Differentially Private Continual Releases of Streaming Frequency Moment Estimations

Alessandro Epasto <aepasto@google.com>, Jieming Mao <maojm@google.com>, Andres Munoz Medina <ammedina@google.com>, Vahab Mirrokni <mirrokni@google.com>,

Sergei Vassilvitskii <sergeiv@google.com> Peilin Zhong <peilinz@google.com>

April 2022

Abstract

The streaming model of computation is a popular approach for working with large-scale data. In this setting, there is a stream of items and the goal is to compute the desired quantities (usually data statistics) while making a single pass through the stream and using as little space as possible.

Motivated by the importance of data privacy, we develop differentially private streaming algorithms under the continual release setting, where the union of outputs of the algorithm at every timestamp must be differentially private. Specifically, we study the fundamental ℓ_p $(p \in [0, +\infty))$ frequency moment estimation problem under this setting, and give an ε -DP algorithm that achieves $(1 + \eta)$ -relative approximation $(\forall \eta \in (0, 1))$ with poly $\log(Tn)$ additive error and uses poly $\log(Tn) \cdot \max(1, n^{1-2/p})$ space, where T is the length of the stream and n is the size of the universe of elements. Our space is near optimal up to poly-logarithmic factors even in the non-private setting.

To obtain our results, we first reduce several primitives under the differentially private continual release model, such as counting distinct elements, heavy hitters and counting low frequency elements, to the simpler, counting/summing problems in the same setting. Based on these primitives, we develop a differentially private continual release level set estimation approach to address the ℓ_p frequency moment estimation problem.

We also provide a simple extension of our results to the harder sliding window model, where the statistics must be maintained over the past W data items.

1 Introduction

Data privacy is a central concern in the deployment of real-world computational systems. In the vast literature on privacy in computation Hsu et al. [2021], the notion of differential privacy (DP) Dwork [2008], Dwork et al. [2014] has remained the *de facto* standard for more than a decade. The classical formulation of differential privacy assumes that the data is static Dwork [2008], and that the data curator is interested obtaining answers to a *predetermined* number of queries on the dataset.

Real world applications, however, often require the analysis of user datasets that are organically and rapidly growing. This is illustrated by the popular *streaming model* of computation Sr. [1978], Alon et al. [1996], where data arrives over time, and at each update, a new solution is output by the algorithm. In such streaming applications, the *continual release* model of differential privacy Dwork et al. [2010a], Hubert Chan et al. [2010] promises a rigours guarantee of privacy: an observer of *all* the outputs of the algorithm is information-theoretically bounded in the ability to learn about the *existence* of an individual data point in the stream.

In this paper, we focus on two fundamental challenges in the field of private streaming algorithms: the insertion only, or streaming, model and the sliding window model. In the former model, a data curator receives a stream of data a_1, a_2, \ldots and at each time t releases a statistical query depending on all data received up to that point. In the latter, the computation depends only on the last W items observed by the data curator.

The sliding window model may be generally more practically relevant compared to the streaming model as it allows to account for information freshness and in some cases it can be a legal or privacy requirement as well. For instance, in some situations, data privacy laws such as the General Data Protection Regulation (GDPR)¹ do not allow unlimited retention of user data.

Our main contribution is to provide private algorithms for a series of foundational streaming problems under both the streaming and sliding window model.

Motivating example: Privacy Sandbox. We present a concrete practical application of our results. As part of the *Privacy Sandbox* initiative, Chrome has developed a series of APIs to reduce cross site tracking while supporting the digital advertising ecosystem. A key part of one of the proposals is a *k*-Anonymity Server.² The server ensures that each ad creative that is reported to advertisers has won its respective auction at least *k* times over a particular time window. Abstracting the specifics, this problem requires computing the number of distinct elements over a sliding window. Moreover, to further strengthen privacy protections, the computation itself should be made differentially private, which is precisely the setting we consider in this work.

The previous example elucidates a concrete motivation for the study of sliding window algorithms for counting distinct element problems with differential privacy in the continual release setting. The rest of the paper proceeds by formalizing this model and our results in this space.

In particular, we study a more general class of statistics of the input data than the problem of counting distinct elements: the ℓ_p frequency moments problem. The ℓ_p frequency moment is the sum of the *p*-th power of the frequencies of the elements. The number of distinct elements is a special case for p = 0. The ℓ_p frequency moment problem is one of the most fundamental problems in the streaming literature. The first non-trivial algorithm for p = 1 is Morris [1978]. Later Flajolet and Martin [1985] is the first to study the case for p = 0. Alon et al. [1996] initiates the study for p = 2 and other $p \in [0, \infty)$. After the developments over several decades, the spaces of the current best ℓ_p frequency moment estimation algorithms are near optimal for all $p \in [0, \infty)$, i.e. they almost match the proven space lower bounds (see e.g., Ganguly [2011], Kane et al. [2011, 2010], Flajolet et al. [2007], Li and Woodruff [2013]).

However the landscape of DP streaming ℓ_p frequency moment is mysterious even in the noncontinual release setting. Most existing work only studied for p = 0, 1, 2 (see more discussion in Section 1.4). The work of Wang et al. [2022] considered general $p \in (0, 1]$. A recent independent work Blocki et al. [2022] studied all $p \in [0, \infty)$. But none of Blocki et al. [2022], Wang et al. [2022]

 $^{^{1}}$ https://gdpr-info.eu/art-17-gdpr/

²https://github.com/WICG/turtledove/blob/main/FLEDGE_k_anonymity_server.md

considered continual release setting. In the DP continual release setting, Bolot et al. [2013] studied the count of distinct elements but not in the low space streaming setting. In the DP streaming continual release setting, existing work Dwork [2008], Hubert Chan et al. [2010] only studied the case for p = 1. No previous algorithm for $p \neq 1$ is known in the DP low space streaming continual release setting.

For ℓ_p frequency moment in the sliding window model, there are known techniques Datar et al. [2002], Braverman and Ostrovsky [2007] which can convert the streaming algorithm into sliding window algorithm using some additional small space. We show how to extend these techniques to convert our DP streaming continual release streaming algorithms to DP sliding window continual release algorithms.

1.1 Computational Model

In this paper, we consider a streaming setting with T timestamps. At each timestamp $t \in [T]$, we get an input $a_t \in \mathcal{U} \cup \{\bot\}$, where \mathcal{U} represents the universe of all possible input elements, and \bot represents empty. We sometimes also consider an input stream of integers, i.e., at each timestamp $t \in [T]$, we get an input $a_t \in \mathbb{Z}$. The goal is to compute some function $g(\cdot)$ based on the inputs. Instead of only computing g at the end of the stream, we consider the continual release model throughout the paper:

• Continual Release Model: At every timestamp $t \in [T]$, we want to output $g(\cdot)$ based on the data a_1, a_2, \dots, a_t .

We consider two different streaming models depending on the range of inputs that g is based on.

- (Insertion-only) Streaming Model: In this model, g depends on all past inputs, i.e. at timestamp $t \in [T]$, we want to compute $g(a_1, ..., a_t)$. Unless otherwise specified, we use streaming model to refer to insertion-only streaming model throughout the paper.
- Sliding Window Model: In this model, we have a parameter $W \in \mathbb{Z}_{\geq 1}$ for window size. g depends on the last W inputs, i.e. at timestamp $t \in [T]$, we want to compute $g(a_{\max(t-W+1,1)}, ..., a_t)$.

In this paper, we are particularly interested in algorithms that use space sub-linear in T. For a (sub-)stream S, we use $\|S\|_p^p$ to denote the ℓ_p frequency moment of S. In particular, for p = 0, $\|S\|_0$ denotes the number of distinct elements. For p = 1, $\|S\|_1$ denotes the number of non-empty elements. We refer readers to preliminaries section (Section 2) for detailed notation and definitions.

1.2 Our Results and Comparison to Prior Work

In this section, we give a brief overview of our results and comparison with prior work. We use n to denote the size of the universe \mathcal{U} . We use (α, γ) -approximation to specify the approximation guarantee with multiplicative factor α and additive error γ . In all of our results, we use $\varepsilon \geq 0$ for the DP parameter, and use $\eta \in (0, 0.5)$ in the relative error.

1.2.1 Differentially Private Streaming Continual Release Algorithms

We developed a series of DP algorithms for solving frequency moments estimation and its related problems in the streaming continual release model.

 ℓ_p Frequency moment estimation $(p \in [0, \infty))$. Our main result is a general ℓ_p frequency moment estimation algorithm which works for all $p \in [0, \infty)$.

Theorem 1.1 (ℓ_p Frequency moment, informal version of Theorem 5.31). There is an ε -DP algorithm in the streaming continual release model such that with probability at least 0.9, it always outputs an $\left(1+\eta, \left(\frac{\log(Tn)}{\eta\varepsilon}\right)^{O(\max(1,p))}\right)$ -approximation to $\|\mathcal{S}\|_p^p$ for every timestamp t, where \mathcal{S} de-

notes the stream up to timestamp t. The algorithm uses space $\max(1, n^{1-2/p}) \cdot \left(\frac{\log(Tn)}{\eta\varepsilon}\right)^{O(\max(1,p))}$.

To the best of our knowledge, we are the first to study the general ℓ_p frequency moment estimation problem in the differentially private streaming continual release setting. Dwork et al. [2010a] and Hubert Chan et al. [2010] studies the summing problem in the same setting, where the summing problem can be seen as a special case for p = 1. Wang et al. [2022] studies the streaming ℓ_p frequency moment estimation for $p \in (0, 1]$ based on the *p*-stable distribution, and a concurrent independent work Blocki et al. [2022] studies the case for $p \in (0, 1]$, but it is not clear how to generalize their techniques to the continual release model, i.e., their approach only provides the differential privacy guarantee of the output at the end of the stream. In addition, the approach of Wang et al. [2022] does not achieve ε -DP for an arbitrarily small $\varepsilon > 0$, and they also mention that their technique might not be easily extended to the case for p > 1.

Our space usage is near optimal up to poly-logarithmic factors even when comparing with the non-private streaming ℓ_p frequency moment estimation algorithms: for $p \leq 2$, the space needed for both our algorithm and previous non-private algorithm (see e.g., Kane et al. [2010]) is poly-logarithmic, for p > 2, the space needed for both our algorithm and previous non-private algorithm and previous non-private algorithm and previous non-private algorithm and previous non-private algorithm Indyk and Woodruff [2005] is $\tilde{O}(n^{1-2/p})^3$. Note that $\Omega(n^{1-2/p})$ space is a proven lower bound for p > 2 even in the non-private case Saks and Sun [2002], Bar-Yossef et al. [2004].

Summing (ℓ_1 frequency moment estimation). The easiest problem that is related to the ℓ_p frequency moment estimation problem would be the summing problem: the goal is to compute the summation of the input numbers. Note that ℓ_1 frequency moment estimation is a special case of the summing of a binary stream, i.e., we regard \perp as 0 and all other elements as 1.

Theorem 1.2 (Summing of a non-negative stream, informal version of Theorem 3.3). There is an ε -DP algorithm for summing problem in the streaming continual release model. If the input numbers are guaranteed to be non-negative, with probability at least 0.9, the output is always a $(1 + \eta, O_{\varepsilon,\eta}(\log T))$ -approximation to the sum of all input numbers at any timestamp $t \in [T]$. The algorithm uses space O(1).

The summing problem was studied by Dwork et al. [2010a], Hubert Chan et al. [2010] in the differentially private streaming continual release model. Their approximation has $O(\log^{2.5} T)$ additive error. In our work, we show that if we allow $(1 + \eta)$ relative error and work on the stream with non-negative numbers only, we can reduce the additive error to $O(\log T)$. This is useful when we

³We use $\tilde{O}(g)$ to denote $g \cdot \text{poly}(\log(g))$

cannot avoid the relative error for some problem (such as the number of distinct elements) in the streaming model but we still need summing as a subroutine.

Counting distinct elements (ℓ_0 frequency moment estimation). Counting distinct elements if one of the fundamental problems in the streaming literature. The goal is to estimate the number of distinct elements that appeared in the stream. We provide a DP streaming continual release algorithm for counting distinct elements.

Theorem 1.3 (Number of distinct elements, informal version of Corollary 4.11). There is an ε -DP algorithm for the number of distinct elements in the streaming continual release model. With probability at least 0.9, the output is always a $(1 + \eta, O_{\varepsilon,\eta}(\log^2(T)))$ -approximation for every timestamp $t \in [T]$. The algorithm uses poly $\left(\frac{\log(T)}{\eta\min(\varepsilon,1)}\right)$ space.

In the non-private streaming setting, counting distinct element can be solved via sketching algorithms of Flajolet and Martin [1985] and its variants e.g., Flajolet et al. [2007]. Some recent work Choi et al. [2020], Smith et al. [2020] extends these sketching techniques for counting distinct element in a DP streaming setting. However, it is not clear how to extend these techniques to the continual release setting. Continual release of counts of distinct elements is studied by Bolot et al. [2013]. However, Bolot et al. [2013] is not in the low space streaming setting.

Estimation of frequencies and ℓ_2 frequency moments. The goal of ℓ_2 frequency moment estimation is to estimate the sum of square of frequencies of elements. We present a DP streaming continual release CountSketch Charikar et al. [2002] algorithm and use it for estimating ℓ_2 frequency moments and the frequency of each element.

Theorem 1.4 (Frequency and ℓ_2 frequency moments, informal version of Theorem 5.4). There is an ε -DP algorithm in the streaming continual release model such that with probability at least 0.9, it always outputs for every timestamp $t \in [T]$:

- 1. \hat{f}_a for every $a \in \mathcal{U}$ such that $|f_a \hat{f}_a| \leq \eta \|\mathcal{S}\|_2 + \tilde{O}_{\varepsilon,\eta} (\log^{3.5}(Tn))$, where \mathcal{S} denotes the stream up to timestamp t and f_a denotes the frequency of a in \mathcal{S} ,
- 2. \hat{F}_2 such that $|\hat{F}_2 \|\mathcal{S}\|_2^2| \le \eta \|\mathcal{S}\|_2^2 + \tilde{O}_{\varepsilon,\eta}(\log^7(Tn))$

The algorithm uses $O\left(\frac{\log(Tn)}{\eta^2} \cdot \log(T)\right)$ space.

Although DP ℓ_2 frequency moment was studied by a line of work (see e.g., Blocki et al. [2012], Sheffet [2017], Bu et al. [2021]), none of them considers the streaming continual release setting, and it is not clear how to extend previous techniques to the the continual release setting.

 ℓ_p Heavy hitters. In the ℓ_p heavy hitters problem, we are given a parameter k, and the goal is to find elements whose frequency to the p-th power is at least 1/k fraction of the ℓ_p frequency moment. By extending our DP streaming continual release CountSketch algorithm, we obtain a DP streaming continual release ℓ_p heavy hitters algorithm.

Theorem 1.5 (ℓ_p Heavy hitters for all $p \in [0, \infty)$, informal version of Theorem 5.10). There is an ε -DP algorithm in the streaming continual release model such that with probability at least 0.9, it always outputs a set $H \subseteq \mathcal{U}$ and a function $\hat{f} : H \to \mathbb{R}$ for every timestamp $t \in [T]$ satisfying

1. $\forall a \in H, \hat{f}(a) \in (1 \pm \eta) \cdot f_a$ where f_a is the frequency of a in the stream S up to timestamp t,

- 2. $\forall a \in \mathcal{U}, \text{ if } f_a \geq \frac{1}{\varepsilon \eta} \cdot \operatorname{poly}\left(\log\left(\frac{T \cdot k \cdot n}{\eta}\right)\right) \text{ and } f_a^p \geq \|\mathcal{S}\|_p^p / k \text{ then } a \in H,$
- 3. The size of H is at most $O(\log(Tn) \cdot 2^p \cdot k)$.

The algorithm uses $\max(1, n^{1-2/p}) \cdot \frac{k^3}{n^2} \cdot \operatorname{poly}\left(\log\left(T \cdot k \cdot n\right)\right)$ space.

To the best of our knowledge, though DP streaming continual release ℓ_1 heavy hitters problem is studied by Chan et al. [2012], ℓ_p (for $p \neq 1$) heavy hitters problem has not been studied in the DP streaming continual release setting before. Note that $\Omega(n^{1-2/p})$ for p > 2 is a lower bound of space needed for ℓ_p heavy hitters even in the non-private setting Saks and Sun [2002], Bar-Yossef et al. [2004].

1.2.2 Differentially Private Sliding Window Continual Release Algorithms

Smooth histogram Braverman and Ostrovsky [2007] is a general algorithmic framework which can convert a relative-approximate streaming algorithm into a relative-approximate sliding window algorithm if the objective function that we want to compute has some nice properties.

We generalize the smooth histogram to make it support converting an approximate streaming algorithm with both relative error and additive error into an approximate sliding window algorithm with both relative error and additive error if the objective function has good properties. In addition, we show that if the streaming algorithm is DP in the continual release setting, then the converted sliding window algorithm is also DP in the continual release setting.

By applying our generalized smooth histogram approach and paying a poly $\left(\frac{\log T}{\eta}\right)$ more factor than our DP streaming continual release algorithms in both additive error and space usage, we show DP sliding window continual release algorithms for

- 1. ℓ_p Moment estimation (see Corollary 6.10),
- 2. Summing (see Corollary 6.7),
- 3. Counting distinct elements (see Corollary 6.8),
- 4. ℓ_2 Moment estimation (see Corollary 6.9).

1.3 Our Techniques

In this section, we briefly discuss the high level ideas of our algorithms. We present a set of techniques to reduce almost all problems that we considered in the DP streaming continual release setting to the summing problem in the DP streaming continual release setting.

Summing with better additive error via grouping. To illustrate the intuition of using grouping for differentially private streaming continual release algorithms, we start with the following simple problem: given a stream of numbers c_1, c_2, \dots, c_T where each c_i is at least $10 \cdot \ln(10 \cdot T)/\varepsilon$, the goal is to output a (1 ± 0.1) -approximation to $\sum_{j=1}^{t} c_j$ for every prefix t with probability at least 0.9, and we want the set of all outputs to be ε -DP, i.e., the continual released results to be ε -DP. A simple way to solve the above problem is that we release a stream of noisy numbers $\hat{c}_1, \hat{c}_2, \dots, \hat{c}_T$ where $\forall t \in [T], \hat{c}_t = c_t + \operatorname{Lap}(1/\varepsilon)$, and we report $\sum_{j=1}^{t} \hat{c}_j$ for every prefix $t \in [T]$. It is easy to see that $(\hat{c}_1, \hat{c}_2, \dots, \hat{c}_T)$ is ε -DP. Since the reported approximate prefix sums only depend on $(\hat{c}_1, \hat{c}_2, \dots, \hat{c}_T)$, the continual released results are ε -DP. Furthermore, with probability at least

 $0.9, \forall t \in [T], |c_t - \hat{c}_t| \leq \ln(10 \cdot T)/\varepsilon$. Since c_t is at least $10 \ln(10 \cdot N)/\varepsilon$, we have $0.9c_t \leq \hat{c}_t \leq 1.1c_t$ which implies that every reported approximate prefix sum is a (1 ± 0.1) -approximation.

To generalize the above idea, we propose a grouping approach in to group the consecutive numbers in the stream in a differentially private way such that the total count of each group is large enough. To implement grouping, we need to apply the sparse vector technique (see e.g., Dwork et al. [2014]) iteratively. The similar idea also appeared in Dwork et al. [2015] which shows a better additive error guarantee for the summing problem than Dwork et al. [2010a] when the stream is sparse. In contrast, our additive error guarantee is always better than Dwork et al. [2010a] and Dwork et al. [2015] while we allow an additional $(1 + \varepsilon)$ relative approximation.

Counting distinct elements. We explain how to reduce counting distinct elements problem to the summing problem. Suppose the element universe is small, we are able to track the set of elements that already appeared during the stream. Then, we can create a binary stream of $\{0,1\}$ where 1 denotes that we see a new element and 0 denotes that the input element already appeared or it is empty. Therefore, the sum of the binary stream at timestamp t is exactly the number of distinct elements. Furthermore, if we change an element in the input stream from a to b, there are only constant number of positions of the binary stream will flip: consider the change $a \to \perp \to b$. If a is not its first appearance in the input stream, changing a to \perp does not cause any change in the binary stream. If a is its first appearance in the input stream, changing a to \perp will make the corresponding 1 in the binary stream be 0 and make the 0 corresponding to the original second appearance of a in the input stream to be 1. Thus, it will affect at most 2 entries of the binary stream. Similarly, changing \perp to b will cause the change of at most 2 entries of the binary stream. Thus, the binary stream has low sensitivity which implies that a DP streaming continual release summing algorithm gives a good approximation to the number of distinct elements with a small additive error. Next, we discuss how to handle the large universe. For large universe, we can try different sampling rate $1/2, 1/4, 1/8 \cdots, 1/T$. There should be a sampling rate such that (1) if we hash the sampled elements into hashing buckets, there is no collision with a good probability, (2) the number of samples is much larger than the additive error caused by the summing subroutine so we can have a good relative approximation of the number of distinct sampled elements. Then we can use the number of distinct sampled elements to estimate the number of distinct elements in the input stream.

