Multidimensional Digital Signal Processing

Or plain: Image processing for dummies.
Version 2013-02-18

1 MD signals and systems

1.1 Periodic sequences

Let $\mathbf{n}, \mathbf{r} \in \mathbb{N}^k$ and $\mathbf{N}, \mathbf{P} \in \mathbb{N}^{k\times k}$. $x(\mathbf{n})$ is peridioc if

$$x(\mathbf{n} + \mathbf{N}\mathbf{r}) = x(\mathbf{n}). \tag{1}$$

The number of samples in one period is

$$|\det(\mathbf{N})|, \tag{2}$$

and $\mathbf{NP}$ is also a valid periodicity matrix.

1.2 Linear shift-invariant systems

Let $a, b \in \mathbb{C}$ and let $v(\mathbf{n}) = au_1(\mathbf{n}) + bu_2(\mathbf{n})$ and $w(\mathbf{n}) = u(\mathbf{n} - \mathbf{m})$. A system $\mathcal{L}$ is linear and shift-invariant (LSI) if

1. $\mathcal{L}[v](\mathbf{n}) = a\mathcal{L}[u_1](\mathbf{n}) + b\mathcal{L}[u_2](\mathbf{n})$,
2. $\mathcal{L}[w](\mathbf{n}) = \mathcal{L}[u](\mathbf{n} - \mathbf{m})$.

If $\mathcal{L}$ is linear and $y(n_1, n_2) := L[x](n_1, n_2)$ then

$$y(n_1, n_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x(k_1, k_2) \underbrace{\mathcal{L}[\delta](n_1 - k_1, n_2 - k_2)}_{h_{k_1 k_2}(n_1, n_2)}. \tag{3}$$

If $\mathcal{L}$ is also shift-invariant then

$$h := h_{00}(n_1 - k_1, n_2 - k_2) = h_{k_1 k_2}(n_1, n_2). \tag{4}$$

1.3 2D convolution

1. $x ** h = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x(k_1, k_2) h(n_1 - k_1, n_2 - k_2)$
2. $x ** h = h ** x$
3. $(x ** h) ** g = x ** (h ** g)$
4. $x ** (h + g) = (x ** h) + (x ** g)$

1.4 Separable systems

If $h(n_1, n_2) = h_1(n_1)h_2(n_2)$ then

$$y(n_1, n_2) = h_1 * (h_2 * x). \tag{5}$$

That is

1. $g(n_1, n_2) = h_2 * x = \sum_{k_2=-\infty}^{\infty} h_2(k_2) x(n_1, n_2 - k_2)$,
2. $y(n_1, n_2) = h_1 * g = \sum_{k_1=-\infty}^{\infty} h_1(k_1) g(n_1 - k_1, n_2)$.

1.5 BIBO stability

A system is BIBO stable if

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} |h(n_1, n_2)| = S_1 < \infty \tag{6}$$

1.6 Fourier transform

The Fourier transform is defined as

$$X(\omega_1, \omega_2) = \sum_{n_1} \sum_{n_2} x(n_1, n_2) \exp(-j\omega_1 n_1 - j\omega_2 n_2). \tag{7}$$

X is periodic with $\mathbf{N} = \begin{bmatrix} 2\pi & 0 \\ 0 & 2\pi \end{bmatrix}$. Its inverse transform is

$$x(n_1, n_2) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(\omega_1, \omega_2) \exp(j\omega_1 n_1 + j\omega_2 n_2) d\omega_1 d\omega_2. \tag{8}$$

Some nice properties:

1. $a, b \in \mathbb{C} : ax_1 + by_1 \circ\!\!-\!\!\bullet\ aX_1 + bX_2$
2. $x(n_1 - k_1, n_2 - k_2) \circ\!\!-\!\!\bullet \exp(-j\omega_1 k_1 - j\omega_2 k_2) X(\omega_1, \omega_2)$
3. $\exp(j\theta_1 n_1 + j\theta_2 n_2) x(n_1, n_2) \circ\!\!-\!\!\bullet\ X(\omega_1 - \theta_1, \omega_2 - \theta_2)$
4. $h ** x \circ\!\!-\!\!\bullet\ HX$

