(n) z^{-\frac{n}{2}} \right) \\ &= \frac{1}{2} \left(\sum_{n=-\infty}^{\infty} x(n) (-z^{\frac{1}{2}})^{-n} + \sum_{n=-\infty}^{\infty} x(n) z^{-\frac{n}{2}} \right) \\ &= \frac{1}{2} (X(-\sqrt{z}) + X(\sqrt{z})). \end{aligned} \quad (38)$$

So in short decimation by 2 is

$$Y(z) = \frac{1}{2} (X(-\sqrt{z}) + X(\sqrt{z})). \quad (39)$$

4.4 Frequency response of a 2-channel filterbank

Let F_0, F_1 denote the analysis filters and G_0, G_1 the synthesis filters. Then

$$\begin{aligned}\hat{X}(\omega) &= \frac{1}{2} (F_0(\omega)G_0(\omega) + F_1(\omega)G_1(\omega)) X(\omega) \\ &+ \frac{1}{2} (F_0(\omega + \pi)G_0(\omega) + F_1(\omega + \pi)G_1(\omega)) X(\omega + \pi).\end{aligned}\quad (40)$$

We get aliasing cancellation if

$$\begin{aligned}G_0(\omega) &= F_1(\omega + \pi), \\ -G_1(\omega) &= F_0(\omega + \pi) \\ &\text{or} \\ -G_0(\omega) &= F_1(\omega + \pi), \\ G_1(\omega) &= F_0(\omega + \pi) \\ &\text{or} \\ g_0(n) &= (-1)^n f_1(n), \\ g_1(n) &= (-1)^{n+1} f_0(n) \\ &\text{or} \\ g_0(n) &= (-1)^{n+1} f_1(n), \\ g_1(n) &= (-1)^n f_0(n).\end{aligned}\quad (41)$$

4.5 Design of a 2-channel perfect reconstruction filter bank

Let z^{-l} be a delay by l . A filter bank with PR and aliasing cancellation has to satisfy

$$F_0(z)G_0(z) + F_1(z)G_1(z) = 2z^{-l} \quad (42)$$

$$F_0(-z)G_0(z) + F_1(-z)G_1(z) = 0 \quad (43)$$

To satisfy (43) let

$$\begin{aligned}G_0(z) &= F_1(-z) \\ G_1(z) &= -F_0(-z).\end{aligned}\quad (44)$$

Let $P_0(z) = G_0(z)F_0(z)$ and $P_1(z) = G_1(z)F_1(z)$. With $P_1(z) = -P_0(z)$, (42) is satisfied if

$$P_0(z) - P_0(-z) = 2z^{-l}. \quad (45)$$

Thus there must be only one odd power l in $P_0(z)$. The design of a 2-channel PR filter bank consists of two steps:

1. Design a low-pass filter $P_0(z)$ satisfying (45).
2. Factor $P_0(z)$ into $F_0(z)G_0(z)$ and get $F_1(z), G_1(z)$ with (44).

5 The lifting implementation of the DWT

5.1 Polyphase representation

Lifting is super duper. To do this we split the calculation of the even and the odd samples for each channel of our filterbank.

$$Y_e(z) = H_e(z)X_e(z) + z^{-1}H_o(z)X_o(z) \quad (46)$$

$Y_e(z)$ includes subsampling implicitly. The polyphase representation of a filter H is

$$H(z) = H_e(z^2) + z^{-1}H_o(z^2) \quad (47)$$

We can determine $H_e(z)$ and $H_o(z)$ with

$$H_e(z^2) = \frac{H(z) + H(-z)}{2} \quad (48)$$

and

$$H_o(z^2) = \frac{H(z) - H(-z)}{2z^{-1}}. \quad (49)$$

Or you just take the even coefficients and throw them in $H_e(z^2)$ and throw the odd ones in $H_o(z^2)$ (don't forget to divide the latter by z^{-1}). Now we can represent our filterbank with the polyphase matrix:

$$\begin{pmatrix} X_L(z) \\ X_H(z) \end{pmatrix} = \tilde{\mathbf{P}}(z) \begin{pmatrix} X_e(z) \\ z^{-1}X_o(z) \end{pmatrix} \quad (50)$$

with

$$\tilde{\mathbf{P}}(z) = \begin{pmatrix} F_{0e}(z) & F_{0o}(z) \\ F_{1e}(z) & F_{1o}(z) \end{pmatrix}. \quad (51)$$

$X_L(z)$ and $X_H(z)$ hereby denote the outputs of the analysis step. Note that as $X_e(z)$ and $X_o(z)$ are subsampled versions of $X(z)$ you also have to subsample the filters. That is, take $F_{0e}(z)$ instead of $F_{0e}(z^2)$ for example. The synthesis matrix is

$$\mathbf{P}(z) = \begin{pmatrix} G_{0e}(z) & G_{1e}(z) \\ G_{0o}(z) & G_{1o}(z) \end{pmatrix}. \quad (52)$$

Perfect reconstruction can be achieved if

$$\tilde{\mathbf{P}}(z^{-1})\mathbf{P}(z) = \mathbf{I}. \quad (53)$$

That leads for example to

$$\begin{aligned}F_{0e}(z) &= G_{1o}(z^{-1}), \\ F_{0o}(z) &= -G_{1e}(z^{-1}), \\ F_{1e}(z) &= -G_{0o}(z^{-1}), \\ F_{1o}(z) &= G_{0e}(z^{-1}).\end{aligned}\quad (54)$$

With (47) it also holds

$$\begin{aligned}F_0(z) &= -z^{-1}G_1(z^{-1}), \\ F_1(z) &= z^{-1}G_0(z^{-1}).\end{aligned}\quad (55)$$

If $F_0 = G_0$ and $F_1 = G_1$ the wavelet transform is orthogonal otherwise it is biorthogonal.