CountSketch and ℓ_p heavy hitters. Let $h : \mathcal{U} \to [k]$ be a hash function which uniformly hash elements into k hash buckets. Let $g: \mathcal{U} \to \{-1, 1\}$ randomly map each element to -1 or 1 with equal probability. The CountSketch is a tuple of k numbers (z_1, z_2, \cdots, z_k) where z_i is the sum of weighted frequencies of elements hashed to the bucket i, and the weight of the frequency of element a is g(a). Changing a to b in the input stream will change at most 2 buckets: the bucket i contains a and the bucket j contains b. Since $|g(a)| \leq 1$, z_i and z_j will be changed by at most 1. We can use DP streaming continual release summing algorithm of Dwork et al. [2010a], Hubert Chan et al. [2010] to estimate (z_1, z_2, \dots, z_k) such that each estimation \hat{z}_i of z_i only has poly $(\log T)$ additive error, and $(\hat{z}_1, \hat{z}_2, \cdots, \hat{z}_k)$ is DP under the streaming continual release model. Suppose the ℓ_2 frequency moment is much larger than $poly(\log T)$, then the additive error becomes the relative error, and we can use $\hat{z}_1, \hat{z}_2, \dots, \hat{z}_k$ to obtain a good relative approximation of the ℓ_2 frequency moment. Similarly, if an element has frequency much larger than poly(log(T)), then poly(log(T))becomes small relative error of the frequency and we are able to check whether it is an ℓ_2 heavy hitter by the standard analysis of CountSketch. Thus, we can use this DP streaming continual release CountSketch to estimate ℓ_2 frequency moment with $(1 + \eta)$ -relative error and poly(log T)additive error, and we can use such CountSketch to find all elements which are at least poly(log T)

and are ℓ_2 heavy hitters.

Note that for $p \leq 2$, if a has the largest frequency and it is an $(1/k)-\ell_p$ heavy hitter, then a must be an $(1/k)-\ell_2$ heavy hitter. For p > 2, if a is an $1/k-\ell_p$ heavy hitter, than a must be an $1/(kn^{1-2/p})-\ell_2$ heavy hitter. Therefore, by some hashing technique, we can use ℓ_2 heavy hitters algorithm to construct ℓ_p heavy hitters algorithm. But since ℓ_2 heavy hitters can only report the elements with frequency larger than poly(log T), the obtained ℓ_p heavy hitters algorithm can only report the elements with frequency larger than poly(log T) as well.

 ℓ_p Frequency moment estimation. In high level we want to simulate the level set estimation idea of Indyk and Woodruff [2005] in the DP streaming continual release setting. In particular, let $\alpha = 1 + \eta$, let f_a denote the frequency of a and let $G_i = \{a \mid f_a \in (\alpha^i, \alpha^{i+1}]\}$. Then $\sum_i |G_i| \cdot (\alpha^i)^p$ is a good approximation to the ℓ_p frequency moment. We say G_i is contributing, if $|G_i| \cdot (\alpha^i)^p$ is at least $\Omega_{\alpha}(1/\log(T))$ fraction of the ℓ_p moment. Since non-contributing elements only contributes a small total amount to the ℓ_p frequency moment. Thus, we only need to estimate the size of each contributing G_i . Due to the definition of contributing, it is easy to see that if G_i is contributing, either α^i is large or $|G_i|$ is large. In fact, as observed by Indyk and Woodruff [2005], for each contributing level set G_i , there must be a proper sampling probability such that after sampling, there are at least poly(log T) elements from G_i sampled and all of the sampled elements from G_i are at least $1/\text{poly}(\log T)$ - ℓ_p heavy hitters among the set of all sampled elements from the universe \mathcal{U} . Ideally, we can try different sampling rate $1, 1/2, 1/4, 1/8, \dots, 1/T$ and use our ℓ_p heavy hitters algorithm to report the heavy hitters and estimate $|G_i|$ for each i. However, for i with $\alpha^i \ll \text{poly}(\log T)$, our DP streaming continual release ℓ_p heavy hitters algorithm does not report any element from G_i . We must find another way to estimate $|G_i|$ instead of using heavy hitters.

Similar to counting distinct elements, let us start with the case that the universe size is small so we can track the sets A_1, A_2, \dots, A_k for some $k = \text{poly}(\log T)$, where A_i is the set of all elements whose frequency is exactly *i*. We can construct streams S_1, S_2, \dots, S_k with numbers in $\{-1, 0, 1\}$. During the stream, when we see an input element *a*, and if *a* is in the set A_i , then we move *a* to the set A_{i+1} due to the increase of the frequency of *a*. At the same time, we append -1 to the stream S_i , append 1 to the stream S_{i+1} and append 0 to S_j for $j \neq i, i+1$. It is easy to check that the sum of S_l is always the same as $|A_l|$, i.e., the number of elements which have frequency exactly *l*. Furthermore, similar to the analysis for counting distinct elements, if we change an element in the input stream, each S_l might be affected by at most 4 entries. Thus, the total sensitivity of (S_1, S_2, \dots, S_k) is at most O(k). Therefore, we can use the DP continual release summing to estimate the sum of each S_l with additive error poly $(\log T)$. Thus, we can estimate $|G_i|$ for each *i* with $\alpha^i \ll \text{poly}(\log T)$ with additive error poly $(\log T)$ and will only introduce at most poly $(\log T)$ additive error in approximating the ℓ_p frequency moment.

Now let us go back to the case that the size of the universe is large. In this case, we can use the similar hashing and subsampling technique discussed for counting distinct elements to estimate $|A_l|$ for each $l \in [k]$.

1.4 Related Work

Dwork et al. [2010a] and Hubert Chan et al. [2010] initiated the study of differential privacy in the continual release model, and proposed the binary tree mechanism for computing summations. Bolot et al. [2013] and Perrier et al. [2019] generalized their results to decayed summations, counting distinct elements without space constraints and summations with real-valued data. Song et al. [2018], Fichtenberger et al. [2021] studied graph problems under the differentially private continual release model. Jain et al. [2012], Smith and Thakurta [2013], Agarwal and Singh [2017] studied differentially private online learning. Jain et al. [2021] gave the first polynomial separation in terms of error between the continual release model and the batch model under differential privacy. Upadhyay [2019] studied heavy hitters in the differentially private sliding window model.

Differentially private frequency moment estimation for p = 0, 1, 2 (without continual releases) has been well-studied Mir et al. [2011], Dwork et al. [2010b], Blocki et al. [2012], Sheffet [2017], Choi et al. [2020], Smith et al. [2020], Bu et al. [2021]. Wang et al. [2022] studied frequency moment estimation (without continual releases) for $p \in (0, 1]$ with low space complexity. Recent concurrent independent work Blocki et al. [2022] studies $p \in [0, \infty)$ with low space complexity but not in continual release setting as well. The differentially private ℓ_1 heavy hitters problem is studied by Mir et al. [2011], Dwork et al. [2010b] in the low space streaming setting but not in the continual release setting. Chan et al. [2012] studied differentially private ℓ_1 heavy hitters problem in the low space continual release streaming setting. But it is not clear how to extend their techniques to l_p case for $p \neq 1$.

 ℓ_p Frequency moment estimation and ℓ_p heavy hitters are heavily studied in the non-private streaming literature. For ℓ_p frequency moment estimation, the problem can be solved by e.g. Flajolet and Martin [1985], Flajolet et al. [2007], Durand and Flajolet [2003] for p = 0, Alon et al. [1996], Charikar et al. [2002], Thorup and Zhang [2004] for p = 2, Kane et al. [2010], Indyk [2006], Li [2008], Kane et al. [2011] for $p \in (0, 2)$ and Indyk and Woodruff [2005], Andoni et al. [2011], Andoni [2017] for p > 2. For ℓ_p heavy hitters, the problem can be solved by e.g., Cormode and Muthukrishnan [2005], Misra and Gries [1982] for p = 1, Charikar et al. [2002] for p = 2, Jowhari et al. [2011] for $p \in (0, 2)$, and Indyk and Woodruff [2005], Andoni et al. [2011] for p > 2.

2 Preliminaries

2.1 Notation

In this paper, for $n \ge 1$, we use [n] to denote the set $\{1, 2, \dots, n\}$. If there is no ambiguity, for $i \le j \in \mathbb{Z}$, we sometimes use [i, j] to denote the set of integers $\{i, i + 1, \dots, j\}$ instead of the set of real numbers $\{a \in \mathbb{R} \mid i \le a \le j\}$. We use $f_a(a_1, a_2, \dots, a_k)$ to denote the frequency of a in the sequence (a_1, a_2, \dots, a_k) , i.e., $f_a(a_1, a_2, \dots, a_k) = |\{i \in [k] \mid a_i = a\}|$. If the sequence (a_1, a_2, \dots, a_k) is clear in the context, we use f_a to denote the frequency of a for short. We use $\mathbf{1}_{\mathcal{E}}$ to denote a indicator, i.e., $\mathbf{1}_{\mathcal{E}} = 1$ if condition \mathcal{E} holds and $\mathbf{1}_{\mathcal{E}} = 0$ if condition \mathcal{E} does not hold.

For $\alpha \geq 1, \gamma \geq 0$, if $\frac{1}{\alpha} \cdot x - \gamma \leq y \leq \alpha \cdot x + \gamma$, then y is an (α, γ) -approximation to x. If y is a $(1, \gamma)$ -approximation to x, we say y is an approximation to x with additive error γ . If y is an $(\alpha, 0)$ -approximation to x, we say y is an α -approximation to x. We use $a \pm b$ to denote the real number interval [a-|b|,a+|b|]. For a set S of real numbers, we use $S \pm b$ to denote the set $\bigcup_{a \in S} a \pm b$, and use $S \cdot c$ to denote the set $\bigcup_{a \in S} a \cdot c$. We use Lap(b) to denote the Laplace distribution with scale b, i.e., Lap(b) has density function given by $\frac{1}{2b} \exp(|x|/b)$

2.2 Functions to Compute

We study several fundamental functions in the streaming literature. When the inputs are integers, we consider the summing problem over a (sub-)stream (a_i, \dots, a_j) :

• Sum of numbers: $\mathbf{Sum}(a_i, \cdots, a_j) := \sum_{k=i}^j a_k$

When the inputs are from $\mathcal{U} \cup \{\bot\}$, we consider the functions $g(a_i, \cdots, a_j)$ that are based on the frequencies of the elements in a (sub-)stream (a_i, \cdots, a_j) .

- Count of non-empty elements: $||(a_i, ..., a_j)||_1 := \sum_{a \in \mathcal{U}} f_a(a_i, ..., a_j)$.
- The number of distinct elements: $||(a_i,...,a_j)||_0 := \sum_{a \in \mathcal{U}} \mathbf{1}_{f_a(a_i,...,a_j)>0}$.
- ℓ_p -Frequency moment: $||(a_i, ..., a_j)||_p^p := \sum_{a \in \mathcal{U}} f_a(a_i, ..., a_j)^p$.
- ℓ_p -Heavy hitters: $(1/k)-\ell_p$ -HH $(a_i,...,a_j) := \{a \in \mathcal{U} \mid f_a(a_i,\cdots,a_j)^p \ge ||(a_i,\cdots,a_j)||_p^p/k\}.$

Note that $||(a_i, ..., a_j)||_1$ is a special case of $\mathbf{Sum}(a_i, \cdots, a_j)$ with binary inputs.

2.3 Differential Privacy

Neighboring streams: Consider two streams $S = (a_1, a_2, \dots, a_T)$ and $S' = (a'_1, a'_2, \dots, a'_T)$. If there is at most one timestamp $t \in [T]$ such that (1). $|a_t - a'_t| \leq 1$ (only required when the inputs are treated as integers) (2). $\forall i \neq t, a_i = a'_i$, then we say S and S' are neighboring streams.

Definition 2.1 (Differential Privacy). We say algorithm A is ε -DP, if for any two neighboring streams S, S', and any output set O,

$$\Pr[A(\mathcal{S}) \in \mathcal{O}] \le e^{\varepsilon} \cdot \Pr[A(\mathcal{S}') \in \mathcal{O}].$$

Note that in the continual release model, the output A(S) mentioned in Definition 2.1 is the entire output history of the algorithm A over stream S at every timestamp.

Definition 2.2 (Distance between streams). Consider two streams S and S'. If d is the minimum number such that there exists a sequence of streams S_0, S_1, \dots, S_d where $S_0 = S, S_d = S'$ and $\forall i \in [d], S_i$ and S_{i-1} are neighboring streams, then the distance between S and S' is dist(S, S') = d.

Definition 2.3 (Sensitivity of a stream mapping). Let \mathcal{F} be a mapping which maps a given input stream \mathcal{S} to a tuple of streams $(\mathcal{F}_1(\mathcal{S}), \mathcal{F}_2(\mathcal{S}), \dots, \mathcal{F}_k(\mathcal{S}))$. The sensitivity of \mathcal{F} is the minimum value s such that for any two neighboring streams \mathcal{S} and $\mathcal{S}', \sum_{i \in [k]} \text{dist}(\mathcal{F}_i(\mathcal{S}), \mathcal{F}_i(\mathcal{S}')) \leq s$.

Theorem 2.4 (Composition Dwork et al. [2014]). Let \mathcal{F} be a mapping which maps a given input stream \mathcal{S} to a tuple of streams $(\mathcal{F}_1(\mathcal{S}), \mathcal{F}_2(\mathcal{S}), \cdots, \mathcal{F}_k(\mathcal{S}))$. Let A_1, A_2, \cdots, A_k be $k \in DP$ algorithms. Let A be an algorithm such that $A(\mathcal{S}) = M(A_1(\mathcal{F}_1(\mathcal{S})), A_2(\mathcal{F}_2(\mathcal{S})), \cdots, A_k(\mathcal{F}_k(\mathcal{S})))$ for some function $M(\cdot)$. Then the algorithm A is $(s \in)$ -DP.

Example usage of Theorem 2.4: Some example usage of Theorem 2.4 in our paper are presented as the following:

- Composition of multiple algorithms over the input stream: Suppose each of A_1, A_2, \dots, A_k is ε -DP, then for any function $M(\cdot), M(A_1(\mathcal{S}), A_2(\mathcal{S}), \dots, A_k(\mathcal{S}))$ is $(k\varepsilon)$ -DP.
- Composition of algorithms on disjoint sub-streams: For an input stream $\mathcal{S} = (a_1, a_2, \dots, a_T)$, we partition \mathcal{S} into $\mathcal{S}_1 = (a_{1,1}, a_{1,2}, \dots, a_{1,T}), \mathcal{S}_2 = (a_{2,1}, a_{2,2}, \dots, a_{2,T}), \dots, \mathcal{S}_k = (a_{k,1}, a_{k,2}, \dots, a_{k,T})$, i.e., $\forall t \in [T]$, there is only one $i \in [k]$ such that $a_{i,t} = a_t$, and $\forall i' \neq i, a_{i,t} = \bot$ (or 0). It is clear that such partitioning has sensitivity 1. Thus, if each of A_1, A_2, \dots, A_k is an ε -DP algorithms, then for any function $M(\cdot), M(A_1(\mathcal{S}_1), A_2(\mathcal{S}_2), \dots, A_k(\mathcal{S}_k))$ is ε -DP.

2.4 Streaming Continual Release Summing and Counting

For summing problem in the streaming continual release model, the binary tree mechanism was proposed in Dwork et al. [2010a], Hubert Chan et al. [2010]. It gets poly-logarithmic additive error and uses logarithmic space. Furthermore, it can handle negative numbers in the stream.

Theorem 2.5 (Dwork et al. [2010a], Hubert Chan et al. [2010]). Let $\varepsilon \ge 0, \xi \in (0, 0.5)$, there is an ε -DP algorithm for summing in the streaming continual release model. With probability $1 - \xi$, the additive error of the output for every timestamp $t \in [T]$ is always at most $O\left(\frac{1}{\varepsilon}\log^{2.5}(T)\log\left(\frac{1}{\xi}\right)\right)$. The algorithm uses $O(\log(T))$ space.

2.5 Probability Tools

Lemma 2.6 (Bellare and Rompel [1994]). Let $\lambda \ge 4$ be an even integer. Let X be the sum of n λ -wise independent random variables which take values in [0,1]. Let $\mu = E[X]$ and A > 0. Then we have

$$\Pr\left[|X - \mu| > A\right] \le 8 \cdot \left(\frac{\lambda \mu + \lambda^2}{A^2}\right)^{\lambda/2}$$

Lemma 2.7 (Median trick to boost success probability). Suppose X is a random estimator of value v such that with probability at least 2/3, X is an (α, γ) -approximation to v. Then, for $\xi \in (0, 0.5)$, if we draw $k = \lceil 50 \log(1/\xi) \rceil$ independent copies X_1, X_2, \dots, X_k of X, with probability at least $1 - \xi$, the median of X_1, X_2, \dots, X_k is an (α, γ) -approximation to v.

Proof. We say X_i is good if X_i is an (α, γ) -approximation to v. If the median is not good, then $\sum_{i \in [k]} \mathbf{1}_{X_i \text{ is good}} < \frac{k}{2}$. By Chernoff bound, $\Pr[\text{median is not good}] \leq \Pr[\sum_{i \in [k]} \mathbf{1}_{X_i \text{ is good}} < \frac{k}{2}] \leq \Pr[E[\sum_{i \in [k]} \mathbf{1}_{X_i \text{ is good}}] - \frac{k}{2} > \frac{k}{6}] \leq e^{-k/48} \leq \xi.$

3 Continual Released Summing with Better Additive Error

In this section, we show that if we allow a relative approximation and the input stream only contains non-negative numbers, then we can have a continual released summing algorithm with additive error better than Theorem 2.5.

Algorithm 1: Grouping Stream of Counts.

Input: A stream of non-negative numbers c_1, c_2, \dots, c_T DP parameter $\varepsilon > 0$, approximation parameter $\eta \in (0, 0.5)$, and failure probability $\xi \in (0, 1)$. Output: A stream of groups with grouped noisy counts $(\hat{c}_1, \hat{c}_2, \dots, \hat{c}_T)$ Let $\varepsilon_0 \leftarrow \varepsilon/2$. Initialize a group index $i \leftarrow 1$, current group $G_1 \leftarrow \emptyset$, and threshold $\tau_1 \leftarrow \left(\frac{1}{\eta} + 1\right) \cdot \frac{\tau}{\varepsilon_0} \cdot \ln(3 \cdot T/\gamma) + \operatorname{Lap}(2/\varepsilon_0)$. // G_i is used for analysis only. for t = 1 to T do $G_i \leftarrow G_i \cup \{t\}$. Let $\nu_t \leftarrow \operatorname{Lap}(4/\varepsilon_0)$. if $\nu_t + \sum_{j \in G_i} c_j \ge \tau_i$ then $\hat{c}_t \leftarrow \operatorname{Lap}(1/\varepsilon_0) + \sum_{j \in G_i} c_j$. $i \leftarrow i + 1$. $\tau_i \leftarrow \left(\frac{1}{\eta} + 1\right) \cdot \frac{\tau}{\varepsilon_0} \cdot \ln(3 \cdot T/\gamma) + \operatorname{Lap}(2/\varepsilon_0)$. else $\hat{c}_i \leftarrow 0$. end end

Lemma 3.1. The output stream $\hat{c}_1, \hat{c}_2, \cdots, \hat{c}_T$ of Algorithm 1 is ε -DP.

The proof idea is to iteratively apply sparse vector technique. We put the proof into Appendix A.1.

Lemma 3.2. Let $\hat{c}_1, \hat{c}_2, \dots, \hat{c}_T$ be the output stream of Algorithm 1. Then with probability at least $1 - \xi, \forall l, r \text{ satisfying } 1 \leq l \leq r \leq T$,

$$(1-\eta)\sum_{j=l}^{r}c_j - \left(\frac{1}{\eta} + 4\right) \cdot \frac{7}{\varepsilon_0} \cdot \ln\left(3 \cdot T/\xi\right) \le \sum_{j=l}^{r}\hat{c}_j \le (1+\eta)\sum_{j=l}^{r}c_j + \left(\frac{1}{\eta} + 4\right) \cdot \frac{7}{\varepsilon_0} \cdot \ln\left(3 \cdot T/\xi\right).$$

We put the proof of Lemma 3.2 into Appendix A.2

Theorem 3.3 (Summing of a non-negative stream). Let $\varepsilon \ge 0, \xi \in (0, 0.5)$, there is an ε -DP algorithm for summing in the streaming continual release model. If the input numbers are guaranteed to be non-negative, with probability at least $1 - \xi$, the output is always a $\left(1 + \eta, O\left(\frac{\log(T/\xi)}{\varepsilon\eta}\right)\right)$ -approximation to the summing problem at any timestamp $t \in [T]$. The algorithm uses space O(1).

Proof. According to Lemma 3.1, the output stream of Algorithm 1 is ε -DP, thus, we only need to solve non-private summing over $(\hat{c}_1, \hat{c}_2, \dots, \hat{c}_T)$. The approximation guarantee is given by Lemma 3.2. Note that we do not need to store G_i , we only need to maintain the sum of numbers in G_i at any timestamp. Thus, the total space needed is O(1).

4 Continual Released Number of Distinct Elements

In this section, we show how to use ε -DP streaming continual release summing to solve ε -DP streaming continual release number of distinct elements. In Section 4.1, we show how to estimate

the number of distinct elements if the universe is small. In Section 4.2, we reduce the number of distinct elements of a large universe to the number of distinct elements of a small universe via subsampling.