2 Discrete fourier transform

2.1 Discrete fourier series

Let x be periodic with with $\mathbf{N} = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}$. Them its 2D fourier series is

$$x(n_1, n_2) = \frac{1}{N_1 N_2} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} X(k_1, k_2) \exp(j\frac{2\pi}{N_1} n_1 k_1 + j\frac{2\pi}{N_2} n_2 k_2), \tag{9}$$

with

$$X(k_1, k_2) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x(n_1, n_2) \exp(-j\frac{2\pi}{N_1} n_1 k_1 - j\frac{2\pi}{N_2} n_2 k_2). \tag{10}$$

It also holds that

$$X(k_1, k_2) = X(\omega_1, \omega_2)\big|_{\omega_1 = 2\pi \frac{k_1}{N_1}, \omega_2 = 2\pi \frac{k_2}{N_2}} \tag{11}$$

if $X(\omega_1, \omega_2)$ is the FT of one period of x.

3 Linear block transforms

3.1 Separable unitary block transforms

The forward transform is

$$\mathbf{Y} = \mathbf{A}\mathbf{X}\mathbf{A}^T \tag{12}$$

and the backward transform is

$$\mathbf{X} = \mathbf{A}^H \mathbf{Y} \mathbf{A}^* \tag{13}$$

Let $\mathbf{A} = (\mathbf{a}_1, \ldots, \mathbf{a}_N)$. Then, $\mathbf{X}$ can be written as a linear combination of basis images.

$$\mathbf{X} = \sum_{k=1}^{N} \sum_{l=1}^{N} y_{kl} \mathbf{a}_k \mathbf{a}_l^T \tag{14}$$

3.2 Separable block transforms

The forward transform is

$$\mathbf{Y} = \mathbf{A}_1 \mathbf{X} \mathbf{A}_2^T. \tag{15}$$

It can be also written as

$$\mathbf{y} = \mathbf{A}_{2D} \mathbf{x} \tag{16}$$

if is $\mathbf{x}$ is extraced row-wise from $\mathbf{X}$ and

$$\mathbf{A}_{2D} = \mathbf{A}_1 \otimes \mathbf{A}_2 = \begin{bmatrix} (\mathbf{A}_1)_{11}\mathbf{A}_2 & \ldots & (\mathbf{A}_1)_{1N}\mathbf{A}_2 \\ \vdots & \ddots & \vdots \\ (\mathbf{A}_1)_{N1}\mathbf{A}_2 & \ldots & (\mathbf{A}_1)_{NN}\mathbf{A}_2 \end{bmatrix}. \tag{17}$$

3.3 Haar transform

$$\mathbf{H}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{18}$$

$$\mathbf{H}_{2^{k+1}} = \begin{bmatrix} \mathbf{H}_{2^k} \otimes \begin{bmatrix} 1 & 1 \end{bmatrix} \\ 2^{\frac{k}{2}} \mathbf{I}_{2^k} \otimes \begin{bmatrix} 1 & -1 \end{bmatrix} \end{bmatrix} \tag{19}$$

$$\mathbf{A}_{2^k}^{Haar} = \frac{1}{\sqrt{2^k}} H_{2^k} \tag{20}$$

3.4 Hadamard transform

$$\mathbf{A}_2^{Had} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{21}$$

$$\mathbf{A}_{2^k}^{Had} = \bigotimes_{l=1}^{k} \mathbf{A}_2^{Had} \tag{22}$$

3.5 Unitary DFT

$$\mathbf{A}_M^{DFT} = \frac{1}{\sqrt{M}} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & e^{-j\frac{2\pi}{M}} & e^{-j\frac{4\pi}{M}} & \dots & e^{-j\frac{2\pi(M-1)}{M}} \\ 1 & e^{-j\frac{4\pi}{M}} & e^{-j\frac{8\pi}{M}} & \dots & e^{-j\frac{4\pi(M-1)}{M}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-j\frac{2\pi(M-1)}{M}} & e^{-j\frac{4\pi(M-1)}{M}} & \dots & e^{-j\frac{2\pi(M-1)^2}{M}} \end{bmatrix} \tag{23}$$