4.1 Number of Distinct Elements for Small Universe

Algorithm 2: Number of Distinct Elements for Small Universe **Input:** A stream S of elements $a_1, a_2, \dots, a_T \in U \cup \{\bot\}$ with guarantee that $|\mathcal{U}| \leq m$. **Parameters** : Relative approximation factor $\alpha \geq 1$ and additive approximation factor $\gamma \geq 0$ depending on the streaming continual release summing algorithm. //See Theorem 2.5. **Output:** Estimation of the number of distinct elements at every timestamp t. Initialize an empty stream \mathcal{C} . Let $S \leftarrow \emptyset$. for each a_t in the stream S do if $a_t \notin S$ and $a_t \neq \perp$ then $S \leftarrow S \cup \{a_t\}.$ Append 1 to the end of the stream C. end else Append 0 to the end of the stream C. end Output an (α, γ) -approximation to the total counts of \mathcal{C} . \mathbf{end}

Lemma 4.1. At the end of any time $t \in [T]$, the output of Algorithm 2 is an (α, γ) -approximation to the number of distinct elements.

Proof. Since we append 1 to the stream C if and only if we see a new non-empty element, the total counts in C is always equal to the number of distinct elements at the end of any time $t \in [T]$. Thus, an (α, γ) -approximation to the total counts of C is an (α, γ) -approximation to the number of distinct elements.

Lemma 4.2. If the algorithm to continually release the approximate total counts of C in Algorithm 2 is ε -DP, Algorithm 2 is 5ε -DP in the continual release model.

Proof. Consider two neighboring stream $S = (a_1, a_2, \dots, a_T)$ and $S' = (a'_1, a'_2, \dots, a'_T)$ of elements in \mathcal{U} where they only differ at timestamp t, i.e., $a_t \neq a'_t$. Let us consider the difference between the generated count stream $\mathcal{C} = (c_1, c_2, \dots, c_T)$ and $\mathcal{C}' = (c'_1, c'_2, \dots, c'_T)$.

Consider any timestamp $i \in [T]$, if $a_i \neq a_t$ and $a_i \neq a'_t$, it is easy to verify that $c_i = c'_i$. Suppose $i \neq t$. If $a_i = a_t = u \in \mathcal{U}$ but a_i is at least the third appearance of u in \mathcal{S} , then $a'_i = a_i = u$ is at least the second appearance of u in \mathcal{S}' which implies that $c_i = c'_i = 0$. Similarly, if $a_i = a'_t = u \in \mathcal{U}$ is at least third appearance of u in \mathcal{S} , we can show $c_i = c'_i = 0$ as well. Thus the sensitivity of the stream \mathcal{C} is at most 5: only when i = t or a_i is the first/second appearance of $a_t, a_{t'}, c_i$ might be different from c'_i . Therefore, if we use an ε -DP algorithm to continually release the total counts of \mathcal{C} , the continually released output of Algorithm 2 is $(5 \cdot \varepsilon)$ -DP.

Theorem 4.3. Let $\varepsilon \ge 0, \xi \in (0, 0.5)$, suppose there is an ε -DP streaming continual release summing algorithm (for stream of non-negative numbers) which uses space J and with probability at least $1-\xi$ always outputs an (α, γ) -approximation for every timestamp. There is a (5ε) -DP algorithm for the number of distinct elements of streams with universe size at most m in the streaming continual release model. With probability at least $1-\xi$, the algorithm always outputs an (α, γ) -approximation for every timestamp terms an (α, γ) -approximation for every timestamp terms an (α, γ) -approximation for every timestamp.

Proof. Consider Algorithm 2. The approximation guarantee is proven by Lemma 4.1. The DP guarantee is proven by Lemma 4.2. In the remaining of the proof, we only need to prove the space usage. Since $|\mathcal{U}| \leq m$, the space needed to maintain set S is at most m. The space needed to continually release an (α, γ) -approximation to the summing problem over \mathcal{C} is at most \mathcal{J} . Thus, the total space needed is at most O(m + J).

By combining the above theorem with Theorem 2.5, we obtain the following corollary.

Corollary 4.4 (Streaming continual release distinct elements for small universe). There is an ε -DP algorithm for the number of distinct elements of streams with universe size at most m in the streaming continual release model. With probability at least $1 - \xi$, the additive error of the output is always at most $O\left(\frac{1}{\varepsilon}\log^{2.5}(T)\log\left(\frac{1}{\xi}\right)\right)$ for every timestamp $t \in [T]$. The algorithm uses $O(m + \log(T))$ space.

By combining Theorem 4.3 with Theorem 3.3, we obtain the following corollary:

Corollary 4.5 (Streaming continual release distinct elements for small universe, better additive error). There is an ε -DP algorithm for the number of distinct elements of streams with universe size at most m in the streaming continual release model. With probability at least $1-\xi$, the additive error of the output is always an $\left(1+\eta, O\left(\frac{\log(T/\xi)}{\varepsilon\eta}\right)\right)$ -approximation to the number of distinct elements for every timestamp $t \in [T]$. The algorithm uses O(m) space.

4.2 Number of Distinct Elements for General Universe

```
Algorithm 3: Number of Distinct Elements via Subsampling
  Input: A stream S of elements a_1, a_2, \dots, a_T \in U \cup \{\bot\}, and a error parameter \eta \in (0, 0.5).
  Parameters : Relative approximation factor \alpha \geq 1 and additive approximation factor \gamma \geq 0
                      depending on the streaming continual release algorithm for number of distinct
                      elements of streams with small universe of elements.
                                                                                                      //See Theorem 4.3.
  Output: Estimation of the number of distinct elements \|S\|_0.
  L \leftarrow \lceil \log \min(|\mathcal{U}|, T) \rceil, \lambda \leftarrow 2 \log(1000L), m \leftarrow 100L \cdot \left(16\alpha \max\left(\gamma/\eta, 32\alpha\lambda/\eta^2\right)\right)^2.
  Let h: \mathcal{U} \to [m] be a pairwise independent hash function.
                   //Here we treat [m] as a universe of elements with size m instead of a set of integers.
  Let g: \mathcal{U} \to [L] \cup \{\bot\} be a \lambda-wise independent hash function and
   \forall a \in \mathcal{U}, i \in [L], \Pr[g(a) = i] = 2^{-i}, \Pr[g(a) = \bot] = 2^{-L}.
  Initialize empty streams S_1, S_2, \cdots, S_L.
  for each a_t in the stream S do
      for i \in [L] do
           if a_t \neq \perp and g(a_t) = i then
            Append h(a_t) to the end of the stream S_i.
           \mathbf{end}
           else
            Append \perp to the end of the stream S_i.
           \mathbf{end}
       \mathbf{end}
      \forall i \in [L], compute \hat{s}_i which is an (\alpha, \gamma)-approximation to \|S_i\|_0.
      Find the largest i \in [L] such that \hat{s}_i \geq \max(\gamma/\eta, 32\alpha\lambda/\eta^2), and output \hat{s}_i \cdot 2^i.
      If such i does not exist, output 0.
  end
```

Lemma 4.6. Consider any timestamp $t \in [T]$. Let v be the output of Algorithm 3. With probability at least 0.9, v is a $((1+O(\eta))\alpha, O(\alpha^2 \max(\gamma/\eta, \alpha \log(L)/\eta^2)))$ -approximation to $||(a_1, a_2, \cdots, a_t)||_0$.

To prove Lemma 4.6, we need following intermediate statements. We consider a timestamp $t \in [T]$. Let S denote the input stream at timestamp t, i.e., $S = (a_1, a_2, \dots, a_t)$. Let $G_i = \{a_j \mid g(a_j) = i, j \leq t\}$ for $i \in [L]$.

Claim 4.7. $\forall i \in [L], if \|\mathcal{S}\|_0 \geq 2^i \cdot 4\lambda/\eta^2, \Pr[|G_i| \in (1 \pm \eta) \cdot \|\mathcal{S}\|_0/2^i] \geq 1 - 0.01/L.$ Otherwise, $\Pr[||G_i| - \|\mathcal{S}\|_0/2^i| \leq 4\lambda/\eta] \geq 1 - 0.01/L.$

Proof. Suppose $\|S\|_0 \ge 2^i \cdot 4\lambda/\eta^2$. Due to Lemma 2.6, we have:

$$\Pr\left[\left||G_i| - \|\mathcal{S}\|_0/2^i\right| > \eta \cdot \|\mathcal{S}\|_0/2^i\right]$$
$$\leq 8 \cdot \left(\frac{\lambda \cdot \|\mathcal{S}\|_0/2^i + \lambda^2}{\left(\eta \cdot \|\mathcal{S}\|_0/2^i\right)^2}\right)^{\lambda/2}$$
$$\leq 0.01/L,$$

where the last inequality follows from that $\lambda = 2 \cdot \log(1000L)$ and $\|S\|_0 \ge 2^i \cdot 4\lambda/\eta^2$. Suppose $\|S\|_0 \le 2^i \cdot 4\lambda/\eta^2$, By applying Lemma 2.6 again, we have:

$$\Pr\left[\left||G_i| - \|\mathcal{S}\|_0/2^i\right| > 4\lambda/\eta\right]$$

$$\leq 8 \cdot \left(\frac{\lambda \cdot \|\mathcal{S}\|_0 / 2^i + \lambda^2}{(4\lambda/\eta)^2}\right)^{\lambda/2} \leq 0.01/L,$$

where the last inequality follows from $\|S\|_0/2^i \leq 4\lambda/\eta^2$ and $\lambda = 2 \cdot \log(1000L)$.

Claim 4.8. For $i \in [L]$, conditioning on $|G_i| \leq 16\alpha \max(\gamma/\eta, 32\alpha\lambda/\eta^2)$, the probability that $|G_i| = \|S_i\|_0$ is at least 1 - 0.01/L.

Proof. Since h is pairwise independent, $\forall a, b \in G_i$, $\Pr[h(a) = h(b)] = 1/m$. Since $m = 100L \cdot (16\alpha \max(\gamma/\eta, 32\alpha\lambda/\eta^2))^2$, $\Pr[\exists a \neq b \in G_i, h(a) = h(b)] \leq |G_i|^2/m \leq 0.01/L$ by a union bound.

Let \mathcal{E} be the event that both of the following hold:

- 1. $\forall i \in [L]$ with $2^i \cdot 4\lambda/\eta^2 \leq \|\mathcal{S}\|_0, |G_i| \in (1 \pm \eta) \cdot \|\mathcal{S}\|_0/2^i$.
- 2. $\forall i \in [L]$ with $2^i \cdot 4\lambda/\eta^2 > \|\mathcal{S}\|_0, \ |G_i| \in \|\mathcal{S}\|_0/2^i \pm 4\lambda/\eta$.

According to Claim 4.7, \mathcal{E} happens with probability at least 0.99. Let \mathcal{E}' be the event that $\forall i \in [L]$ with $|G_i| \leq 16\alpha \max(\gamma/\eta, 32\alpha\lambda/\eta^2)$, $|G_i| = \|\mathcal{S}_i\|_0$. According to Claim 4.8, \mathcal{E}' happens with probability at least 0.99.

Next, we are going to prove Lemma 4.6.

Proof of Lemma 4.6. In this proof, we condition on both events \mathcal{E} and \mathcal{E}' . Note that the probability that both \mathcal{E} and \mathcal{E}' happen is at least 0.98.

Consider the case that $\|\mathcal{S}\|_0 \geq 8\alpha \cdot \max(\gamma/\eta, 32\alpha\lambda/\eta^2)$. Let $i^* \in [L]$ be the largest value such that $\|\mathcal{S}\|_0/2^{i^*} \geq 4\alpha \cdot \max(\gamma/\eta, 32\alpha\lambda/\eta^2)$. According to event \mathcal{E} , we have $|G_{i^*}| \in (1 \pm \eta) \cdot \|\mathcal{S}\|_0/2^{i^*}$. Due to our choice of i^* , we have $\|\mathcal{S}\|_0/2^{i^*} \leq 8\alpha \cdot \max(\gamma/\eta, 32\alpha\lambda/\eta^2)$. Thus, $|G_{i^*}| \leq 16\alpha \max(\gamma/\eta, 32\alpha\lambda/\eta^2)$. According to event \mathcal{E}' , we have $\|\mathcal{S}_{i^*}\|_0 = |G_{i^*}|$. Therefore, we have $\hat{s}_{i^*} \geq \|\mathcal{S}_{i^*}\|_0/\alpha - \gamma \geq |G_{i^*}|/\alpha - \gamma$. Since $|G_{i^*}| \geq 2\alpha \cdot \max(\gamma/\eta, 32\alpha\lambda/\eta^2)$, we have $\hat{s}_{i^*} \geq \max(\gamma/\eta, 32\alpha\lambda/\eta^2)$. Therefore, Algorithm 3 will output $\hat{s}_{i'} \cdot 2^{i'}$ for some $i' \geq i^*$. Due to event \mathcal{E} , we know $|G_{i'}| \leq \max(2 \cdot \|\mathcal{S}\|_0/2^{i^*}, \|\mathcal{S}\|_0/2^{i^*} + 4\lambda/\eta) \leq 16\alpha \max(\gamma/\eta, 32\alpha\lambda/\eta^2)$. According to event \mathcal{E}' , we have $\|\mathcal{S}_{i'}\|_0 = |G_{i'}|$. Since the algorithm outputs $\hat{s}_{i'} \cdot 2^{i'}$, we know that $\hat{s}_{i'} \geq \max(\gamma/\eta, 32\alpha\lambda/\eta^2)$ which implies that

$$|G_{i'}| = \|\mathcal{S}_{i'}\|_0$$

$$\geq (\hat{s}_{i'} - \gamma)/\alpha$$

$$\geq (1 - \eta)\hat{s}_{i'}/\alpha$$

$$\geq 16\lambda/\eta^2.$$

According to event \mathcal{E} , we have $|G_{i'}| \in (1 \pm \eta) \cdot ||\mathcal{S}||_0/2^{i'}$. Thus, we have

$$\hat{s}_{i'} \leq \alpha \|\mathcal{S}_{i'}\|_0 + \gamma$$
$$= \alpha |G_{i'}| + \gamma$$
$$\leq \frac{\alpha}{1 - \eta} \cdot |G_{i'}|$$

$$\leq \frac{1+\eta}{1-\eta} \cdot \alpha \cdot \|\mathcal{S}\|_0 / 2^{i'}$$

$$\leq (1+4\eta)\alpha \|\mathcal{S}\|_0 / 2^{i'},$$

where the second inequality follows from $\hat{s}_{i'} \ge \gamma/\eta$ and the last inequality follows from $\eta \le 0.5$. Similarly, we have:

$$\begin{split} \hat{s}_{i'} &\geq \|\mathcal{S}_{i'}\|_0/\alpha - \gamma \\ &= |G_{i'}|/\alpha - \gamma \\ &\geq |G_{i'}|/((1+\eta)\alpha) \\ &\geq \frac{1-\eta}{1+\eta} \cdot \frac{1}{\alpha} \cdot \|\mathcal{S}\|_0/2^{i'} \\ &\geq (1-4\eta)/\alpha \cdot \|\mathcal{S}\|_0/2^{i'}, \end{split}$$

where the second inequality follows from $\hat{s}_{i'} \geq \gamma/\eta$.

Next, consider the case that $\|S\|_0 < 8\alpha \cdot \max(\gamma/\eta, 32\alpha\lambda/\eta^2)$. Algorithm 3 either outputs 0 or outputs $\hat{s}_{i'} \cdot 2^{i'}$ for some $i' \in [L]$. Suppose it outputs $\hat{s}_{i'} \cdot 2^{i'}$. We have $\hat{s}_{i'} \ge \max(\gamma/\eta, 32\alpha\lambda/\eta^2)$, which implies that

$$|G_{i'}| \ge \|\mathcal{S}_{i'}\|_0 \ge (\hat{s}_{i'} - \gamma)/\alpha \ge (1 - \eta)\hat{s}_{i'}/\alpha \ge 16\lambda/\eta^2.$$

According to event \mathcal{E} , we have $|G_{i'}| \leq 2 \cdot ||\mathcal{S}||_0 / 2^{i'}$. Therefore

$$\begin{split} \hat{s}_{i'} \cdot 2^{i'} &\leq (\alpha \| \mathcal{S}_{i'} \|_0 + \gamma) \cdot 2^{i'} \\ &\leq \frac{\alpha}{1 - \eta} \cdot \| \mathcal{S}_{i'} \|_0 \cdot 2^{i'} \\ &\leq \frac{\alpha}{1 - \eta} \cdot |G_{i'}| \cdot 2^{i'} \\ &\leq 4\alpha \| \mathcal{S} \|_0 \\ &\leq 32\alpha^2 \max(\gamma/\eta, 32\alpha\lambda/\eta^2). \end{split}$$

Theorem 4.9. Let $\varepsilon \ge 0, \xi, \xi' \in (0, 0.5), \eta \in (0, 0.5)$, suppose there is an ε -DP algorithm for the number of distinct elements of streams with element universe size at most $100 \log(\min(|\mathcal{U}|, T)) \cdot (16\alpha \max(\gamma/\eta, 32\alpha \cdot 2\log(1000 \log(\min(|\mathcal{U}|, T)))/\eta^2))^2$ in the streaming continual release model which uses space J and with probability at least $1 - \xi$ always outputs an (α, γ) -approximation for every timestamp. There is an $(\varepsilon' = \lceil 50 \log(T/\xi') \rceil \varepsilon)$ -DP algorithm for the number of distinct elements of streams with element universe \mathcal{U} in the streaming continual release model. With probability at least $1 - \xi' - \log(\min(|\mathcal{U}|, T)) \cdot \lceil 50 \log(T/\xi') \rceil \cdot \xi$, the algorithm always outputs an $((1 + O(\eta))\alpha, O(\alpha^2 \max(\gamma/\eta, \alpha \log \log(\min(|\mathcal{U}|, T))/\eta^2)))$ -approximation for every timestamp $t \in [T]$. The algorithm uses $O(J \cdot \log(\min(|\mathcal{U}|, T)) \cdot \log(T/\xi'))$ space.

Proof. According to our construction of S_1, S_2, \dots, S_L and the definition of the sensitivity (Definition 2.3), (S_1, S_2, \dots, S_L) has sensitivity 1. Since the algorithm to report \hat{s}_i for each $i \in [L]$ is

 ε -DP, Algorithm 3 is ε -DP. According to Lemma 4.6, for any $t \in [T]$, condition on that \hat{s}_i is an (α, γ) approximation to $\|\mathcal{S}\|_0$, with probability at least 0.9, the output is a $((1+O(\eta))\alpha, O(\alpha^2 \max(\gamma/\eta, \alpha \log \log(\min(|\mathcal{U}|, T))/\eta^2 \log(\gamma/\eta, \alpha)))$ approximation. To boost the probability to $1 - \xi'$ such that $\forall t \in [T]$ the approximation guarantee
always holds, we need to run $[50 \log(T/\xi')]$ independent copies of Algorithm 3 and take the median
of the outputs at every timestamp. Thus, the overall algorithm is $([50 \log(T/\xi')] \cdot \varepsilon)$ -DP.

Next, consider the success probability that every \hat{s}_i is an (α, β) -approximation to $\|S_i\|_0$. By taking a union bound over all *i* and all independent copies of Algorithm 3, the success probability is at least $1 - \log(\min(|\mathcal{U}|, T)) \cdot [50 \log(T/\xi')] \cdot \xi$.

Finally, consider the space usage. Consider each running copy of Algorithm 3. Hashing function $h(\cdot)$ takes O(1) space. Hashing function $g(\cdot)$ takes $O(\lambda) = O(\log \log(\min(|\mathcal{U}|, T)))$ space. To continually release \hat{s}_i for all i, we need to use $O(J \cdot \log(\min(|\mathcal{U}|, T)))$ space. Thus, the total space needed for all copies is at most $O(J \cdot \log(\min(|\mathcal{U}|, T)) \cdot \log(T/\xi'))$.

By plugging Corollary 4.4 into above theorem with $\xi = \frac{\xi'/2}{\log(\min(|\mathcal{U}|,T))\cdot[50\log(2T/\xi')]}, \varepsilon = \frac{\varepsilon'}{[50\log(2T/\xi')]}, \varepsilon = \frac{\varepsilon'}{[50\log(2T/\xi')]}, \varepsilon = \frac{\varepsilon'}{[50\log(2T/\xi')]}, \alpha = 1, \gamma = O(\frac{1}{\varepsilon}\log^{2.5}(T)\log(1/\xi)) \text{ and } J = O(\log(T) + 100\log(\min(|\mathcal{U}|,T)) \cdot (16\alpha\max(\gamma/\eta, 32\alpha \cdot 2\log(1000\log(\min(|\mathcal{U}|,T)))/\eta^2))^2), we get the following corollary.$

Corollary 4.10 (Streaming continual release distinct elements). For $\eta \in (0, 0.5)$, there is an ε -DP algorithm for the number of distinct elements of streams with element universe \mathcal{U} in the streaming continual release model. With probability at least $1 - \xi$, the output is always a $(1 + \eta, O\left(\max\left(\frac{\log(T/\xi)\log^{2.5}(T)\log(1/\xi)\log\log(T/\xi)}{\eta\varepsilon}, \frac{\log\log T}{\eta^2}\right)\right))$ -approximation for every timestamp $t \in [T]$. The algorithm uses poly $\left(\frac{\log(T/\xi)}{\eta\min(\varepsilon,1)}\right)$ space.

By plugging Corollary 4.5 into Theorem 4.9 with $\xi = \frac{\xi'/2}{\log(\min(|\mathcal{U}|, T)) \cdot \lceil 50 \log(2T/\xi') \rceil}, \varepsilon = \frac{\varepsilon'}{\lceil 50 \log(2T/\xi') \rceil}, \varepsilon = \frac{\varepsilon'}{\lceil 50 \log(2T/\xi') \rceil}, \varepsilon = 1 + \eta, \gamma = O\left(\frac{\log(T/\xi)}{\varepsilon \eta}\right)$ and

 $J = O(100\log(\min(|\mathcal{U}|, T)) \cdot (16\alpha \max(\gamma/\eta, 32\alpha \cdot 2\log(1000\log(\min(|\mathcal{U}|, T)))/\eta^2))^2),$

we get the following corollary.