3.6 Unitary DCT

$$(\mathbf{A}_M^{DCT})_{kn} = \begin{cases} \frac{1}{\sqrt{M}} & k = 0 \\ \sqrt{\frac{2}{M}} \cos\frac{\pi(2n+1)k}{2M} & k \neq 0 \end{cases} \tag{24}$$

$$\mathbf{A}_M^{DCT} = \sqrt{\frac{2}{M}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \dots & \frac{1}{\sqrt{2}} \\ \cos\frac{\pi}{2M} & \cos\frac{3\pi}{2M} & \dots & \cos\frac{(2M-1)\pi}{2M} \\ \cos\frac{2\pi}{2M} & \cos\frac{6\pi}{2M} & \dots & \cos\frac{2(2M-1)\pi}{2M} \\ \vdots & \vdots & \ddots & \vdots \\ \cos\frac{(M-1)\pi}{2M} & \cos\frac{(M-1)3\pi}{2M} & \dots & \cos\frac{(M-1)(2M-1)\pi}{2M} \end{bmatrix} \tag{25}$$

3.7 KLT

Let $\mathbf{U}\mathbf{\Lambda}\mathbf{U}^H$ be the eigendecomposition of the covariance matrix $\mathrm{E}\left[\mathbf{X}\mathbf{X}^T\right] - \boldsymbol{\mu}\boldsymbol{\mu}^T$ or the autocorrelation matrix $\mathrm{E}\left[\mathbf{X}\mathbf{X}^T\right]$. Then the KLT is defined as

$$\mathbf{A}^{KLT} = \mathbf{U}^H \mathbf{x}. \tag{26}$$

The KLT minimises the energy packing coefficient

$$\eta_e(m) = \frac{\sum_{k=0}^{m-1} \mathrm{E}\left[y_k^2\right]}{\mathrm{E}\left[\mathbf{y}^T\mathbf{y}\right]}. \tag{27}$$

The decorrelation efficiency η_c is 1 for the KLT, with

$$\eta_c = 1 - \frac{\sum_{k=0}^{M-1} \sum_{l=0,k\neq l}^{M-1} |r_{yy}(k,l)|}{\sum_{k=0}^{M-1} \sum_{l=0,k\neq l}^{M-1} |r_{xx}(k,l)|}. \tag{28}$$

4 Multiresolution theory

4.1 Resolution pyramids

The total number of pixels of a $M \times N$ signal in a $P+1$ level pyramid is

$$MN\left(1 + \frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^P}\right) \leqslant \frac{4}{3}MN. \tag{29}$$

4.2 Critically sampled filterbank

A filterbank is called critically samples if the amount of samples stays the same before analysis and after synthesis. That is, suppose you have M subsamplers with a subsampling ratio of $1 : k_i$. Then it should hold that

$$\frac{1}{k_0} + \frac{1}{k_1} + \cdots + \frac{1}{k_{M-1}} = 1 \tag{30}$$

4.3 Effect of up-/downsampling

4.3.1 Continuous fourier transform

If we decimate $X(\omega)$ by the the factor m, then

$$Y(\omega) = \frac{1}{m} \sum_{l=0}^{m-1} X\left(\frac{\omega}{m} + 2\pi\frac{l}{m}\right). \tag{31}$$

4.3.2 Z-transform

4.3.2.1 Upsampling

Upsampling $x(n)$ by the factor L yields

$$y(n) = \begin{cases} x\left(\frac{n}{L}\right) & : \quad n \bmod L = 0 \\ 0 & : \quad \text{else} \end{cases}. \tag{32}$$