Corollary 4.11 (Streaming continual release distinct elements, better dependence in $\log(T)$). For $\eta \in (0, 0.5)$, there is an ε -DP algorithm for the number of distinct elements of streams with element universe \mathcal{U} in the streaming continual release model. With probability at least $1 - \xi$, the output is always a $(1 + O(\eta), O\left(\frac{\log^2(T/\xi)}{\eta^2 \varepsilon}\right))$ -approximation for every timestamp $t \in [T]$. The algorithm uses poly $\left(\frac{\log(T/\xi)}{\eta \min(\varepsilon, 1)}\right)$ space.

5 Continual Released ℓ_p Heavy Hitters and Frequency Moment Estimation

In this section, we present ε -DP streaming continual release algorithms for ℓ_p heavy hitters and frequency moment estimation. In Section 5.1, we present an algorithm for ε -DP CountSketch Charikar et al. [2002] in the streaming continual release model. The CountSketch is used for ℓ_2 heavy hitters and ℓ_2 moment estimation. In Section 5.2, we show how to use ℓ_2 heavy hitters to solve ℓ_p heavy hitters. In Section 5.3, we show how to estimate the number of elements which have low frequencies. In Section 5.4, we show how to use ℓ_p heavy hitters and the estimator of low frequency elements to estimate the ℓ_p frequency moment.

5.1 Continual Released CountSketch

Algorithm 4: Continual Released CountSketch **Input:** A stream S of elements $a_1, a_2, \dots, a_T \in U \cup \{\bot\}$, a parameter $k \in \mathbb{Z}_{>1}$. **Parameters** : Relative approximation factor $\alpha \geq 1$ and additive approximation factor $\gamma \geq 0$ depending on the streaming continual release summing algorithm. //See Theorem 2.5. **Output:** A tuple (z_1, z_2, \cdots, z_k) at every timestamp t. Let $h: \mathcal{U} \to [k]$ be a 4-wise independent hash function, s.t., $\forall a \in \mathcal{U}, i \in [k], \Pr[h(a) = i] = \frac{1}{k}$. Let $g: \mathcal{U} \to \{-1, 1\}$ be a 4-wise independent hash function, s.t., $\forall a \in \mathcal{U}, \Pr[g(a) = 1] = \frac{1}{2}$. Initialize empty streams S_1, S_2, \cdots, S_k . for each a_t in the stream S do if $a_t = \perp$ then Append 0 to the end of every stream S_1, S_2, \cdots, S_k . end else Append $g(a_t)$ to the end of the stream $\mathcal{S}_{h(a_t)}$ and append 0 to the end of every stream \mathcal{S}_i for $i \neq h(a_t)$. end Output a tuple (z_1, z_2, \cdots, z_k) where z_i is an estimation of the total counts of \mathcal{S}_i with additive error at most γ . end

Lemma 5.1 (DP guarantee). If the subroutine of continually releasing the approximate total counts of S_i for every $i \in [k]$ in Algorithm 4 is ε -DP, Algorithm 4 is 2ε -DP.

Proof. Consider two neighboring streams S and S' where the only difference is the *t*-th element, i.e., $a_t \neq a'_t$. Consider their corresponding streams S_1, S_2, \dots, S_k and S'_1, S'_2, \dots, S'_k in Algorithm 4. For $i \in [k]$ and $j \neq t$, the *j*-th number in S_i should be the same as the *j*-th number in S'_i . Thus, we only need to consider the *t*-th number in S_i and S'_i for every $i \in [k]$. Since a_t can only make the *t*-th number of $S_{h(a_t)}$ be non-zero, a'_t can only make the *t*-th number of $S'_{h(a'_t)}$ be non-zero, and the non-zero number can be only ± 1 , the sensitivity is at most 2. Thus, if we use ε -DP algorithm to continually release the total counts of S_1, S_2, \dots, S_k , the continually released output of Algorithm 4 is 2ε -DP.

Lemma 5.2 (Good approximation for frequent elements). Consider any $a \in \mathcal{U}$ and any timestamp $t \in [T]$. Let f_a be the frequency of a in a_1, a_2, \dots, a_t . Let (z_1, z_2, \dots, z_k) be the output of Algorithm 4 at timestamp t. Then $\forall \eta \in (0, 0.5)$, with probability at least $1 - 1/(k\eta^2)$, $|f_a - g(a) \cdot z_{h(a)}| \leq \eta \cdot \sqrt{\sum_{b \in \mathcal{U}} f_b^2} + \gamma$.

Proof. Let $\hat{z}_{h(a)}$ be the true total counts of stream $S_{h(a)}$ at timestamp t. According to the original CountSketch Charikar et al. [2002], we have $\Pr\left[|f_a - g(a) \cdot \hat{z}_{h(a)}| \le \eta \sqrt{\sum_{b \in \mathcal{U}} f_b^2}\right] \ge 1 - 1/(k\eta^2)$. Since

$$|f_a - g(a) \cdot z_{h(a)}| \le |f_a - g(a) \cdot \hat{z}_{h(a)}| + |g(a)| \cdot |z_{h(a)} - \hat{z}_{h(a)}| \le |f_a - g(a) \cdot \hat{z}_{h(a)}| + \gamma,$$

we have with probability at least $1 - 1/(k\eta^2)$, $|f_a - g(a) \cdot z_{h(a)}| \le \eta \cdot \sqrt{\sum_{b \in \mathcal{U}} f_b^2} + \gamma$.

Lemma 5.3 (ℓ_2 Frequency moment estimation). Consider any timestamp $t \in [T]$. For $a \in \mathcal{U}$, let f_a be the frequency of a in a_1, a_2, \dots, a_t . Let (z_1, z_2, \dots, z_k) be the output of Algorithm 4 at timestamp t. Then $\forall \eta \in (0, 0.5)$, with probability at least $1 - 100/(k\eta^2)$, $|\sum_{i=1}^k z_i^2 - \sum_{a \in \mathcal{U}} f_a^2| \leq \eta \sum_{a \in \mathcal{U}} f_a^2 + 4k\gamma^2/\eta$

Proof. Let $F_2 = \sum_{a \in \mathcal{U}} f_a^2$. Let $Z = \sum_{i \in [k]} z_i^2$. For $i \in [k]$, let \hat{z}_i be the true total counts of stream S_i at timestamp t. Let $\hat{Z} = \sum_{i \in [k]} \hat{z}_i^2$. According to the analysis of CountSketch Charikar et al. [2002], Thorup and Zhang [2004], We have $\Pr\left[|F_2 - \hat{Z}| \leq \eta/4 \cdot F_2\right] \geq 1 - 100/(k\eta^2)$. In the following, we condition on $|F_2 - \hat{Z}| \leq \eta/4 \cdot F_2$.

We have

$$|F_2 - Z| \le |F_2 - \hat{Z}| + |\hat{Z} - Z| \le \eta/4 \cdot F_2 + \sum_{i=1}^k |z_i^2 - \hat{z}_i^2|$$

Denote $z_i = \hat{z}_i + v_i$ for $i \in [k]$. We have $|v_i| \leq \gamma$. Due to convexity, we have:

$$(1 - \eta/4)\hat{z}_i^2 - 4v_i^2/\eta \le (\hat{z}_i + v_i)^2 \le \hat{z}_i^2/(1 - \eta/4) + 4v_i^2/\eta$$

Since $\eta \in (0, 0.5), 1/(1 - \eta/4) \le (1 + \eta/2)$. Therefore, $|z_i^2 - \hat{z}_i^2| \le \eta/2 \cdot \hat{z}_i^2 + 4v_i^2/\eta$. We have:

$$|F_2 - Z| \le \eta/4 \cdot F_2 + \eta/2 \cdot \hat{Z} + 4k\gamma^2/\eta \le \eta/4 \cdot F_2 + \eta/2 \cdot 3/2 \cdot F_2 + 4k\gamma^2/\eta \le \eta F_2 + 4k\gamma^2/\eta.$$

Theorem 5.4 (Streaming continual release ℓ_2 frequency estimators). Let $\varepsilon > 0, \eta \in (0, 0.5), \xi \in (0, 0.5)$. There is an ε -DP algorithm in the streaming continual release model such that with probability at least $1 - \xi$, it always outputs for every timestamp $t \in [T]$:

1. \hat{f}_a for every $a \in \mathcal{U}$ such that $|f_a - \hat{f}_a| \leq \eta \|\mathcal{S}\|_2 + O\left(\frac{\log(T/\xi) + \log(|\mathcal{U}|)}{\varepsilon} \cdot \log^{2.5}(T) \cdot \log\left(\frac{\log(T/\xi) + \log(|\mathcal{U}|)}{\xi\eta}\right)\right)$, where \mathcal{S} denotes the stream (a_1, a_2, \cdots, a_t) and f_a denotes the frequency of a in \mathcal{S} ,

2.
$$\hat{F}_2$$
 such that $|\hat{F}_2 - \|\mathcal{S}\|_2^2| \le \eta \|\mathcal{S}\|_2^2 + O\left(\frac{(\log(T/\xi) + \log(|\mathcal{U}|))^2}{\varepsilon^2 \eta^3} \cdot \log^5(T) \cdot \log^2\left(\frac{\log(T/\xi) + \log(|\mathcal{U}|)}{\xi\eta}\right)\right)$

The algorithm uses $O\left(\frac{\log(T/\xi) + \log(|\mathcal{U}|)}{\eta^2} \cdot \log(T)\right)$ space.

Proof. Suppose we set $k = 400/\eta^2$. Due to Lemma 5.2 and Lemma 5.3, the approximation guarantees hold with probability at least 2/3 for each particular timestamp $t \in [T]$ and $a \in \mathcal{U}$. To boost the success probability to $1 - \xi/2$ for the approximation guarantees and simultaneously for all $t \in [T]$ and all $a \in |\mathcal{U}|$, according to Lemma 2.7, we run $\lceil 50(\log(2T/\xi) + \log(|\mathcal{U}|)) \rceil$ copies of Algorithm 4 and take the median of each estimator.

We apply Theorem 2.5 for the summing problem of S_i for each $i \in [k]$. Since we run $\lceil 50(\log(2T/\xi) + \log(|\mathcal{U}|)) \rceil$ copies of Algorithm 4, if we desire ε -DP algorithm in the end, we need each summing

subroutine to be $\varepsilon/(2 \cdot \lceil 50(\log(2T/\xi) + \log(|\mathcal{U}|)) \rceil)$ -DP according to Lemma 5.1. To simultaneously make the call of each run of the summing subroutine succeeds with probability at least $1 - \xi/2$, we need to apply union bound over all calls of summing and thus each run of the summing subroutine should success with probability at least $1 - \xi/(2 \cdot \lceil 50(\log(2T/\xi) + \log(|\mathcal{U}|)) \rceil \cdot k) = 1 - \xi/(2 \cdot \lceil 50(\log(2T/\xi) + \log(|\mathcal{U}|)) \rceil \cdot 400/\eta^2)$. Thus, according to Theorem 2.5, we have $\alpha = 1$ and $\gamma = O\left(\frac{\log(T/\xi) + \log(|\mathcal{U}|)}{\varepsilon} \cdot \log^{2.5}(T) \cdot \log\left(\frac{\log(T/\xi) + \log(|\mathcal{U}|)}{\xi\eta^2}\right)\right)$.

Finally, let us consider the total space usage, since we call $\left(\left\lceil 50(\log(2T/\xi) + \log(|\mathcal{U}|))\right\rceil \cdot 400/\eta^2\right)$ times of summing subroutine, the space needed for executing them is $O\left(\frac{\log(T/\xi) + \log(|\mathcal{U}|)}{\eta^2} \cdot \log(T)\right)$. Additional space needed is O(1). Thus, the total space required is $O\left(\frac{\log(T/\xi) + \log(|\mathcal{U}|)}{\eta^2} \cdot \log(T)\right)$

5.2 Continual Released ℓ_p Heavy Hitters

By applying the CountSketch, we are able to develop ℓ_p heavy hitters.

Algorithm 5: Continual Released ℓ_p Heavy Hitters $(p \in [0, \infty))$ **Input:** A stream S of elements $a_1, a_2, \dots, a_T \in U \cup \{\bot\}$, a parameter $k \in \mathbb{Z}_{>1}$, an error parameter $\eta \in (0, 0.5)$. **Parameters** : Additive error parameters $\gamma_1, \gamma_2 \geq 0$ depending on the streaming continual release CountSketch algorithm. //See Theorem 5.4. **Output:** A set $H \subseteq \mathcal{U}$ of elements and their estimated frequencies $\hat{f}: H \to \mathbb{R}_{>0}$ at every timestamp t. Let $\phi \ge \max\left(|\mathcal{U}|^{1-2/p}, 1\right)$. Let $m = 10k^2$. Let $h: \mathcal{U} \to [m]$ be a pairwise independent hash function where $\forall a \in \mathcal{U}, i \in [m], \Pr[h(a) = i] = 1/m.$ Initialize empty streams S_1, S_2, \dots, S_m . for each a_t in the stream S do if $a_t = \perp$ then Append \perp to the end of every stream S_1, S_2, \cdots, S_m . end else Append a_t to the end of the stream $\mathcal{S}_{h(a_t)}$ and append \perp to the end of every stream \mathcal{S}_i for $i \neq h(a_t).$ end For $i \in [m]$, compute $\hat{F}_{2,i}$ which is a $(1.1, \gamma_1)$ -approximation to $\|S_i\|_2^2$. For $a \in \mathcal{U}$, compute \hat{f}_a which is a $(1, (\eta/16)/(10\sqrt{\phi k}) \cdot \|\mathcal{S}_{h(a)}\|_2 + \gamma_2)$ -approximation to f_a , the frequency of a in \mathcal{S} (or equivalently in $\mathcal{S}_{h(a)}$). For $a \in \mathcal{U}$, if $\hat{f}_a^2 \ge \frac{\hat{F}_{2,h(a)} + \gamma_1}{25\phi k} + \frac{512\gamma_2^2}{\eta^2}$, add a into \hat{H} . Let $H \subseteq \hat{H}$ only keep the elements a such that \hat{f}_a is one the top- $\left(\left(\frac{1+\eta}{1-\eta}\right)^p \cdot k\right)$ values among $\{\hat{f}_b \mid b \in \hat{H}\}$. For each $a \in H$, report $\hat{f}(a) \leftarrow \hat{f}_a$. \mathbf{end}

Lemma 5.5 (DP guarantee). If $\forall i \in [m]$ the subroutine in Algorithm 5 of continually releasing $\hat{F}_{2,i}$ and \hat{f}_a for all a satisfying h(a) = i is ε -DP, Algorithm 5 is 2ε -DP.

Proof. Consider two neighboring streams S and S' where the only difference is the *t*-th element, i.e., $a_t \neq a'_t$. If $a_t \neq \perp$, it only causes the difference of at most one element between $S_{h(a_t)}$ and $S'_{h(a_t)}$. Similarly, if $a'_t \neq \perp$, it only causes the difference of at most one element between $S_{h(a'_t)}$ and

 $\mathcal{S}'_{h(a'_{t})}$. Thus if for each $i \in [m]$, the continual release algorithm which releases $\hat{F}_{2,i}$ and \hat{f}_{a} for every $a \in \mathcal{U}$ with h(a) = i is ε -DP, the overall algorithm is 2ε -DP.

Lemma 5.6. At any timestamp $t \in [T]$, $\forall a \in \mathcal{U}$, if $a \in \hat{H}$, $(1 - \eta)f_a^2 \leq \hat{f}^2(a) \leq (1 + \eta)f_a^2$ where f_a is the frequency of a in a_1, a_2, \cdots, a_t .

Proof. Since $a \in \hat{H}$, we have:

$$\hat{f}_a^2 \ge \frac{\hat{F}_{2,h(a)} + \gamma_1}{25\phi k} + \frac{512\gamma_2^2}{\eta^2}.$$

Thus:

$$\begin{aligned} \frac{\eta}{16} \cdot \hat{f}_{a}^{2} &\geq \frac{16}{\eta} \cdot \left(\frac{2 \cdot (\eta/16)^{2}}{100\phi k} \cdot \left(2\hat{F}_{2,h(a)} + 2\gamma_{1} \right) + 2\gamma_{2}^{2} \right) \\ &\geq \frac{16}{\eta} \cdot \left(\frac{2 \cdot (\eta/16)^{2}}{100\phi k} \cdot \|\mathcal{S}_{h(a)}\|_{2}^{2} + 2\gamma_{2}^{2} \right) \\ &= \frac{16}{\eta} \cdot \left(2 \cdot \left(\frac{\eta/16}{10\sqrt{\phi k}} \cdot \|\mathcal{S}_{h(a)}\|_{2} \right)^{2} + 2\gamma_{2}^{2} \right) \\ &\geq \frac{16}{\eta} \cdot \left(\frac{\eta/16}{10\sqrt{\phi k}} \cdot \|\mathcal{S}_{h(a)}\|_{2} + \gamma_{2} \right)^{2} \end{aligned}$$

By convexity:

$$\hat{f}_a^2 \ge (1 - \eta/16) \cdot f_a^2 - \frac{16}{\eta} \cdot \left(\frac{\eta/16}{10\sqrt{\phi k}} \cdot \|\mathcal{S}_{h(a)}\|_2 + \gamma_2\right)^2 \ge (1 - \eta/16) \cdot f_a^2 - \eta/16 \cdot \hat{f}_a^2$$

and

$$\hat{f}_a^2 \le 1/(1-\eta/16) \cdot f_a^2 + \frac{16}{\eta} \cdot \left(\frac{\eta/16}{10\sqrt{\phi k}} \cdot \|\mathcal{S}_{h(a)}\|_2 + \gamma_2\right)^2 \le 1/(1-\eta/16) \cdot f_a^2 + \eta/16 \cdot \hat{f}_a^2.$$

Since $\eta \in (0, 0.5)$, we have $(1 - \eta)f_a^2 \le \hat{f}_a^2 \le (1 + \eta)f_a^2$.

Lemma 5.7. At any timestamp t, the output H of Algorithm 5 has size at most $\left(\frac{1+\eta}{1-\eta}\right)^p \cdot k$. *Proof.* Note that H only keeps the top- $\left(\left(\frac{1+\eta}{1-\eta}\right)^p \cdot k\right)$ values from \hat{H} .

Lemma 5.8. At any timestamp t, consider any $a \in \mathcal{U}$. Let f_a be the frequency of a in a_1, a_2, \cdots, a_t . If $f_a \geq 4\sqrt{\gamma_1/(\phi k) + 512\gamma_2^2/\eta^2}$ and $f_a^p \geq \|\mathcal{S}\|_p^p/k$, with probability at least 0.9, $a \in \hat{H}$.

Proof. In this proof, we consider all streams and variables at the timestamp t. Suppose $f_a^p \ge ||\mathcal{S}||_p^p/k$. Let $B = \{b \in \mathcal{U} \mid f_b^p \ge ||\mathcal{S}||_p^p/k\}$. Then with probability at least 0.9, $\forall b \in B \setminus \{a\}, h(a) \ne h(b)$. In the remaining of the proof, we condition on $\forall b \in B \setminus \{a\}, h(a) \ne h(b)$.

Case 1, $p \leq 2$: In this case, we have $\phi = 1$. $\forall x \in \mathcal{U}$ with h(x) = h(a), we have $f_x \leq f_a$ which implies that $f_x^{p-2} \geq f_a^{p-2}$ since $p \leq 2$. Since $\forall x \in \mathcal{U}, \frac{f_x^p}{f_x^2} \leq 1$, we have

$$\frac{\|\mathcal{S}_{h(a)}\|_{p}^{p}}{\|\mathcal{S}_{h(a)}\|_{2}^{2}} = \frac{\sum_{x \in \mathcal{U}: h(x) = h(a)} f_{x}^{p}}{\sum_{x \in \mathcal{U}: h(x) = h(a)} f_{x}^{2}} \ge \min_{x \in \mathcal{U}: h(x) = h(a)} \frac{f_{x}^{p}}{f_{x}^{2}} = \frac{f_{a}^{p}}{f_{a}^{2}},$$

which implies that $f_a^2 / \|\mathcal{S}_{h(a)}\|_2^2 \ge f_a^p / \|\mathcal{S}_{h(a)}\|_p^p \ge f_a^p / \|\mathcal{S}\|_p^p \ge 1/k$ and thus $f_a^2 \ge \|\mathcal{S}_{h(a)}\|_2^2 / (\phi k)$. **Case 2,** p > 2: In this case, we have $\phi = |\mathcal{U}|^{1-2/p}$. Since $f_a^p \ge ||\mathcal{S}||_p^p/k$, we have:

$$f_a \ge \|\mathcal{S}\|_p / k^{1/p} \ge \|\mathcal{S}\|_p / k^{1/2} \ge \|\mathcal{S}\|_2 / (k^{1/2} \cdot |\mathcal{U}|^{1/2 - 1/p}),$$

where the second inequality follows from $k^{1/2} \ge k^{1/p}$ for p > 2, and the third inequality follows from Holder's inequality that $\|\mathcal{S}\|_2 \le |\mathcal{U}|^{1/2-1/p} \cdot \|\mathcal{S}\|_p$. Therefore, $f_a^2 \ge \|\mathcal{S}\|_2^2/(\phi k)$. Therefore, in both above cases, we always have $f_a^2 \ge \|\mathcal{S}\|_2^2/(\phi k)$.

By convexity, we have

$$\hat{f}_{a}^{2} \geq (1 - \eta/16) \cdot f_{a}^{2} - \frac{16}{\eta} \cdot \left(\frac{\eta/16}{10\sqrt{\phi k}} \cdot \|\mathcal{S}_{h(a)}\|_{2} + \gamma_{2}\right)^{2}$$
$$\geq (1 - \eta/16) \cdot f_{a}^{2} - \frac{16}{\eta} \cdot \left(\frac{2(\eta/16)^{2}}{100\phi k} \cdot \|\mathcal{S}_{h(a)}\|_{2}^{2} + 2\gamma_{2}^{2}\right)$$

Thus we have:

$$\begin{split} \hat{f}_{a}^{2} &\geq \frac{1}{2} \cdot \|\mathcal{S}_{h(a)}\|_{2}^{2} / (\phi k) - \frac{16}{\eta} \cdot \left(\frac{2(\eta/16)^{2}}{100\phi k} \cdot \|\mathcal{S}_{h(a)}\|_{2}^{2} + 2\gamma_{2}^{2}\right) \\ &\geq \frac{1}{2} \cdot \frac{1}{2} \cdot (\hat{F}_{2,h(a)} - \gamma_{1}) / (\phi k) - \frac{16}{\eta} \cdot \left(\frac{2(\eta/16)^{2}}{100\phi k} \cdot 2 \cdot (\hat{F}_{2,h(a)} + \gamma_{1}) + 2\gamma_{2}^{2}\right) \\ &= \left(\frac{1}{4} - \frac{\eta/16}{25}\right) \cdot \frac{\hat{F}_{2,h(a)}}{\phi k} - \left(\frac{1}{4} + \frac{\eta/16}{25}\right) \cdot \frac{\gamma_{1}}{\phi k} - \frac{32}{\eta} \cdot \gamma_{2}^{2} \\ &\geq \frac{1}{5} \cdot \frac{\hat{F}_{2,h(a)}}{\phi k} - \frac{1}{3} \cdot \frac{\gamma_{1}}{\phi k} - \frac{32}{\eta^{2}} \cdot \gamma_{2}^{2} \\ &\geq 2 \cdot \left(\frac{\hat{F}_{2,h(a)} + \gamma_{1}}{25\phi k} + \frac{512\gamma_{2}^{2}}{\eta^{2}}\right) - \left(\frac{2048\gamma_{2}^{2}}{\eta^{2}} + \frac{2}{3} \cdot \frac{\gamma_{1}}{\phi k} - \frac{3}{25\phi k} \cdot \hat{F}_{2,h(a)}\right) \end{split}$$
(1)

On the other hand, by convexity, we have:

$$\hat{f}_{a}^{2} \geq (1 - \eta/16) f_{a}^{2} - \frac{16}{\eta} \cdot \left(\frac{\eta/16}{10\sqrt{\phi k}} \|\mathcal{S}_{h(a)}\|_{2} + \gamma_{2}\right)^{2}$$
$$\geq (1 - \eta/16) f_{a}^{2} - \frac{16}{\eta} \cdot \left(\frac{2(\eta/16)^{2}}{100\phi k} \|\mathcal{S}_{h(a)}\|_{2}^{2} + 2\gamma_{2}^{2}\right)$$

and thus

$$\begin{aligned} \hat{f}_{a}^{2} &\geq \frac{1}{2} \cdot 16 \cdot \left(\gamma_{1}/(\phi k) + 512\gamma_{2}^{2}/\eta^{2}\right) - \frac{16}{\eta} \cdot \left(\frac{2(\eta/16)^{2}}{100\phi k} \cdot 2 \cdot \left(\hat{F}_{2,h(a)} + \gamma_{1}\right) + 2\gamma_{2}^{2}\right) \\ &= \left(8 - \frac{\eta/16}{25k}\right) \cdot \frac{\gamma_{1}}{\phi k} + \left(\frac{4096}{\eta^{2}} - \frac{32}{\eta}\right) \gamma_{2}^{2} - \frac{\eta/16}{25\phi k} \cdot \hat{F}_{2,h(a)} \\ &\geq \frac{2}{3} \cdot \frac{\gamma_{1}}{\phi k} + \frac{2048}{\eta^{2}} \cdot \gamma_{2}^{2} - \frac{3}{25\phi k} \cdot \hat{F}_{2,h(a)} \end{aligned}$$
(2)

By looking at Equation (1) + Equation (2), we have $\hat{f}_a^2 \ge \frac{\hat{F}_{2,h(a)} + \gamma_1}{25\phi k} + \frac{512\gamma_2^2}{\eta^2}$. Thus, $a \in \hat{H}$. **Lemma 5.9.** At any timestamp t, if $f_a^p \ge \|\mathcal{S}\|_p^p/k$ and $a \in \hat{H}$, then $a \in H$.

Proof. We prove the statement by contradiction. Suppose $a \notin H$, there is a subset $Q \subseteq H$ with $|Q| \geq \left(\frac{1+\eta}{1-\eta}\right)^p \cdot k$ such that $\forall b \in Q, \hat{f}_b \geq \hat{f}_a$. According to Lemma 5.6, we have $\forall b \in Q, f_b^p \geq \left(\frac{1-\eta}{1+\eta}\right)^p \cdot f_a^p \geq \left(\frac{1-\eta}{1+\eta}\right)^p \cdot \|\mathcal{S}\|_p^p / k$. Then we have $\sum_{b \in Q} f_b^p \geq \|\mathcal{S}\|_p^p$ which leads to a contradiction. \Box

Theorem 5.10 $(\ell_p$ Heavy hitters for all $p \in [0, \infty)$). Let $\varepsilon > 0, \eta \in (0, 0.5), k \ge 1, \xi \in (0, 0.5)$. Let $\phi = \max(1, |\mathcal{U}|^{1-2/p})$. There is an ε -DP algorithm in the streaming continual release model such that with probability at least $1 - \xi$, it always outputs a set $H \subseteq \mathcal{U}$ and a function $\hat{f} : H \to \mathbb{R}$ for every timestamp $t \in [T]$: such that

- 1. $\forall a \in H, \ \hat{f}(a) \in (1 \pm \eta) \cdot f_a$ where f_a is the frequency of a in the stream $\mathcal{S} = (a_1, a_2, \cdots, a_t),$
- 2. $\forall a \in \mathcal{U}, \text{ if } f_a \geq \frac{1}{\varepsilon \eta} \cdot \log^C \left(\frac{T \cdot k \cdot |\mathcal{U}|}{\xi \eta} \right)$ for some sufficiently large constant C > 0 and $f_a^p \geq \|\mathcal{S}\|_p^p / k$ then $a \in H$,

3. The size of H is at most
$$O\left(\left(\log(T/\xi) + \log(|\mathcal{U}|)\right) \cdot \left(\frac{1+\eta}{1-\eta}\right)^p \cdot k\right)$$
.

The algorithm uses $\frac{\phi k^3}{\eta^2} \cdot \operatorname{poly}\left(\log\left(\frac{T \cdot k \cdot |\mathcal{U}|}{\xi}\right)\right)$ space.

Proof. The first property follows from Lemma 5.6.

According to Lemma 5.8 and Lemma 5.9, the guarantee holds with probability 0.9 for any particular $t \in [T]$ and $a \in \mathcal{U}$. To boost the probability to make the guarantee holds simultaneously for all $t \in [T]$ and $a \in \mathcal{U}$ with probability at least $1 - \xi/2$, we need to repeat Algorithm 5 $[50(\log(2T/\xi) + \log(|\mathcal{U}|))]$ times, and let final H at timestamp t be the union of all output H at timestamp t, and let final $\hat{f}(a)$ at timestamp t be any output $\hat{f}(a)$ at timestamp t. According to Lemma 5.7, the output H of a single running copy of Algorithm 5 is at most $\left(\frac{1+\eta}{1-\eta}\right)^p \cdot k$. Thus, the size of the final output H is at most $O\left(\left(\log(T/\xi) + \log(|\mathcal{U}|)\right) \cdot \left(\frac{1+\eta}{1-\eta}\right)^p \cdot k\right)$ which proves the second property.

We apply Theorem 5.4 for the frequency and ℓ_2 frequency moment estimators of S_i for each $i \in [m]$. Since we run $\lceil 50(\log(2T/\xi) + \log(|\mathcal{U}|)) \rceil$ copies of Algorithm 5, if we desire ε -DP algorithm in the end, we need each frequency and ℓ_2 frequency moment subroutine to be $\varepsilon/(4 \cdot \lceil 50(\log(2T/\xi) + \log(|\mathcal{U}|)) \rceil)$ -DP according to Lemma 5.5. To simultaneously make the call of each run of the frequency and ℓ_2 frequency moment subroutine succeeds with probability at least $1 - \xi/2$, we need to apply union bound over all calls of the subroutines and thus each run of the subroutine should succeed with probability at least $1 - \xi/(4 \cdot \lceil 50(\log(T/\xi) + \log(|\mathcal{U}|)) \rceil \cdot m) = 1 - \xi/(4 \cdot \lceil 50(\log(T/\xi) + \log(|\mathcal{U}|)) \rceil \cdot 10k^2)$. Notice that we also need to re-scale η in Theorem 5.4 to be $(\eta/16)/(10\sqrt{\phi k})$ used in Algorithm 5 for estimation of the frequency of each element and set $\eta = 0.01$ in Theorem 5.4 for the estimation of the ℓ_2 frequency moment. Thus, according to Theorem 5.4, we have

$$\gamma_1 = \frac{1}{\varepsilon^2} \cdot \operatorname{poly}\left(\log\left(\frac{T \cdot k \cdot |\mathcal{U}|}{\xi}\right)\right)$$

 $\gamma_2 = \frac{1}{\varepsilon} \cdot \operatorname{poly}\left(\log\left(\frac{T \cdot k \cdot |\mathcal{U}|}{\xi\eta}\right)\right)$

Therefore, the second property follows from Lemma 5.8 and Lemma 5.9 and our probability boosting argument.

Finally, let us consider the space usage. Since we run $O(\log(T/\xi) + \log(|\mathcal{U}|))$ copies of Algorithm 5, and each copy calls $O(k^2)$ frequency estimator and ℓ_2 frequency moment estimator using the parameters discussed above. Due to Theorem 5.4, the total space usage is at most $\frac{\phi k^3}{\eta^2} \cdot \operatorname{poly}\left(\log\left(\frac{T \cdot k \cdot |\mathcal{U}|}{\xi}\right)\right)$.

5.3 Differentially Private Continual Released Counting of Low Frequency Elements

In this section, we show a differentially private continual released algorithm for counting the number of elements that have a certain (low) frequency. Similar to our counting distinct elements algorithm, we first consider the case where the universe of the elements is small.

5.3.1 Number of Low Frequency Elements for Small Universe

Algorithm 6: Number of Low Frequency Elements for Small Universe
Input: A stream S of elements $a_1, a_2, \dots, a_T \in U \cup \{\bot\}$ with gaurantee that $ \mathcal{U} \leq m$, and a
target frequency k .
Parameters : Relative approximation factor $\alpha \geq 1$ and additive approximation factor $\gamma \geq 0$
depending on the streaming continual release summing algorithm.
//See Theorem 2.5.
Output: Estimation of the number of elements with frequency exactly <i>i</i> for each $i \in [k]$ at every timestamp <i>t</i> .
Initialize empty streams $\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_k$.
For each $a \in \mathcal{U}$, initialize frequency $f(a) \leftarrow 0$
for each a_t in the stream S do
If $a_t \neq \perp$, $f(a_t) \leftarrow f(a_t) + 1$.
for $each \ i \in [k]$ do
if $a_t \neq \perp$ and $f(a_t) = i + 1$ then
Append -1 at the end of C_i .
end
else if $a_t \neq \perp$ and $f(a_t) = i$ then
Append 1 at the end of C_i .
end
else
Append 0 at the end of C_i .
end
end
For each $i \in [k]$, output \hat{s}_i which is an (α, γ) -approximation to the total counts of C_i .
end

Lemma 5.11. At the end of any time $t \in [T]$, $\forall i \in [k]$, the output \hat{s}_i of Algorithm 6 is an (α, γ) -approximation to $|\{a \in \mathcal{U} \mid f_a = i\}|$, the size of the set of elements of which the frequency is exact *i*.

and

Proof. It is easy to observe that Algorithm 6 maintains f(a) such that $f(a) = f_a$ for any $a \in \mathcal{U}$. For $i \in [k]$, if before adding a_t the frequency of a_t is i, we append -1 to \mathcal{C}_i . If after adding a_t the frequency of a_t becomes i, we append 1 to \mathcal{C}_i . Note that if the frequency of a_t has the value which is not in above two cases, it does not affect the number of elements with frequency is i. Therefore, the total counts of \mathcal{C}_i is always the number of elements of which frequency is i. Thus, \hat{s}_i is an (α, γ) -approximation to the number of elements of which frequency is i.

Lemma 5.12. If the algorithm that continually release the approximate total counts of C_i for every $i \in [k]$ is ε -DP, Algorithm 6 is $(8k\varepsilon)$ -DP in the continual release model.

Proof. Consider two neighboring stream $S = (a_1, a_2, \dots, a_T)$ and $S' = (a'_1, a'_2, \dots, a'_T)$ of elements in \mathcal{U} where they only differ at timestamp t, i.e., $a_t \neq a'_t$. Fix any $i \in [k]$, let us consider the differences between the corresponding count streams $C_i = (c_1, c_2, \dots, c_T)$ and $C'_i = (c'_1, c'_2, \dots, c'_T)$.

Consider an intermediate neighboring stream $S'' = (a_1, a_2, \dots, a_{t-1}, \bot, a_{t+1}, \dots, a_T)$. Let $C''_i = (c''_1, c''_2, \dots, c''_T)$ be the count stream corresponding to S''. The total difference between C_i and C'_i is bounded by the sum of the total difference between C_i and C''_i and the total difference between C'_i and C''_i .

Consider the difference between C_i and C''_i . If $a_t = \bot$, C''_i is exactly the same as C_i . Suppose $a_t = a \in \mathcal{U}$ is the *j*-th appearance of *a* in S. If j > i, change a_t to \bot does not affect C_i . Suppose $j \leq i$. Let $a_{t_1}, a_{t_2}, a_{t_3}$ be the *i*-th, (i + 1)-th, (i + 2)-th appearances of *a* in S respectively. Then it is easy to verify that $c_{t_1} = 1, c''_{t_1} = 0, c_{t_2} = -1, c''_{t_2} = 1, c_{t_3} = 0, c''_{t_3} = -1$, and for any other $p \neq t_1, t_2, t_3$, we have $c_p = c''_p$. Thus, the total difference between C_i and C''_i is at most 4.

Similarly, the total difference between C'_i and C''_i is at most 4. Thus, the total difference between C_i and C'_i is at most 8. Therefore, the total sensitivity of C_1, C_2, \dots, C_k is at most 8k. If we use an ε -DP algorithm to continually release the total counts of C_i for every $i \in [k]$, the continually released outputs of Algorithm 6 is $(8k\varepsilon)$ -DP.

Theorem 5.13 (Streaming continual release count of low frequency elements for small universe). Let $k \geq 1, \varepsilon \geq 0, \xi \in (0, 0.5)$. Suppose the universe \mathcal{U} has size at most m. There is an ε -DP algorithm in the streaming continual release model such that with probability at least $1-\xi$, it always outputs k numbers $\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_k$ for every timestamp t, such that $\forall i \in [k], \hat{s}_i$ is an approximation to $|\{a \in \mathcal{U} \mid f_a = i\}|$ with additive error $O\left(\frac{k}{\varepsilon} \cdot \log^{2.5}(T) \log\left(\frac{k}{\xi}\right)\right)$. The algorithm uses $O(m+k \log(T))$ space.

Proof. Consider Algorithm 6, we use Theorem 2.5 as the summing subroutine. According to Lemma 5.12, if we want the final algorithm to be ε -DP, then each subroutine must be $(\varepsilon/(8k))$ -DP. Furthermore, if we want the over success probability to be $1 - \xi$, the success probability of each summing subroutine should be at least $1 - \xi/k$. By applying Theorem 2.5, we have $\alpha = 1$ and $\gamma = O\left(\frac{k}{\varepsilon} \cdot \log^{2.5}(T) \log\left(\frac{k}{\xi}\right)\right)$.

We need O(m) space to maintain the frequency of the elements. We need $O(k \log(T))$ space to run k summing subroutines. Therefore, the total space usage is $O(m + k \log(T))$.

5.3.2 Number of Low Frequency Elements for General Universe

Algorithm 7: Number of Low Frequency Elements via Subsampling
Input: A stream S of elements $a_1, a_2, \dots, a_T \in U \cup \{\bot\}$, an error parameter $\eta \in (0, 0.5)$, and a
parameter $k \ge 1$.
Parameters : Additive approximation factor γ_1 depending on the streaming continual release
number of distinct elements, relative approximation factor $\alpha_2 \geq 1$ and additive
approximation factor $\gamma_2 \geq 0$ depending on the streaming continual release count of
low frequency elements for small universe.
//See Corollary 4.10 and Theorem 5.13.
Output: Estimation of the number of elements with frequency exactly i for each $i \in [k]$ at every
timestamp t .
$L \leftarrow \log\min(\mathcal{U} , T) , \lambda \leftarrow 2\log(1000k), m \leftarrow 100 \cdot (25600\lambda/\eta^2)^2.$
Let $h: \mathcal{U} \to [m]$ be a pairwise independent hash function.
Let $g: \mathcal{U} \to [L] \cup \{\bot\}$ be a λ -wise independent hash function and
$\forall a \in \mathcal{U}, i \in [L], \Pr[g(a) = i] = 2^{-i}, \Pr[g(a) = \bot] = 2^{-L}.$
Initialize empty streams S_1, S_2, \cdots, S_L .
for each a_t in the stream S do
for $i \in L$ do
if $a_t \neq \perp$ and $g(a_t) = i$ then
Append $h(a_t)$ to the end of the stream S_i .
end
Append \perp to the end of the stream \mathcal{S}_i .
end
end
Compute d which is a $(1.1, \gamma_1)$ -approximation to the number of distinct elements in \mathcal{S} .
For $i \in [L]$, compute $\hat{s}_{i,1}, \hat{s}_{i,2}, \dots, \hat{s}_{i,k}$ where $\hat{s}_{i,j}$ is an (α_2, γ_2) -approximation to the number of
elements in S_i where each element has frequency exact j .
If $d \le \max(3\gamma_1, 64\lambda/\eta^2)$, set $\hat{s}_1 = \hat{s}_2 = \dots = \hat{s}_k = 0$.
Otherwise, find the largest $i^* \in [L]$ such that $2^{i^*} \cdot (64\lambda/\eta^2) \leq d$, and $\forall j \in [k]$, set $\hat{s}_j = \hat{s}_{i^*,j} \cdot 2^i$.
Output $\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_k$.
end

Lemma 5.14. If the subroutine of continually releasing \hat{d} in Algorithm 7 is ε -DP and the subroutine of continually releasing $\{\hat{s}_{i,1}, \hat{s}_{i,2}, \dots, \hat{s}_{i,k}\}$ for each $i \in [k]$ is also ε -DP, Algorithm 7 is 3ε -DP.

Proof. Consider two neighboring streams S and S' where the only difference is the *t*-th element, i.e., $a_t \neq a'_t$. Consider their corresponding streams S_1, S_2, \dots, S_L and S'_1, S'_2, \dots, S'_L in Algorithm 7. For $i \in [L]$ and $j \neq t$, the *j*-th element in S_i should be the same as the *j*-th element in S'_i . Since a_t can only make the *t*-th element of $S_{h(a_t)}$ be different from the *t*-th element of $S'_{h(a_t)}$, and a'_t can only make the *t*-th element of $S_{h(a'_t)}$ be different from the *t*-th element of $S'_{h(a'_t)}$, the total sensitivity of $\{S_1, S_2, \dots, S_L\}$ is at most 2. Thus, if the continual release of \hat{d} is ε -DP and for every $i \in [L]$, the continual release of $\hat{s}_{i,1}, \hat{s}_{i,2}, \dots, \hat{s}_{i,k}$ is also ε -DP, the continual release of $\hat{s}_1, \hat{s}_2, \dots, \hat{s}_k$ is (3ε) -DP.

Lemma 5.15. Consider an arbitrary timestamp $t \in [T]$. $\forall a \in \mathcal{U}$, let f_a denote the frequency of a in a_1, a_2, \dots, a_t . With probability at least 0.9, $\forall j \in [k]$, the output \hat{s}_j of Algorithm 7 is a

 $\left(\alpha_2, \left(\alpha_2\eta + \frac{\eta^2\gamma_2}{32\lambda}\right) \cdot \|\mathcal{S}\|_0 + 6\gamma_1 + 128\lambda/\eta^2\right) \text{-approximation to } |\{a \in \mathcal{U} \mid f_a = j\}|, \text{ where } \|\mathcal{S}\|_0 \text{ is the number of distinct elements in } a_1, a_2, \cdots, a_t.$

To prove Lemma 5.15, we need following intermediate statements. Consider timestamp $t \in [T]$. $\forall a \in \mathcal{U}$, let f_a denote the frequency of a in a_1, a_2, \dots, a_t . If i^* exists in Algorithm 7, we define $G = \{a_j \mid g(a_j) = i^*, j \in [t]\}$. Let $\|\mathcal{S}\|_0$ denote the number of distinct elements in a_1, a_2, \dots, a_t . For $l \in [k]$, define $G_l = \{a \in G \mid f_a = l\}$.