The z-transform of $y(n)$ is

$$Y(z) = \sum_{k=-\infty}^{\infty} y(k) z^{-k} \tag{33}$$

As $y(n)$ is only nonzero for $k = Ln$ it holds that

$$Y(z) = \sum_{n=-\infty}^{\infty} y(Ln) z^{-Ln} = \sum_{n=-\infty}^{\infty} x(n)(z^L)^{-n} = X(z^L). \tag{34}$$

4.3.2.2 Downsampling

Let

$$q_L(k) = \begin{cases} 1 & : \quad k \bmod L = 0 \\ 0 & : \quad \text{else} \end{cases}. \tag{35}$$

Let $y(k) = x(Lk)$ and let $l = \frac{n}{L}$. Then

$$Y(z) = \sum_{l=-\infty}^{\infty} x(Ll) z^{-l} = \sum_{n=-\infty}^{\infty} x(n) q_L(n) z^{-\frac{n}{L}}. \tag{36}$$

For $L = 2$ one can write $q_2(k)$ as

$$q_2(k) = \frac{1}{2}((-1)^{-k} + 1). \tag{37}$$

Thus

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} x(n) q_2(n) z^{-\frac{n}{2}} \\ &= \sum_{n=-\infty}^{\infty} x(n) \tfrac{1}{2}((-1)^{-n} + 1) z^{-\frac{n}{2}} \\ &= \tfrac{1}{2}\left(\sum_{n=-\infty}^{\infty} (-1)^{-n} x(n) z^{-\frac{n}{2}} + \sum_{n=-\infty}^{\infty} x(n) z^{-\frac{n}{2}}\right) \\ &= \tfrac{1}{2}\left(\sum_{n=-\infty}^{\infty} x(n)(-z^{\frac{1}{2}})^{-n} + \sum_{n=-\infty}^{\infty} x(n) z^{-\frac{n}{2}}\right) \\ &= \tfrac{1}{2}\left(X(-\sqrt{z}) + X(\sqrt{z})\right). \end{aligned} \tag{38}$$

So in short decimation by 2 is

$$Y(z) = \frac{1}{2}\left(X(-\sqrt{z}) + X(\sqrt{z})\right). \tag{39}$$

4.4 Frequency response of a 2-channel filterbank

Let F_0, F_1 denote the analysis filters and G_0, G_1 the synthesis filters. Then

$$\begin{array}{rcl} \widehat{X}(\omega) & = & \frac{1}{2}\left(F_0(\omega)G_0(\omega) + F_1(\omega)G_1(\omega)\right)X(\omega) \\ & + & \frac{1}{2}\left(F_0(\omega+\pi)G_0(\omega) + F_1(\omega+\pi)G_1(\omega)\right)X(\omega+\pi). \end{array} \tag{40}$$

We get aliasing cancellation if

$$\begin{array}{rcl} G_0(\omega) & = & F_1(\omega+\pi), \\ -G_1(\omega) & = & F_0(\omega+\pi) \\ & \text{or} & \\ -G_0(\omega) & = & F_1(\omega+\pi), \\ G_1(\omega) & = & F_0(\omega+\pi) \\ & \text{or} & \\ g_0(n) & = & (-1)^n f_1(n), \\ g_1(n) & = & (-1)^{n+1} f_0(n) \\ & \text{or} & \\ g_0(n) & = & (-1)^{n+1} f_1(n), \\ g_1(n) & = & (-1)^n f_0(n). \end{array} \tag{41}$$

4.5 Design of a 2-channel perfect reconstruction filter bank

Let z^{-l} be a delay by l. A filter bank with PR and aliasing cancellation has to satisfy

$$F_0(z)G_0(z) + F_1(z)G_1(z) = 2z^{-l} \tag{42}$$

$$F_0(-z)G_0(z) + F_1(-z)G_1(z) = 0 \tag{43}$$

To satisfy (43) let

$$\begin{array}{rcl} G_0(z) & = & F_1(-z) \\ G_1(z) & = & -F_0(-z). \end{array} \tag{44}$$