Claim 5.16. With probability at least 0.98, $\forall a \neq a' \in G, h(a) \neq h(a')$.

Proof. Since $E[|G|] = ||\mathcal{S}||_0/2^{i^*}$, we have $|G| \leq 100 \cdot ||\mathcal{S}||_0/2^{i^*}$ with probability at least 0.99 by Markov's inequality. Since i^* can be found by Algorithm 7, we have $||\mathcal{S}||_0 \leq 1.1 \cdot (\hat{d} + \gamma_1) \leq 2\hat{d}$. By the choice of i^* , we have $2^{i^*} \geq \hat{d}/(128\lambda/\eta^2)$. Therefore, $||\mathcal{S}||_0/2^{i^*} \leq 256\lambda/\eta^2$. With probability at least 0.99, $|G| \leq 25600\lambda/\eta^2$. By union bound and Markov's inequality, $\Pr[\forall a \neq a' \in G, h(a) \neq h(a') \mid |G| \leq 25600\lambda/\eta^2] \geq 1 - |G|^2/m \geq 0.99$.

Claim 5.17. $\forall l \in [k], \Pr[2^{i^*} \cdot |G_l| \in |\{a \in U \mid f_a = l\}| \pm \eta ||\mathcal{S}||_0] \ge 1 - 0.01/k.$

Proof. Let $s_l = |\{a \in U \mid f_a = l\}|$. Since i^* can be found, we have $\hat{d} \ge 3\gamma_1$. Note that $s_l \le ||\mathcal{S}||_0 \le 1.1(\hat{d} + \gamma_1) \le 2\hat{d} \le 2^{i^*} \cdot 4 \cdot 64\lambda/\eta^2$. By applying Lemma 2.6, $\forall l \in [k]$ we have:

$$\Pr\left[\left||G_l| - s_l/2^{i^*}\right| > 32\lambda/\eta\right]$$
$$\leq 8 \cdot \left(\frac{\lambda \cdot s_l/2^{i^*} + \lambda^2}{(32\lambda/\eta)^2}\right)^{\lambda/2}$$
$$\leq 0.01/k,$$

where the last inequality follows from $s_l/2^{i^*} \leq 4 \cdot 64\lambda/\eta^2$ and $\lambda = 2 \cdot \log(1000k)$. Since $\|\mathcal{S}\|_0 \geq (\hat{d} - \gamma_1)/1.1 \geq \hat{d}/2 \geq 2^{i^*} \cdot 32\lambda/\eta^2$, we have

$$\Pr\left[\left||G_l| - s_l/2^{i^*}\right| > \eta \|\mathcal{S}\|_0/2^{i^*}\right] \le 0.01/k.$$

E.		

Next we show the proof of Lemma 5.15

Proof of Lemma 5.15. Let \mathcal{E} denote the event that $\forall a \neq a' \in G, h(a) \neq h(a')$. Let $\forall l \in [k], s_l = |\{a \in U \mid f_a = l\}|$. Let \mathcal{E}' denote the event that $\forall l \in [k], 2^{i^*} \cdot |G_l| \in s_l \pm \eta ||\mathcal{S}||_0$. According to Claim 5.16 and Claim 5.17, the probability that both \mathcal{E} and \mathcal{E}' happen is at least 0.97. In the remaining of the proof, we condition on both events \mathcal{E} and \mathcal{E}' .

First, consider the case that $\hat{d} < \max(3\gamma_1, 64\lambda/\eta^2)$. Since \hat{d} is a $(1.1, \gamma_1)$ -approximation to $\|\mathcal{S}\|_0$, we have $\|\mathcal{S}\|_0 \leq 6\gamma_1 + 128\lambda/\eta^2$. In this case, $\forall l \in [k], \hat{s}_l = 0$ and $|\{a \in \mathcal{U} \mid f_a = l\}| \leq \|\mathcal{S}\|_0 \leq 6\gamma_1 + 128\lambda/\eta^2$.

Next, consider the case that $\hat{d} \geq \max(3\gamma_1, 64\lambda/\eta^2)$. For each $j \in [k]$, let $s_{i^*,l}$ denote the number of elements in \mathcal{S}_l where each element has frequency exact j. According to event \mathcal{E} , we have $\forall l \in [k], s_{i^*,l} = |G_l|$. According to event $\mathcal{E}', \forall l \in [k]$, we have:

$$\hat{s}_l \ge 2^{i^*} \cdot \left(\frac{1}{\alpha_2} \cdot |G_l| - \gamma_2\right) \ge \frac{1}{\alpha_2} \cdot \left(s_l - \eta \|\mathcal{S}\|_0\right) - 2^{i^*} \cdot \gamma_2$$

$$\geq \frac{1}{\alpha_2} \cdot s_l - \left(\eta + \frac{\eta^2 \gamma_2}{32\lambda}\right) \cdot \|\mathcal{S}\|_0,$$

where the first inequality follows from that $\hat{s}_{i^*,l}$ is an (α_2, γ_2) -approximation to $s_{i^*,l}$ and $s_{i^*,l} = |G_l|$, the second inequality follows from event \mathcal{E}' , and the third inequality follows from $\alpha_2 \geq 1$ and $\|\mathcal{S}\|_0 \geq (\hat{d} - \gamma_1)/1.1 \geq \hat{d}/2 \geq 2^{i^*} \cdot 32\lambda/\eta^2$. Similarly, $\forall l \in [k]$, we have:

$$\hat{s}_{l} \leq 2^{i^{+}} \cdot (\alpha_{2} \cdot |G_{l}| + \gamma_{2}) \leq \alpha_{2} \cdot (s_{l} + \eta \|\mathcal{S}\|_{0}) + 2^{i^{+}} \cdot \gamma_{2}$$
$$\leq \alpha_{2} \cdot s_{l} + \left(\alpha_{2}\eta + \frac{\eta^{2}\gamma_{2}}{32\lambda}\right) \cdot \|\mathcal{S}\|_{0}.$$

Theorem 5.18 (Streaming continual release of count of low frequency elements). Let $k \ge 1, \varepsilon \ge 0, \xi \in (0, 0.5), \eta \in (0, 0.5)$. There is an ε -DP algorithm in the streaming continual release model such that with probability at least $1 - \xi$, it always outputs k numbers $\hat{s}_1, \hat{s}_2, \dots, \hat{s}_k$ for every timestamp t such that $\forall i \in [k], \hat{s}_i$ is an approximation to $|\{a \in \mathcal{U} \mid f_a = i\}|$ with additive error:

$$\left(\eta + \eta^2 \cdot \frac{k}{\varepsilon} \cdot \operatorname{poly}\left(\log\left(\frac{Tk}{\xi}\right)\right)\right) \cdot \|\mathcal{S}\|_0 + \frac{1}{\varepsilon} \cdot \operatorname{poly}\left(\log\left(\frac{T}{\xi}\right)\right) + O\left(\frac{\log k}{\eta^2}\right)$$

The algorithm uses $\frac{1}{\eta^4} \cdot \operatorname{poly}\left(\frac{\log(T \cdot k/\xi)}{\min(\varepsilon, 1)}\right)$ space.

Proof. According to Lemma 5.15, the approximation guarantee holds with probability at least 0.9. To boost the success probability to $1 - \xi/3$ for all $t \in [T]$, we need to run $[50 \log(3T/\xi)]$ independent copies of Algorithm 7. Since we run $[50 \log(3T/\xi)]$ copies of Algorithm 7 and according to Lemma 5.14, if we want the final algorithm to be ε -DP, we need each subroutine of the streaming continual release of $\{\hat{s}_{i,1}, \hat{s}_{i,2}, \dots, \hat{s}_{i,k}\}$ for each $i \in [k]$ to be $(\varepsilon/(3 \cdot [50 \log(2T/\xi))])$ -DP and we need the subroutine of the streaming continual release of \hat{d} to be also $(\varepsilon/(3 \cdot [50 \log(3T/\xi))])$ -DP. To simultaneously make the call of each subroutine of the streaming continual release of $\{\hat{s}_{i,1}, \hat{s}_{i,2}, \cdots, \hat{s}_{i,k}\}$ over all independent copies of Algorithm 7 satisfy the desired (α_2, γ_2) -approximation with probability at least $1 - \xi/3$, we need to make $\{\hat{s}_{i,1}, \hat{s}_{i,2}, \cdots, \hat{s}_{i,k}\}$ satisfy the approximation guarantee for each particular $i \in [L]$ and a particular copy of Algorithm 7 with probability at least $1 - \xi/(3 \cdot [50 \log(3T/\xi))] \cdot L)$. To simultaneously make the call of each subroutine of the streaming continual release of \hat{d} over all independent copies of Algorithm 7 satisfy the desired $(1.1, \gamma_1)$ approximation with probability at least $1-\xi/3$, we need to make \hat{d} satisfy the approximation guarantee for each particular copy of Algorithm 7 with probability at least $1 - \xi/(3 \cdot [50 \log(3T/\xi))])$. Then, according to Corollary 4.10, we have $\gamma_1 = \frac{1}{\varepsilon} \cdot \operatorname{poly}\left(\log\left(\frac{T}{\xi}\right)\right)$. According to Theorem 5.13, we have $\alpha_2 = 1$ and $\gamma_2 = \frac{k}{\varepsilon} \cdot \text{poly}\left(\log\left(\frac{Tk}{\xi}\right)\right)$. By plugging above parameters into Lemma 5.15, we have $\forall i \in [k], \hat{s}_i$ is an approximation to $|\{a \in \mathcal{U} \mid f_a = i\}|$ with additive error:

$$\left(\eta + \eta^2 \cdot \frac{k}{\varepsilon} \cdot \operatorname{poly}\left(\log\left(\frac{Tk}{\xi}\right)\right)\right) \cdot \|\mathcal{S}\|_0 + \frac{1}{\varepsilon} \cdot \operatorname{poly}\left(\log\left(\frac{T}{\xi}\right)\right) + O\left(\frac{\log k}{\eta^2}\right).$$

Next, consider the space usage. According to Corollary 4.10, the total space needed to release \hat{d} for all copies of Algorithm 7 needs poly $\left(\frac{\log(T/\xi)}{\min(\varepsilon,1)}\right)$. According to Theorem 5.13, the total space needed to release $\{\hat{s}_{i,1}, \hat{s}_{i,2}, \cdots, \hat{s}_{i,k}\}$ for all $i \in [L]$ over all copies of Algorithm 7 needs $O(\log(T/\xi) \cdot L \cdot (m + \log(T))) = \frac{1}{\eta^4} \cdot \operatorname{poly}\left(\log\left(\frac{T \cdot k}{\xi}\right)\right)$. Thus, the overall space is at most $\frac{1}{\eta^4} \cdot \operatorname{poly}\left(\frac{\log(T \cdot k/\xi)}{\min(\varepsilon,1)}\right)$.

5.4 ℓ_p Moment Estimation

In this section, we show how to use our ℓ_p heavy hitters and the estimator of number of low frequency elements to solve ℓ_p moment estimation problem.

```
Algorithm 8: \ell_p Frequency Moment Estimation
  Input: A stream S of elements a_1, a_2, \dots, a_T \in \mathcal{U} \cup \{\bot\}, an error parameter \eta \in (0, 0.5).
  Parameters : A threshold parameter \tau depending on the heavy hitter algorithm, a relative
                          approximation factor \alpha and an additive approximation factor \gamma depending on the
                          algorithm of estimating count of low frequency elements. //See Theorem 5.10 and
                          Theorem 5.18.
  Output: Estimation of the \ell_p frequency moment at every timestamp t.
  Let \beta' be drawn uniformly at random from [1/2, 1] and let \beta \in \beta' \pm (\eta/T)^C for some sufficiently
    large constant C > 0.
                                                                             //Thus \beta can be represented by \Theta(\log(T/\eta)) bits.
  Let q_1^* be the smallest integer that \beta(1+\eta)^{q_1^*} > \tau and let q_2^* be the smallest integer that
    \beta(1+\eta)^{q_2^*+1} \geq T, and for any q \in [q_1^*, q_2^*], define the interval I_q = (\beta(1+\eta)^q, \beta(1+\eta)^{q+1}].
  \lceil L \leftarrow \log(|\mathcal{U}|) \rceil, \lambda \leftarrow 2 \cdot \log(1000(L+1)\log(4T)/\eta).
  Let k be the largest integer such that k \leq \beta (1+\eta)^{q_1^*}. Let B \leftarrow \left(\frac{\log(4T)}{\eta}\right) \cdot 100(L+1) \cdot \frac{32\lambda}{\eta^3} \cdot (1+\eta)^p.
  Let g: \mathcal{U} \to [L] \cup \{\bot\} be a \lambda-wise independent hash function and
   \forall a \in \mathcal{U}, i \in [L], \Pr[g(a) = i] = 2^{-i}, \Pr[g(a) = \bot] = 2^{-L}.
  Initialize empty streams S_0, S_1, S_2, \cdots, S_L.
  for each a_t in the stream S do
        if a_t \neq \perp then
              Append a_t at the end of \mathcal{S}_0.
             For i \in [L], if g(a_t) = i, append a_t to \mathcal{S}_i, otherwise append \perp to \mathcal{S}_i.
        end
        else
         For i \in [L] \cup \{0\}, append \perp at the end of S_i.
        end
        For each i \in [L] \cup \{0\}, compute a set H_i \subseteq \mathcal{U} together with a function \hat{f}_i : H_i \to \mathbb{R}_{\geq 0} satisfying:
             1. \forall a \in H_i, (1 - \eta') \cdot f_a \leq \hat{f}_i(a) \leq (1 + \eta') \cdot f_a, where f_a is the frequency of a in a_1, a_2, \cdots, a_t, and \eta' satisfies \eta' \leq \frac{\eta}{1000(L+1)|H_i|}.
             2. \forall a \in \mathcal{U} that appears in \mathcal{S}_i, if f_a \geq \tau and f_a^p \geq ||\mathcal{S}_i||_p^p / B, a \in H_i.
        Compute \hat{s}_1, \hat{s}_2, \dots, \hat{s}_k where \forall l \in [k], \hat{s}_l is an (\alpha, \gamma)-approximation to |\{a \in \mathcal{U} \mid f_a = l\}|.
        for q \in [q_1^*, q_2^*] do
              Initialize \hat{z}_q = 0.
              for i \in [L] \cup \{0\} do
                   \begin{aligned} & \text{if } |\{a \in H_i \mid \hat{f}_i(a) \in I_q\}| \ge 8\lambda/\eta^2 \text{ or } i = 0 \text{ then} \\ & | \hat{z}_q \leftarrow \max(\hat{z}_q, |\{a \in H_i \mid \hat{f}_i(a) \in I_q\}| \cdot 2^i). \end{aligned}
end
                   \mathbf{end}
             \mathbf{end}
        \mathbf{end}
       Output \hat{F}_p = \sum_{l \in [k]} \hat{s}_l \cdot l^p + \sum_{q \in [q_1^*, q_2^*]} \hat{z}_q \cdot (\beta (1+\eta)^q)^p
  end
```

Lemma 5.19. Consider the subroutines in Algorithm 8. If the algorithm that continually release $\hat{s}_1, \hat{s}_2, \dots, \hat{s}_k$ is ε -DP and for every $i \in [L] \cup \{0\}$ the algorithm that continually release (H_i, \hat{f}_i) is ε -DP. Algorithm 8 is 4ε -DP in the continual release model.

Proof. Consider two neighboring stream $\mathcal{S} = (a_1, a_2, \cdots, a_T)$ and $\mathcal{S}' = (a'_1, a'_2, \cdots, a'_T)$ where they only differ at timestamp t, i.e., $a_t \neq a'_t$. Consider the corresponding streams S_0, S_1, \dots, S_L and $\mathcal{S}'_0, \mathcal{S}'_1, \dots, \mathcal{S}'_L$. $\forall i \in [L] \cup \{0\}, j \neq t$, the *j*-th element of \mathcal{S}_i should be the same as the *j*th element of \mathcal{S}'_i . Furthermore, $\forall i \in [L]$ with $i \neq g(a_t), i \neq g(a'_t)$, the *t*-th element of \mathcal{S}_i is also the same as the *t*-th element of \mathcal{S}'_i . Thus, at most 3 streams $\mathcal{S}_0, \mathcal{S}_{g(a_t)}$ and $\mathcal{S}_{g(a'_t)}$ might be different from $\mathcal{S}'_0, \mathcal{S}'_{g(a_t)}$ and $\mathcal{S}'_{g(a'_t)}$ respectively. Thus, the output $\{(H_0, \hat{f}_0), (H_1, \hat{f}_1), \cdots, (H_L, \hat{f}_L)\}$ is 3ε -DP. Since $(\hat{s}_1, \hat{s}_2, \dots, \hat{s}_k)$ is ε -DP and the final output only depends on $(\hat{s}_1, \hat{s}_2, \dots, \hat{s}_k)$ and $\{(H_0, \hat{f}_0), (H_1, \hat{f}_1), \cdots, (H_L, \hat{f}_L)\}$, the final output is 4ε -DP. П

In the remaining of the section, let us analyze the approximation guarantee of Algorithm 8. Let us consider a timestamp $t \in [T]$ and $\forall a \in \mathcal{U}$, let f_a denote the frequency of a in a_1, a_2, \cdots, a_t . Let $\mathcal{S}, \mathcal{S}_0, \mathcal{S}_1, \cdots, \mathcal{S}_L \text{ denote the streams up to timestamp } t. \text{ Let } \mathcal{I} = \{[1, 1], [2, 2], \cdots, [k, k], I_{q_1^*}, I_{q_1^*+1}, \cdots, I_{q_2^*}\}.$ By our choice of k, q_1^*, q_2^* , we have the following observation:

Observation 5.20. $\sum_{I \in \mathcal{I}} \sum_{a \in \mathcal{U}: f_a \in I} f_a^p = \|\mathcal{S}\|_p^p$.

Definition 5.21. For any $I \in \mathcal{I}$, if $\sum_{a \in \mathcal{U}: f_a \in I} f_a^p \geq \eta \|\mathcal{S}\|_p^p/(q_2^* - q_1^* + 1)$ or I is $\{i\}$ for some $i \in [k]$, then interval I is contributing.

5.4.1Analysis of High Frequency Elements

In this section, we show that $\sum_{q \in [q_1^*, q_2^*]} \hat{z}_q \cdot (\beta(1+\eta)^q)^p$ is a good approximation to $\sum_{q \in [q_1^*, q_2^*]} \sum_{a \in \mathcal{U}: f_a \in I_q} f_a^p$ and $\sum_{q \in [q_1^*, q_2^*]: I_q \text{ is contributing }} \sum_{a \in \mathcal{U}: f_a \in I_q} f_a^p$. The following lemma says that the frequency of any particular sampled element should be

sufficiently far away from the boundary of intervals $I_{q_1^*}, I_{q_1^*+1}, \cdots, I_{q_2^*}$ with a good probability.

Lemma 5.22 (Indyk and Woodruff [2005]). Consider β in Algorithm 8. Consider any $f \in [T]$ and any $r \geq 2(\eta/T)^{C-1}$.

$$\Pr_{\beta'} \left[\min_{q \in \{q_1^*, q_1^* + 1, \cdots, q_2^*, q_2^* + 1\}} |f - \beta (1 + \eta)^q| < r \right] \le \frac{100r}{\eta \cdot f}$$

By applying above lemma, we show that with high probability, we can use $\hat{f}_i(a)$ to correctly classify a into right interval in $I_{q_1^*}, I_{q_1^*+1}, I_{q_1^*+2}, \cdots, I_{q_2^*}$.

Lemma 5.23. With probability at least 0.99, $\forall i \in [L] \cup \{0\}, \forall a \in H_i, and \forall q \in [q_1^*, q_2^*], \hat{f}_i(a) \in I_q$ if and only if $f_a \in I_q$.

Proof. According to Lemma 5.22, for any $i \in [L] \cup \{0\}$ and any $a \in H_i$, we have:

$$\Pr\left[\min_{q\in\{q_1^*,q_1^*+1,\cdots,q_2^*+1\}} |f_a - \beta(1+\eta)^q| < \eta' f_a\right] \le \frac{1}{100 \cdot (L+1) \cdot |H_i|},$$

where the inequality follows from $\eta' \leq \frac{\eta}{10000(L+1)|H_i|}$. By taking a union bound over all $i \in [L] \cup \{0\}$ and all $a \in H_i$, with probability at least 0.99, the following event happens: $\forall i \in [L] \cup \{0\}, \forall a \in [L]$ $H_i, \forall q \in \{q_1^*, q_1^* + 1, \cdots, q_2^* + 1\},\$

1. if $f_a < \beta(1+\eta)^q$, then $\hat{f}_i(a) < f_a + \eta' f_a < \beta(1+\eta)^q$,

2. if $f_a > \beta (1+\eta)^q$, then $\hat{f}_i(a) \ge f_a - \eta' f_a > \beta (1+\eta)^q$.

Therefore, with probability at least 0.99, $\forall i \in [L] \cup \{0\}$, $\forall a \in H_i$, and $\forall q \in [q_1^*, q_2^*]$, $\hat{f}_i(a) \in I_q$ if and only if $f_a \in I_q$.