Let $P_0(z) = G_0(z)F_0(z)$ and $P_1(z) = G_1(z)F_1(z)$. With $P_1(z) = -P_0(z)$, (42) is satisifed if

$$P_0(z) - P_0(-z) = 2z^{-l}. \tag{45}$$

Thus there must be only one odd power l in $P_0(z)$. The design of a 2-channel PR filter bank consists of two steps:

1. Design a low-pass filter $P_0(z)$ satisfying (45).
2. Factor $P_0(z)$ into $F_0(z)G_0(z)$ and get $F_1(z), G_1(z)$ with (44).

5 The lifting implementation of the DWT

5.1 Polyphase representation

Lifting is super duper. To do this we split the calculation of the even and the odd samples for each channel of our filterbank.

$$Y_e(z) = H_e(z)X_e(z) + z^{-1}H_0(z)X_0(z) \tag{46}$$

$Y_e(z)$ includes subsampling implicitly. The polyphase representation of a filter H is

$$H(z) = H_e(z^2) + z^{-1}H_o(z^2) \tag{47}$$

We can determine $H_e(z)$ and $H_o(z)$ with

$$H_e(z^2) = \frac{H(z) + H(-z)}{2} \tag{48}$$

and

$$H_o(z^2) = \frac{H(z) - H(-z)}{2z^{-1}}. \tag{49}$$

Or you just take the even coefficients and throw them in $H_e(z^2)$ and throw the odd ones in $H_o(z^2)$ (don't forget to divide the latter by z^{-1}). Now we can represent our filterbank with the polyphase matrix:

$$\begin{pmatrix} X_L(z) \\ X_H(z) \end{pmatrix} = \widetilde{\boldsymbol{P}}(z) \begin{pmatrix} X_e(z) \\ z^{-1}X_o(z) \end{pmatrix} \tag{50}$$

with

$$\widetilde{\boldsymbol{P}}(z) = \begin{pmatrix} F_{0e}(z) & F_{0o}(z) \\ F_{1e}(z) & F_{1o}(z) \end{pmatrix}. \tag{51}$$

$X_L(z)$ and $X_H(z)$ hereby denote the outputs of the analysis step. Note that as $X_e(z)$ and $X_o(z)$ are subsampled versions of $X(z)$ you also have to subsample the filters. That is, take $F_{0e}(z)$ instead of $F_{0e}(z^2)$ for example.
The synthesis matrix is

$$\boldsymbol{P}(z) = \begin{pmatrix} G_{0e}(z) & G_{1e}(z) \\ G_{0o}(z) & G_{1o}(z) \end{pmatrix}. \tag{52}$$

Perfect reconstruction can be achieved if

$$\widetilde{\boldsymbol{P}}(z^{-1})\boldsymbol{P}(z) = \mathbf{I}. \tag{53}$$

That leads for example to

$$\begin{array}{rcl} F_{0e}(z) & = & G_{1o}(z^{-1}), \\ F_{0o}(z) & = & -G_{1e}(z^{-1}), \\ F_{1e}(z) & = & -G_{0o}(z^{-1}), \\ F_{1o}(z) & = & G_{0e}(z^{-1}). \end{array} \tag{54}$$

With (47) it also holds

$$\begin{array}{rcl} F_0(z) & = & -z^{-1}G_1(z^{-1}), \\ F_1(z) & = & z^{-1}G_0(-z^{-1}). \end{array} \tag{55}$$

If $F_0 = G_0$ and $F_1 = G_1$ the wavelet transform is orthogonal otherwise it is biorthogonal. If $\det(\boldsymbol{P}(z)) = 1$ then the filter pair (G_0, G_1) is called complementary. If (G_0, G_1) is complementary, so is (F_0, F_1).

5.2 Laurent polynomials/series

A Laurent series is

$$h(z) = \sum_{k=p}^{q} h_k z^{-k}. \tag{56}$$

The degree of a LP is

$$|h| = q - p. \tag{57}$$

For two LPs $a(z)$ and $b(z)$ with $|a(z)| > |b(z)|$ there always exists a $q(z)$ with $|q(z)| = |a(z)| - |b(z)|$ and $r(z)$ with $|r(z)| < |b(z)|$ such that

$$a(z) = b(z)q(z) + r(z) \tag{58}$$

5.3 The euclidean algorithm

Let $|a(z)| \geqslant |b(z)|$, $b(z) \neq 0$, $a_0(z) = a(z)$ and $b_0(z) = b(z)$. Then

$$\begin{array}{rcl} a_{i+1}(z) & = & b_i(z) \\ b_{i+1}(z) & = & a_i(z) \bmod b_i(z) \end{array} \tag{59}$$

There exists some n with $b_n(z) = 0$. Then $a_n(z) = \gcd(a(z), b(z))$. With $q_{i+1}(z) = \frac{a_i(z)}{b_i(z)}$ (without remainder) it holds that

$$a_i(z) = q_{i+1}(z)b_i(z) + b_{i+1}(z). \tag{60}$$

Rewriting that in a matrix yields

$$\begin{pmatrix} a_i(z) \\ b_i(z) \end{pmatrix} = \begin{pmatrix} q_{i+1}(z) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{i+1}(z) \\ b_{i+1}(z) \end{pmatrix}. \tag{61}$$

Thus

$$\begin{pmatrix} a(z) \\ b(z) \end{pmatrix} = \prod_{i=1}^{n} \begin{pmatrix} q_i(z) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n(z) \\ 0 \end{pmatrix}. \tag{62}$$

5.4 Lifting theorem

If (G_0, G_1) is complementary, so is (G_1', G_0) only if

$$G_1'(z) = G_1(z) + G_0(z)s(z^2). \tag{63}$$

Similar statements hold for G_0', F_0' and F_1':

$$G_0'(z) = G_0(z) + G_1(z)t(z^2), \tag{64}$$

$$F_0'(z) = F_0(z) - F_1(z)s(z^{-2}), \tag{65}$$

$$F_1'(z) = F_1(z) - F_0(z)t(z^{-2}). \tag{66}$$

5.5 From the lifting theorem to lifting steps

Consider again (63). Using (47) G'_{1e} and G'_{1o} are

$$\begin{array}{l} G'_{1e} = G_{1e}(z) + G_{0e}(z)s(z) \\ G'_{1o} = G_{1o}(z) + G_{0o}(z)s(z) \end{array} . \tag{67}$$

Rewriting this in a matrix together with G_{0e} and G_{0o} yields

$$\begin{pmatrix} G_{0e}(z) & G'_{1e}(z) \\ G_{0o}(z) & G'_{1o}(z) \end{pmatrix} = \begin{pmatrix} G_{0e}(z) & G_{1e}(z) \\ G_{0o}(z) & G_{1o}(z) \end{pmatrix} \begin{pmatrix} 1 & s(z) \\ 0 & 1 \end{pmatrix} . \tag{68}$$

One can identify the polyphase matrix and rewrite this to

$$\boldsymbol{P}'(z) = \boldsymbol{P}(z) \begin{pmatrix} 1 & s(z) \\ 0 & 1 \end{pmatrix} . \tag{69}$$

Similarly holds

$$G'_0 : \boldsymbol{P}'(z) = \boldsymbol{P}(z) \begin{pmatrix} 1 & 0 \\ t(z) & 1 \end{pmatrix} , \tag{70}$$

$$F'_0 : \widetilde{\boldsymbol{P}}'(z) = \begin{pmatrix} 1 & -s(z^{-1}) \\ 0 & 1 \end{pmatrix} \widetilde{\boldsymbol{P}}(z), \tag{71}$$

$$F'_1 : \widetilde{\boldsymbol{P}}'(z) = \begin{pmatrix} 1 & 0 \\ -t(z^{-1}) & 1 \end{pmatrix} \widetilde{\boldsymbol{P}}(z). \tag{72}$$