Let G_i denote the set of elements that appears in the stream S_i and let $G_{i,q}$ denote the set of elements that appear in the stream S_i and whose frequency is in the interval I_q . Formally, $\forall i \in [L]$, let $G_i = \{a_j \mid g(a_j) = i, j \leq t\}, G_0 = \{a_j \mid j \leq t\}$, and $\forall q \in \{q_1^*, q_1^* + 1, \cdots, q_2^*\}, \forall i \in [L] \cup \{0\}$, let $G_{i,q} = \{a \in G_i \mid f_a \in I_q\}$. For $q \in [q_1^*, q_2^*]$, let $z_q = |G_{0,q}|$, i.e., the number of elements that are in the stream S and have frequency in the range I_q .

Lemma 5.24. $\forall i \in [L] \cup \{0\}, q \in [q_1^*, q_2^*], if i = 0 \text{ or } z_q \ge 2^i \cdot 4\lambda/\eta^2, \Pr[|G_{i,q}| \in (1 \pm \eta) \cdot z_q/2^i] \ge 1 - 0.01/((L+1) \cdot \log(4T)/\eta).$ Otherwise, $\Pr[||G_{i,q}| - z_q/2^i| \le 4\lambda/\eta] \ge 1 - 0.01/((L+1) \cdot \log(4T)/\eta).$

Proof. Consider any $i \in [L] \cup \{0\}$ and $q \in [q_1^*, q_2^*]$. If i = 0, by definition $z_q = |G_{0,q}|$. Suppose $z_q \ge 2^i \cdot 4\lambda/\eta^2$. Due to Lemma 2.6, we have:

$$\Pr\left[\left||G_{i,q}| - z_q/2^i\right| > \eta \cdot z_q/2^i$$
$$\leq 8 \cdot \left(\frac{\lambda \cdot z_q/2^i + \lambda^2}{\left(\eta \cdot z_q/2^i\right)^2}\right)^{\lambda/2}$$
$$\leq 0.01/\left((L+1) \cdot \log(4T)/\eta\right),$$

where the last inequality follows from that $\lambda = 2 \cdot \log(1000(L+1)\log(4T)/\eta)$ and $z_q \ge 2^i \cdot 4\lambda/\eta^2$. Suppose $z_q \le 2^i \cdot 4\lambda/\eta^2$, By applying Lemma 2.6 again, we have:

$$\Pr\left[\left||G_{i,q}| - z_q/2^i\right| > 4\lambda/\eta\right]$$

$$\leq 8 \cdot \left(\frac{\lambda \cdot z_q/2^i + \lambda^2}{(4\lambda/\eta)^2}\right)^{\lambda/2}$$

$$\leq 0.01/\left((L+1) \cdot \log(4T)/\eta\right),$$

where the last inequality follows from $z_q/2^i \leq 4\lambda/\eta^2$ and $\lambda = 2 \cdot \log(1000(L+1)\log(4T)/\eta)$.

We define events \mathcal{E}_1 and \mathcal{E}_2 as the following: \mathcal{E}_1 denotes the event $\forall i \in [L] \cup \{0\}, \forall a \in H_i, \forall q \in [q_1^*, q_2^*], \hat{f}_i(a) \in I_q$ if and only if $f_a \in I_q$. \mathcal{E}_2 denotes the event:

1. $\forall i \in [L] \cup \{0\}, \forall q \in [q_1^*, q_2^*], \text{ if } z_q \ge 2^i \cdot 4\lambda/\eta^2 \text{ or } i = 0, |G_{i,q}| \cdot 2^i \in (1 \pm \eta)z_q.$ 2. $\forall i \in [L], \forall q \in [q_1^*, q_2^*], \text{ if } z_q < 2^i \cdot 4\lambda/\eta^2, |G_{i,q}| \in z_q/2^i \pm 4\lambda/\eta.$

According to Lemma 5.22, \mathcal{E}_1 happens with probability at least 0.99. According to Lemma 5.24, since $\forall q \in [q_1^*, q_2^*], z_q = |G_{0,q}|$ and $q_2^* - q_1^* + 1 \leq \log(4T)/\eta$, by taking a union bound over all $i \in [L] \cup \{0\}$ and all $q \in [q_1^*, q_2^*], \mathcal{E}_2$ happens with probability at least 0.99.

Lemma 5.25 (Upper bound of estimation of high frequency moment). Condition on \mathcal{E}_1 and \mathcal{E}_2 ,

$$\sum_{q \in [q_1^*, q_2^*]} \hat{z}_q \cdot (\beta (1+\eta)^q)^p \le (1+\eta) \cdot \sum_{q \in [q_1^*, q_2^*]} \sum_{a \in \mathcal{U}, f_a \in I_q} f_a^p$$

Proof. Due to event \mathcal{E}_1 , we have $\forall i \in [L] \cup \{0\}, \forall q \in [q_1^*, q_2^*], \{a \in H_i \mid \hat{f}_i(a) \in I_q\} \subseteq G_{i,q}$. Note that for $q \in [q_1^*, q_2^*], \hat{z}_q$ is either 0, or $\hat{z}_q = 2^{i'} \cdot |\{a \in H_{i'} \mid \hat{f}_{i'}(a) \in I_q\}|$ for some i' satisfying i' = 0 or $|\{a \in H_{i'} \mid \hat{f}_{i'}(a) \in I_q\}| \geq 8\lambda/\eta^2$. Since $\forall q \in [q_1^*, q_2^*], \forall i \in [L] \cup \{0\}, |\{a \in H_i \mid \hat{f}_i(a) \in I_q\}| \leq |G_{i,q}|, \forall q \in [q_1^*, q_2^*], \text{ if } \hat{z}_q \neq 0$, there is some $i' \in [L] \cup \{0\}$ such that:

$$\begin{aligned} \hat{z}_{q} &= 2^{i'} \cdot |\{a \in H_{i'} \mid \hat{f}_{i'}(a) \in I_{q}\}| \\ &\leq 2^{i'} \cdot |G_{i',q}| \\ &\leq (1+\eta) \cdot z_{q}, \end{aligned}$$

where the second inequality follows from that $|G_{i',q}| \ge 8\lambda/\eta^2$ which implies that $|G_{i',q}| \cdot 2^{i'} \in (1 \pm \eta)z_q$ according to event \mathcal{E}_2 .

Therefore,

$$\sum_{q \in [q_1^*, q_2^*]} \hat{z}_q \cdot (\beta (1+\eta)^q)^p$$

$$\leq (1+\eta) \sum_{q \in [q_1^*, q_2^*]} z_q \cdot (\beta (1+\eta)^q)^p$$

$$\leq (1+\eta) \sum_{q \in [q_1^*, q_2^*]} \sum_{a \in \mathcal{U}: f_a \in I_q} (\beta (1+\eta)^q)^p$$

$$\leq (1+\eta) \sum_{q \in [q_1^*, q_2^*]} \sum_{a \in \mathcal{U}: f_a \in I_q} f_a^p,$$

where the first inequality follows from $\hat{z}_q \leq (1+\eta)z_q$, the second inequality follows from the definition of z_q , i.e., $z_q = |\{a \in \mathcal{U} \mid f_a \in I_q\}|$, and the last inequality follows from that $f_a \geq \beta(1+\eta)^q$ if $f_a \in I_q$.

Define the event \mathcal{E}_3 as: $\forall i \in [L] \cap \{0\}, \|\mathcal{S}_i\|_p^p \leq 100(L+1) \cdot \|\mathcal{S}\|_p^p/2^i$.

Lemma 5.26. \mathcal{E}_3 happens with probability at least 0.99.

Proof. Consider $i \in [L] \cup \{0\}$. We have $E\left[\|\mathcal{S}_i\|_p^p\right] = \sum_{a \in \mathcal{U}} \Pr[g(a) = i] \cdot f_a^p = \|\mathcal{S}\|_p^p / 2^i$. By Markov's inequality, with probability at least 1 - 1/(100(L+1)), $\|\mathcal{S}_i\|_p^p \leq 100(L+1) \cdot \|\mathcal{S}\|_p^p / 2^i$. By taking a union bound over all $i \in [L] \cup \{0\}$, \mathcal{E}_3 happens with probability at least 0.99.

In the remaining of the analysis, we condition on \mathcal{E}_3 as well.

Lemma 5.27 (Lower bound of estimation of contributing high frequency moment). Condition on $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$.

$$\sum_{q\in[q_1^*,q_2^*]} \hat{z}_q \cdot (\beta(1+\eta)^q)^p \geq (1-\eta)^{p+1} \cdot \sum_{q\in[q_1^*,q_2^*]:I_q \text{ is contributing } a\in\mathcal{U}, f_a\in I_q} f_a^p$$

Proof. Consider any contributing $q \in [q_1^*, q_2^*]$.

Case 1: $z_q \leq 16\lambda/\eta^2$. According to Definition 5.21, we have $\sum_{a \in \mathcal{U}: f_a \in I_q} f_a^p \geq \eta \|\mathcal{S}\|_p^p/(q_2^* - q_1^* + 1)$ which implies that $\eta \|\mathcal{S}_0\|_p^p/(q_2^* - q_1^* + 1) \leq z_q \cdot (\beta(1+\eta)^{q+1})^p \leq 16\lambda/\eta^2 \cdot (1+\eta)^p \cdot (\beta(1+\eta)^q)^p$.

Since $B \ge (q_2^* - q_1^* + 1) \cdot 16\lambda \cdot (1 + \eta)^p / \eta^3$, $\forall a \in \mathcal{U}$ with $f_a \in I_q$, $f_a^p \ge \|\mathcal{S}_0\|_p^p / B$. Therefore, $\forall a \in \mathcal{U}$ with $f_a \in I_q$, we have $a \in H_0$. According to event \mathcal{E}_1 , we have $|\{a \in H_0 \mid \hat{f}_0(a) \in I_q\}| = |G_{0,q}| = z_q$. Thus, we have $\hat{z}_q \ge |\{a \in H_0 \mid \hat{f}_0(a) \in I_q\}| \ge z_q$.

Case 2: $z_q > 16\lambda/\eta^2$. Let $i^* \in \{0\} \cup [L]$ be the largest value such that $z_q/2^{i^*} \ge 16\lambda/\eta^2$. According to event \mathcal{E}_2 , we have $|G_{i^*,q}| \ge 8\lambda/\eta^2$.

Since q is contributing, we have:

$$z_{q} \cdot (\beta(1+\eta)^{q+1})^{p}$$

$$\geq \sum_{a \in \mathcal{U}: f_{a} \in I_{q}} f_{a}^{p}$$

$$\geq \eta \|\mathcal{S}\|_{p}^{p} / (q_{2}^{*} - q_{1}^{*} + 1)$$

$$\geq 2^{i^{*}} \cdot \eta \|\mathcal{S}_{i^{*}}\|_{p}^{p} / ((q_{2}^{*} - q_{1}^{*} + 1) \cdot 100(L+1)),$$

where the last inequality follows from event \mathcal{E}_3 . Therefore, we have

$$(\beta(1+\eta)^{q})^{p} \geq \eta \|\mathcal{S}_{i^{*}}\|_{p}^{p}/((q_{2}^{*}-q_{1}^{*}+1)\cdot 100(L+1)\cdot (z_{q}/2^{i^{*}})\cdot (1+\eta)^{p}) \geq \eta \|\mathcal{S}_{i^{*}}\|_{p}^{p}/((q_{2}^{*}-q_{1}^{*}+1)\cdot 100(L+1)\cdot (32\lambda/\eta^{2})\cdot (1+\eta)^{p})$$

where the last inequality follows from $z_q/2^{i^*} \leq 32\lambda/\eta^2$. Since $B \geq ((q_2^* - q_1^* + 1) \cdot 100(L+1) \cdot (32\lambda/\eta^2) \cdot (1+\eta)^p)/\eta$, we have $\forall a \in \mathcal{U}$ with $a \in G_{i^*,q}$, $f_a^p \geq \|\mathcal{S}_{i^*}\|_p^p/B$ and $f_a > \beta(1+\eta)^q > \tau$ which implies that $a \in H_{i^*}$. According to event \mathcal{E}_1 , we have $\{a \in H_{i^*} \mid \hat{f}_{i^*}(a) \in I_q\} = G_{i^*,q}$. Therefore, $|\{a \in H_{i^*} \mid \hat{f}_{i^*}(a) \in I_q\}| \geq 8\lambda/\eta^2$. Finally, in addition, according to event \mathcal{E}_2 , we have $\hat{z}_q \geq |\{a \in H_{i^*} \mid \hat{f}_{i^*}(a) \in I_q\}| \cdot 2^{i^*} \geq (1-\eta)z_q$.

Therefore, in any case, we have $\hat{z}_q \ge (1-\eta)z_q$, and we have:

$$\begin{split} &\sum_{q\in[q_1^*,q_2^*]} \hat{z}_q \cdot (\beta(1+\eta)^q)^p \\ \geq &\sum_{q\in[q_1^*,q_2^*]:I_q \text{ is contributing}} \hat{z}_q \cdot (\beta(1+\eta)^q)^p \\ \geq &\sum_{q\in[q_1^*,q_2^*]:I_q \text{ is contributing}} (1-\eta)z_q \cdot (\beta(1+\eta)^q)^p \\ = &\frac{1-\eta}{(1+\eta)^p} \sum_{q\in[q_1^*,q_2^*]:I_q \text{ is contributing } a\in\mathcal{U}:f_a\in I_q} (\beta(1+\eta)^{q+1})^p \\ \geq &\frac{1-\eta}{(1+\eta)^p} \sum_{q\in[q_1^*,q_2^*]:I_q \text{ is contributing } a\in\mathcal{U}:f_a\in I_q} f_a^p \\ \geq &(1-\eta)^{p+1} \sum_{q\in[q_1^*,q_2^*]:I_q \text{ is contributing } a\in\mathcal{U}:f_a\in I_q} f_a^p, \end{split}$$

where the equality follows from the definition that $z_q = |\{a \in \mathcal{U} \mid f_a \in I_q\}|.$

5.4.2 Analysis of Low Frequency Elements

In this section, we show that $\sum_{l \in [k]} \hat{s}_l \cdot l^p$ is a good approximation to $\sum_{l \in [k]} \sum_{a \in \mathcal{U}: f_a = l} f_a^p$.

Lemma 5.28 (Approximation of low frequency moments). $\sum_{l \in [k]} \hat{s}_l \cdot l^p$ is a $(\alpha, \gamma \cdot (2\tau)^{p+1})$ approximation to $\sum_{l \in [k]} \sum_{a \in \mathcal{U}: f_a = l} f_a^p$.

Proof. For $l \in [k]$, let $s_l = |\{a \in \mathcal{U} \mid f_a = l\}|$. We have:

$$\sum_{l \in [k]} \hat{s}_l \cdot l^p$$

$$\geq \sum_{l \in [k]} \left(\frac{1}{\alpha} \cdot s_l - \gamma\right) \cdot l^p$$

$$\geq \frac{1}{\alpha} \cdot \sum_{l \in [k]} \sum_{a \in \mathcal{U}: f_a = l} f_a^p - k \cdot \gamma \cdot k^p$$

$$\geq \frac{1}{\alpha} \cdot \sum_{l \in [k]} \sum_{a \in \mathcal{U}: f_a = l} f_a^p - \gamma \cdot (2\tau)^{p+1},$$

where the last inequality follows from that our choice of k implies that $k \leq 2\tau$. Similarly, we have:

$$\sum_{l \in [k]} \hat{s}_l \cdot l^p$$

$$\leq \sum_{l \in [k]} (\alpha \cdot s_l + \gamma) \cdot l^p$$

$$\leq \alpha \cdot \sum_{l \in [k]} \sum_{a \in \mathcal{U}: f_a = l} f_a^p - \gamma \cdot (2\tau)^{p+1}.$$

5.4.3 Putting High Frequency Moments and Low Frequency Moments Together Lemma 5.29. $\sum_{contributing I \in \mathcal{I}} \sum_{a \in \mathcal{U}: f_a \in I} f_a^p \ge (1 - \eta) \cdot \|\mathcal{S}\|_p^p$. Proof.

$$\sum_{\text{contributing } I \in \mathcal{I}} \sum_{a \in \mathcal{U}: f_a \in I} f_a^p$$

$$= \sum_{I \in \mathcal{I}} \sum_{a \in \mathcal{U}: f_a \in I} f_a^p - \sum_{\text{non-contributing } I \in \mathcal{I}} \sum_{a \in \mathcal{U}: f_a \in I} f_a^p$$

$$\geq \|\mathcal{S}\|_p^p - (q_2^* - q_1^* + 1) \cdot \eta \cdot \|\mathcal{S}\|_p^p / (q_2^* - q_1^* + 1)$$

$$\geq (1 - \eta) \cdot \|\mathcal{S}\|_p^p.$$

Lemma 5.30. Consider any timestamp $t \in [T]$. $\forall a \in \mathcal{U}$, let f_a denote the frequency of a in a_1, a_2, \cdots, a_t . With probability at least 0.9, the output \hat{F}_p of Algorithm 8 is $a\left(\max\left(\frac{\alpha}{1-\eta}, (1+2\eta)^{p+2}\right), \gamma \cdot (2\tau)^{p+1}\right)$ -approximation to $\|\mathcal{S}\|_p^p$.

Proof. According to Lemma 5.25 and Lemma 5.27, we have:

$$(1-\eta)^{p+1} \sum_{q \in [q_1^*, q_2^*]: I_q \text{ is contributing } a \in \mathcal{U}, f_a \in I_q} \sum_{q \in [q_1^*, q_2^*]} \hat{z}_q \cdot (\beta(1+\eta)^q)^p \le (1+\eta) \sum_{q \in [q_1^*, q_2^*]} \sum_{a \in \mathcal{U}, f_a \in I_q} f_a^p \sum_{q \in [q_1^*, q_2^*]} \sum_{a \in \mathcal{U}, f_a \in I_q} f_a^p \sum_{q \in [q_1^*, q_2^*]} \sum_{a \in \mathcal{U}, f_a \in I_q} f_a^p \sum_{q \in [q_1^*, q_2^*]} \sum_{a \in \mathcal{U}, f_a \in I_q} f_a^p \sum_{q \in [q_1^*, q_2^*]} \sum_{a \in \mathcal{U}, f_a \in I_q} f_a^p \sum_{q \in [q_1^*, q_2^*]} \sum_{a \in \mathcal{U}, f_a \in I_q} f_a^p \sum_{q \in [q_1^*, q_2^*]} f_a$$

According to Lemma 5.28, we have:

$$\frac{1}{\alpha} \cdot \sum_{l \in [k]} \sum_{a \in \mathcal{U}: f_a = l} f_a^p - \gamma \cdot (2\tau)^{p+1} \le \sum_{l \in [k]} \hat{s}_l \cdot l^p \le \alpha \cdot \sum_{l \in [k]} \sum_{a \in \mathcal{U}: f_a = l} f_a^p + \gamma \cdot (2\tau)^{p+1}$$

Therefore, we have:

$$\hat{F}_{p} \geq \min\left(\frac{1}{\alpha}, (1-\eta)^{p+1}\right) \sum_{I \in \mathcal{I}:I \text{ is contributing } a \in \mathcal{U}: f_{a} \in I} \sum_{f_{a} = I} f_{a}^{p} - \gamma \cdot (2\tau)^{p+1}$$
$$\geq \min\left(\frac{1-\eta}{\alpha}, (1-\eta)^{p+2}\right) \|\mathcal{S}\|_{p}^{p} - \gamma \cdot (2\tau)^{p+1}$$
$$\geq \min\left(\frac{1-\eta}{\alpha}, \frac{1}{(1+2\eta)^{p+2}}\right) \|\mathcal{S}\|_{p}^{p} - \gamma \cdot (2\tau)^{p+1}$$

where the second step follows from Lemma 5.29. Similarly, we have:

$$\hat{F}_p \le \max(\alpha, 1+\eta) \sum_{I \in \mathcal{I}} \sum_{a \in \mathcal{U}: f_a \in I} f_a^p + \gamma \cdot (2\tau)^{p+1}$$
$$= \max(\alpha, 1+\eta) \cdot \|\mathcal{S}\|_p^p + \gamma \cdot (2\tau)^{p+1}$$