We call

$$\boldsymbol{\mathcal{U}} = \begin{pmatrix} 1 & s(z) \\ 0 & 1 \end{pmatrix} \tag{73}$$

an update matrix,

$$\boldsymbol{\mathcal{P}} = \begin{pmatrix} 1 & 0 \\ t(z) & 1 \end{pmatrix} \tag{74}$$

a prediction matrix and

$$\boldsymbol{\mathcal{L}} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \tag{75}$$

a lazy wavelet transform. We can factorize $\boldsymbol{P}$ as

$$\boldsymbol{P} = \boldsymbol{\mathcal{L}} \ldots \boldsymbol{\mathcal{U}}_2 \boldsymbol{\mathcal{P}}_2 \boldsymbol{\mathcal{U}}_1 \boldsymbol{\mathcal{P}}_1 \tag{76}$$

and $\widetilde{\boldsymbol{P}}$ as

$$\widetilde{\boldsymbol{P}} = \ldots \boldsymbol{\mathcal{U}}_2 \boldsymbol{\mathcal{P}}_2 \boldsymbol{\mathcal{U}}_1 \boldsymbol{\mathcal{P}}_1 \boldsymbol{\mathcal{L}}. \tag{77}$$

6 Geometric wavelets

6.1 Radon transform

The radon transform is

$$r(\rho, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos(\theta) + y \sin(\theta) - \rho) dx dy \tag{78}$$

with $\rho \in (-\infty, \infty), \theta \in [0, 2\pi)$. Its is symmetric with

$$\begin{array}{rcl} r(\rho, \theta + \pi) & = & r(-\rho, \theta), \\ r(-\rho, \theta + \pi) & = & r(\rho, \theta). \end{array} \tag{79}$$

The fourier transform for a fixed θ is

$$R_\theta(\omega) = \int_{-\infty}^{\infty} r(\rho, \theta) e^{-j2\pi\omega\rho} d\rho. \tag{80}$$

Inserting the definition of the radon transform yields

$$R_\theta(\omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi\omega(x\cos(\theta) + y \sin(\theta))} dx dy. \tag{81}$$

By substituting $u = \omega \cos(\theta)$ and $v = \omega \sin(\theta)$ it holds that

$$R_\theta(\omega) = F(\omega \cos(\theta), \omega \sin(\theta)). \tag{82}$$

If we do some stuff we end up with

$$f(x, y) = \int_0^\pi \left[\int_{-\infty}^{\infty} |\omega| R_\theta(\omega) e^{j2\pi\omega\rho} d\omega \right]_{\rho = x\cos(\theta) + y\sin(\theta)} d\theta. \tag{83}$$

As is this is not integrable it needs to be multiplied with a window function, e.g.

$$h(\omega) = \begin{cases} 1 & : \quad |\omega| \in [0, \omega_{max}] \\ 0 & : \quad \text{else} \end{cases} \tag{84}$$

or

$$h(\omega) = \begin{cases} c + (c-1)\cos(\frac{2\pi\omega}{M-1}) & : \quad \omega \in [0, M) \\ 0 & : \quad \text{else} \end{cases} . \tag{85}$$

The latter is called Hamming window for $c = 0.54$ or Hann window for $c = 0.5$. So to summarise the inverse radon transform consists of these steps:

1. Calculate $R_\theta(\omega)$.
2. Multiply $R_\theta(\omega)$ by $h(\omega)|\omega|$.
3. Calculate the inverses of $h(\omega)|\omega|R_\theta(\omega)$ for each θ.
4. Sum all the inverses.

6.2 Discrete radon transform

The discrete radon transform is

$$r(\rho, \theta) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \delta(x \cos(\theta) + y \sin(\theta) - \rho) \tag{86}$$