Theorem 5.31 (Streaming continual release ℓ_p frequency moment estimation). Let $p > 0, \varepsilon \ge 0, \xi \in (0, 0.5), \eta \in (0, 0.5)$. There is an ε -DP algorithm in the streaming continual release model such that with probability at least $1 - \xi$, it always outputs an $\left(1 + \eta, \left(\frac{\log(T|\mathcal{U}|/\xi)}{\eta\varepsilon}\right)^{O(\max(1,p))}\right)$ -approximation to $\|\mathcal{S}\|_p^p$. The algorithm uses space at most

$$\phi \cdot \left(\frac{\log(T|\mathcal{U}|/\xi)}{\eta\varepsilon}\right)^{O(\max(1,p))}$$

where $\phi = \max(1, |\mathcal{U}|^{1-2/p}).$

Proof. To boost the probability of the approximation guarantee of Lemma 5.30 to $1 - \xi/3$ and simultaneously for all timestamps $t \in T$, we run $\lceil 50 \log(3T/\xi) \rceil$ independent copies of Algorithm 8 and take the median of the outputs. Since we run $\lceil 50 \log(3T/\xi) \rceil$ independent copies of Algorithm 8 and according to Lemma 5.19, if we want the final algorithm to be ε -DP, we need each subroutine of the streaming continual release of (H_i, \hat{f}_i) for $i \in [L] \cup \{0\}$ to be $(\varepsilon/(4 \cdot \lceil 50 \log(3T/\xi) \rceil))$ -DP and we need the subroutine of the streaming continual release of $\{\hat{s}_1, \hat{s}_2, \dots, \hat{s}_k\}$ to be $(\varepsilon/(4 \cdot \lceil 50 \log(3T/\xi) \rceil))$ -DP as well. To simultaneously make the call of each subroutine of the streaming continual release of (H_i, \hat{f}_i) over all independent copies of Algorithm 8 satisfy the desired properties stated in Algorithm 8 with probability at least $1 - \xi/3$, we need to make (H_i, \hat{f}_i) satisfy the property for each particular $i \in [L] \cup \{0\}$ and a particular copy of Algorithm 8 with probability at least $1 - \xi/(4 \cdot \lceil 50 \log(3T/\xi) \rceil \cdot (L+1))$. To simultaneously make the call of each subroutine of the streaming continual release of $\{\hat{s}_1, \hat{s}_2, \dots, \hat{s}_k\}$ over all independent copies of Algorithm 8 satisfy the desired property stated in Algorithm 8 with probability at least $1 - \xi/(4 \cdot \lceil 50 \log(3T/\xi) \rceil \cdot (L+1))$. To simultaneously make the call of each subroutine of the streaming continual release of $\{\hat{s}_1, \hat{s}_2, \dots, \hat{s}_k\}$ over all independent copies of Algorithm 8 satisfy the desired property stated in Algorithm 8 with probability at least $1 - \xi/3$, we need to make $\{\hat{s}_1, \hat{s}_2, \dots, \hat{s}_k\}$ satisfy the desired property for each particular $i \in [L] \cup \{0\}$ and a particular copy of Algorithm 8 with probability at least $1 - \xi/3$, we need to make $\{\hat{s}_1, \hat{s}_2, \dots, \hat{s}_k\}$ satisfy the desired property for each particular $i \in [L] \cup \{0\}$ and a particular copy of Algorithm 8 with probability at least $1 - \xi/(3 \cdot \lceil 50 \log(3T/\xi) \rceil)$. According to Algorithm 8, we have

$$B = \Theta\left(\frac{\log(T)\log(|\mathcal{U}|)\log(\log(T|\mathcal{U}|)/\eta) \cdot (1+\eta)^p}{\eta^4}\right)$$

Thus, according to Theorem 5.10, the size of $|H_i|$ in Algorithm 8 for $i \in [L] \cup \{0\}$ is at most $\operatorname{poly}\left(\frac{\log(T|\mathcal{U}|/\xi)}{\eta}\right) \cdot 2^{O(p)}$. Thus, we choose $\eta' = 1/\left(\operatorname{poly}\left(\frac{\log(T|\mathcal{U}|/\xi)}{\eta}\right) \cdot 2^{O(p)}\right)$. Then according to Theorem 5.10, we have

$$\tau = \frac{1}{\varepsilon} \cdot \operatorname{poly}\left(\frac{\log(T \cdot |\mathcal{U}|/\xi)}{\eta}\right) \cdot 2^{O(p)}.$$

According to Theorem 5.18, we have $\alpha = 1$, and we choose γ to be

$$\left(\eta'' + \eta''^2 \cdot \frac{\tau}{\varepsilon} \cdot \operatorname{poly}\left(\log\left(\frac{T\tau}{\xi}\right)\right)\right) \cdot \|\mathcal{S}\|_0 + \frac{1}{\varepsilon} \cdot \operatorname{poly}\left(\log\left(\frac{T}{\xi}\right)\right) + O\left(\frac{\log\tau}{\eta''^2}\right),$$

where we choose η'' to be

$$\frac{\varepsilon\eta}{\tau^{O(\max(1,p))}\operatorname{poly}\left(\frac{\log(T|\mathcal{U}|/\xi)}{\eta}\right)} = \frac{1}{\left(\frac{\log(T|\mathcal{U}|/\xi)}{\eta\varepsilon}\right)^{O(\max(1,p))}}$$

such that

$$\gamma \cdot (2\tau)^{p+1} \le \eta \|\mathcal{S}\|_0 + \left(\frac{\log(T|\mathcal{U}|/\xi)}{\eta\varepsilon}\right)^{O(\max(1,p))} \\ \le \eta \|\mathcal{S}\|_p^p + \left(\frac{\log(T|\mathcal{U}|/\xi)}{\eta\varepsilon}\right)^{O(\max(1,p))}$$

Note that $\eta \|\mathcal{S}\|_p^p$ becomes the relative error. Thus, according to Lemma 5.30, the output \hat{F}_p is an $\left((1+\eta)^{O(\max(1,p))}, \left(\frac{\log(T|\mathcal{U}|/\xi)}{\eta\varepsilon}\right)^{O(\max(1,p))}\right)$ -approximation.

Next, consider the space usage. According to Theorem 5.10, the total space needed to run all heavy hitters subroutines is at most

$$\phi \cdot 2^{O(\max(1,p))} \cdot \operatorname{poly}\left(\frac{\log(T|\mathcal{U}|/\xi)}{\eta}\right).$$

According to Theorem 5.18, the total space needed for computing $\{\hat{s}_1, \hat{s}_2, \dots, \hat{s}_k\}$ for all running copies of Algorithm 8 is at most

$$\left(\frac{\log(T|\mathcal{U}|/\xi)}{\eta\varepsilon}\right)^{O(\max(1,p))}$$

Therefore, the overall space needed is at most

$$\phi \cdot \left(\frac{\log(T|\mathcal{U}|/\xi)}{\eta\varepsilon}\right)^{O(\max(1,p))}$$

6 Extension to Sliding Window Continual Release Algorithms

In this section, we briefly review the smooth histogram Braverman and Ostrovsky [2007] technique which converts any (non-private) streaming algorithm into (non-private) sliding window algorithm. The original framework only supports the approximation algorithm which only has relative error and no additive error. In this section, we show how to extend it to support the additive error as well.

Suppose there are two streams $\mathcal{A} = (a_1, a_2, \cdots, a_{t_1})$ and $\mathcal{B} = (b_1, b_2, \cdots, b_{t_2})$. We use $\mathcal{A} \cup \mathcal{B}$ to denote the concatenation of two streams, i.e., $\mathcal{A} \cup \mathcal{B} = (a_1, a_2, \cdots, a_{t_1}, b_1, b_2, \cdots, b_{t_2})$. If \mathcal{B} is a suffix of \mathcal{A} , i.e., $\exists i \in [t_1]$ such that $a_i = b_1, a_{i+1} = b_2, \cdots, a_{t_1} = b_{t_2}$, then we denote it as $\mathcal{B} \subseteq_r \mathcal{A}$.

Definition 6.1 (Smooth function Braverman and Ostrovsky [2007]). Let $g(\cdot)$ be a function over streams. Function $g(\cdot)$ is (ζ, β) -smooth if:

1. $\forall \mathcal{A}, 0 \leq g(\mathcal{A}) \leq \operatorname{poly}(T).$

2.
$$\forall \mathcal{A}, \mathcal{B} \text{ with } \mathcal{B} \subseteq_r \mathcal{A}, \ g(\mathcal{A}) \geq g(\mathcal{B})$$

- 3. For any $\eta \in (0,1)$, there exists $\zeta(\eta,g)$ and $\beta(\eta,g)$ such that
 - (a) $0 < \beta \le \zeta < 1$.

(b) If
$$\mathcal{B} \subseteq_r \mathcal{A}$$
 and $(1-\beta)g(\mathcal{A}) \leq g(\mathcal{B})$ then $(1-\zeta)g(\mathcal{A}\cup\mathcal{C}) \leq g(\mathcal{B}\cup\mathcal{C})$ for any stream \mathcal{C} .

Lemma 6.2 (Smoothness of frequency moments Braverman and Ostrovsky [2007]). For p > 1, $\|S\|_p^p$ is $\left(\eta, \left(\frac{\eta}{p}\right)^p\right)$ -smooth. For $0 , <math>\|S\|_p^p$ is (η, η) -smooth. $\|S\|_0$ is (η, η) -smooth.

Lemma 6.3. Let $0 \le Z \le \operatorname{poly}(T)$. If $g(\mathcal{S}) := \|\mathcal{S}\|_p^p + Z$, then $g(\cdot)$ is $\left(\eta, \left(\frac{\eta}{p}\right)^p\right)$ -smooth if p > 1 and $g(\cdot)$ is (η, η) -smooth if $0 . If <math>g(\mathcal{S}) = \|\mathcal{S}\|_0 + Z$, $g(\cdot)$ is (η, η) -smooth.

Proof. For $g(\cdot)$ stated in the lemma statement, the first two requirements in Definition 6.1 are satisfied obviously. Therefore, we only need to justify the third requirement of Definition 6.1.

Let us consider \mathcal{A} and \mathcal{B} such that $\mathcal{B} \subseteq_r \mathcal{A}$ and $(1-\beta)g(\mathcal{A}) \leq g(\mathcal{B})$. Consider any stream \mathcal{C} . We are going to prove $(1-\zeta)g(\mathcal{A}\cup\mathcal{C}) \leq g(\mathcal{B}\cup\mathcal{C})$. Consider the case $g(\mathcal{S}) = \|\mathcal{S}\|_p^p + Z$. We construct an auxiliary stream \mathcal{X} such that the elements of \mathcal{X} do not appear in any of $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\|\mathcal{X}\|_p^p = Z$. Then, we have $(1-\beta)\|\mathcal{A}\cup\mathcal{X}\|_p^p = (1-\beta)g(\mathcal{A}) \leq g(\mathcal{B}) = \|\mathcal{B}\cup\mathcal{X}\|_p^p$. Due to the smoothness of $\|\mathcal{S}\|_p^p$

(Lemma 6.2), we have $(1-\zeta)g(\mathcal{A}\cup\mathcal{C}) = (1-\zeta)\|\mathcal{A}\cup\mathcal{X}\cup\mathcal{C}\|_p^p \le \|\mathcal{B}\cup\mathcal{X}\cup\mathcal{C}\|_p^p = g(\mathcal{B}\cup\mathcal{C})$. According to Lemma 6.2, $g(\cdot)$ is $\left(\eta, \left(\frac{\eta}{p}\right)^p\right)$ -smooth if p > 1 and $g(\cdot)$ is (η, η) -smooth if 0 . $Similarly, using the similar argument by constructing <math>\|\mathcal{X}\|_0 = Z$, we can show that $g(\mathcal{S}) =$

 $\|\mathcal{S}\|_0 + Z$ is (η, η) -smooth.

Lemma 6.4. Let $\alpha \geq 1, \gamma \geq 0$. If g' is an (α, γ) -approximation to g, then g' + Z is an α -approximation to g + Z if $Z \geq \frac{\alpha}{\alpha-1} \cdot \gamma$.

Proof. We have:

$$g' + Z \ge \frac{1}{\alpha} \cdot g - \gamma + Z$$
$$\ge \frac{1}{\alpha} \cdot g - \frac{\alpha - 1}{\alpha} \cdot Z + Z$$
$$\ge \frac{1}{\alpha} \cdot (g + Z).$$

On the other hand,

$$g' + Z \le \alpha \cdot g + \gamma + Z$$

$$\le \alpha \cdot g + (\alpha - 1)Z + Z$$

$$\le \alpha \cdot (g + Z).$$

г		1
L		
L		

Theorem 6.5 (Smooth histogram algorithmic framework Braverman and Ostrovsky [2007]). Let $\eta \in (0, 0.5)$. Let $g(\cdot)$ be an (ζ, β) -smooth function. If there exists a streaming algorithm Λ which maintains an $(\frac{1}{1-n})$ -approximation of $g(\cdot)$ simultaneously for all timestamps $t \in [T]$ with probability at least $1 - \xi$, using space $h(\eta, \xi)$, then there is a sliding window algorithm Λ' that maintains a $\left(\frac{1}{1-\eta-\zeta}\right)$ -approximation of $g(\cdot)$ over sliding windows simultaneously for all timestamps $t \in [T]$ with probability at least $1 - \xi$ and uses space $O\left(\frac{\log T}{\beta} \cdot h(\eta, \xi\beta/\log(T))\right)$.

Furthermore, at any timestamp $t \in [T]$, Λ' starts a new instance of Λ which regards the t-th element in the stream as the beginning of the stream, and Λ' only keeps at most $O(\log(T)/\beta)$ past instances of Λ (started from different timestamps). The output of Λ' at any timestamp t only depends on the outputs of its maintained instances Λ , and the decision of whether keeping an instance Λ to timestamp t+1 only depends on the outputs of its maintained instances Λ at timestamp t as well.

We are able to extend the above smooth histogram framework to the differentially private continual release setting.

Theorem 6.6 (Smooth histogram for differentially private continual release model). Let $q(\cdot)$ be an (ζ, β) -smooth function. If there exists a ε' -DP streaming continual release algorithm Λ which maintains an $\left(\frac{1}{1-\eta}\right)$ -approximation of $g(\cdot)$ simultaneously for all timestamps $t \in [T]$ with probability at least $1-\xi$, using space $h(\eta,\xi)$, then there is a ε -DP sliding window continual release algorithm $\Lambda' \text{ with } \varepsilon = O(\varepsilon'\beta/\log(T)) \text{ which maintains a } \left(\frac{1}{1-\eta-\zeta}\right) \text{-approximation of } g(\cdot) \text{ over sliding windows}$ for all timestamps $t \in [T]$ with probability at least $1 - \xi$ and uses space $O\left(\frac{\log T}{\beta} \cdot h(\eta, \xi\beta/\log(T))\right)$. *Proof.* The approximation guarantee, space guarantee and success probability follows from Theorem 6.5 directly. In the remaining of the proof, we prove the DP guarantee.

Let $\Lambda_1, \Lambda_2, \dots, \Lambda_T$ be instances of Λ where Λ_t is started at timestamp t. Let $o_{t,1}, o_{t,2}, \dots, o_{t,T}$ be the outputs of Λ_t , if at timestamp j, Λ_t is not started yet or is already kicked out by Λ' , then $o_{t,j} = \bot$. According to Theorem 6.5, the outputs of Λ' over all timestamps $t \in [T]$ is determined by $\{o_{i,j} \mid i, j \in [T]\}$. Thus, we only need to show that $\{o_{i,j} \mid i, j \in [T]\}$ is ε -DP. Consider two neighboring streams S and S' where only the r-th elements are different. Let us fix a possible configuration $\{o_{i,j} \mid i, j \in [T]\}$. According to Theorem 6.5, there are at most $O(\log(T)/\beta)$ different $i \in [T]$ such that $o_{i,r} \neq \bot$. Let such set of i to be I. Since each Λ_i for $i \in I$ is ε' -DP in the streaming continual releasing setting, we have:

$$\Pr\left[\forall i \in I, j \in [T], o_{i,j}(\mathcal{S}) = o_{i,j}\right] \leq \exp(\varepsilon' \cdot |I|) \cdot \Pr\left[\forall i \in I, j \in [T], o_{i,j}(\mathcal{S}') = o_{i,j}\right]$$
$$\leq \exp(\varepsilon) \cdot \Pr\left[\forall i \in I, j \in [T], o_{i,j}(\mathcal{S}') = o_{i,j}\right].$$

On the other hand, we have:

$$\Pr\left[\forall i \notin I, j \in [T], o_{i,j}(\mathcal{S}) = o_{i,j} \mid o_{i,j}(\mathcal{S}) = o_{i,j} \forall i \in I, j \in [T]\right]$$
$$= \Pr\left[\forall i \notin I, j \in [T], o_{i,j}(\mathcal{S}') = o_{i,j} \mid o_{i,j}(\mathcal{S}') = o_{i,j} \forall i \in I, j \in [T]\right]$$

Therefore, the algorithm is ε -DP.

By combining Theorem 3.3 with Lemma 6.3, Lemma 6.4 and Theorem 6.6, we are able to obtain the following sliding window continual release algorithm for summing non-negative numbers.

Corollary 6.7 (Sliding window summing of a non-negative numbers). Let $\eta \in (0, 0.5), \varepsilon \ge 0, \xi \in (0, 0.5)$, there is an ε -DP algorithm for summing in the sliding window continual release model. If the input numbers are guaranteed to be non-negative, with probability at least $1 - \xi$, the output is always a $\left(1 + \eta, O\left(\frac{\log(T/(\eta\xi))\log(T)}{\varepsilon\eta^3}\right)\right)$ -approximation to the summing problem at any timestamp $t \in [T]$. The algorithm uses space $O(\log(T)/\eta)$.

By combining Corollary 4.11 with Lemma 6.3, Lemma 6.4 and Theorem 6.6, we are able to obtain the following sliding window continual release algorithm for number of distinct elements.

Corollary 6.8 (Sliding window continual release distinct elements). For $\eta \in (0, 0.5), \varepsilon \ge 0, \xi \in (0, 0.5)$ there is an ε -DP algorithm for the number of distinct elements of streams with element universe \mathcal{U} in the sliding window continual release model. With probability at least $1 - \xi$, the output is always a $(1 + \eta, O\left(\frac{\log^2(T/(\eta\xi))\log(T)}{\eta^{4}\varepsilon}\right))$ -approximation for every timestamp $t \in [T]$. The algorithm uses poly $\left(\frac{\log(T/\xi)}{\eta\min(\varepsilon,1)}\right)$ space.

By combining Theorem 5.4 with Lemma 6.3, Lemma 6.4 and Theorem 6.6, we are able to obtain the following sliding window continual release algorithm for ℓ_2 frequency moment.

Corollary 6.9 (Sliding window continual release ℓ_2 frequency moments). Let $\varepsilon > 0, \eta \in (0, 0.5), \xi \in (0, 0.5)$. There is an ε -DP algorithm in the sliding window continual release model such that with probability at least $1 - \xi$, it always outputs \hat{F}_2 for every timestamp $t \in [T]$ such that $|\hat{F}_2 - ||\mathcal{S}||_2^2| \leq \eta ||\mathcal{S}||_2^2 + O\left(\frac{(\log(T/(\xi\eta)) + \log(|\mathcal{U}|))^2 \log^2(T)}{\varepsilon^2 \eta^8} \cdot \log^5(T) \cdot \log^2\left(\frac{\log(T/\xi) + \log(|\mathcal{U}|)}{\xi\eta}\right)\right)$, where \mathcal{S} denotes the substream corresponding to the latest W elements at timestamp t. The algorithm uses $O\left(\frac{\log(T/(\xi\eta)) + \log(|\mathcal{U}|)}{\eta^4} \cdot \log^2(T)\right)$ space.

By combining Theorem 5.31 with Lemma 6.3, Lemma 6.4 and Theorem 6.6, we are able to obtain the following sliding window continual release algorithm for ℓ_p frequency moment.

Corollary 6.10 (Sliding window continual release ℓ_p frequency moments). Let $p > 0, \varepsilon \ge 0, \xi \in (0, 0.5), \eta \in (0, 0.5)$. There is an ε -DP algorithm in the sliding window continual release model such that with probability at least $1 - \xi$, it always outputs an $\left(1 + \eta, \left(\frac{\log(T|\mathcal{U}|/\xi)}{\eta\varepsilon}\right)^{O(p)}\right)$ -approximation to $\|\mathcal{S}\|_p^p$, where \mathcal{S} denotes the sub-stream corresponding to the latest W elements at timestamp t. The algorithm uses space at most

$$\phi \cdot \left(\frac{\log(T|\mathcal{U}|/\xi)}{\eta\varepsilon}\right)^{O(p)}$$

where $\phi = \max(1, |\mathcal{U}|^{1-2/p}).$

References

- Naman Agarwal and Karan Singh. The price of differential privacy for online learning. In Doina Precup and Yee Whye Teh, editors, Proceedings of the 34th International Conference on Machine Learning, volume 70 of Proceedings of Machine Learning Research, pages 32-40. PMLR, 06-11 Aug 2017. URL https://proceedings.mlr.press/v70/agarwal17a.html.
- Noga Alon, Yossi Matias, and Mario Szegedy. The space complexity of approximating the frequency moments. In *Proceedings of the twenty-eighth annual ACM symposium on Theory of computing*, pages 20–29, 1996.
- Alexandr Andoni. High frequency moments via max-stability. In 2017 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), pages 6364–6368. IEEE, 2017.
- Alexandr Andoni, Robert Krauthgamer, and Krzysztof Onak. Streaming algorithms via precision sampling. In 2011 IEEE 52nd Annual Symposium on Foundations of Computer Science, pages 363–372. IEEE, 2011.
- Ziv Bar-Yossef, Thathachar S Jayram, Ravi Kumar, and D Sivakumar. An information statistics approach to data stream and communication complexity. *Journal of Computer and System Sciences*, 68(4):702–732, 2004.
- Mihir Bellare and John Rompel. Randomness-efficient oblivious sampling. In Proceedings 35th Annual Symposium on Foundations of Computer Science, pages 276–287. IEEE, 1994.
- Jeremiah Blocki, Avrim Blum, Anupam Datta, and Or Sheffet. The johnson-lindenstrauss transform itself preserves differential privacy. In 53rd Annual IEEE Symposium on Foundations of Computer Science, FOCS 2012, New Brunswick, NJ, USA, October 20-23, 2012, pages 410-419. IEEE Computer Society, 2012. doi: 10.1109/FOCS.2012.67. URL https://doi.org/10.1109/FOCS.2012.67.
- Jeremiah Blocki, Elena Grigorescu, Tamalika Mukherjee, and Samson Zhou. How to make your approximation algorithm private: A black-box differentially-private transformation for tunable approximation algorithms of functions with low sensitivity. *arXiv preprint arXiv:2210.03831*, 2022.

- Jean Bolot, Nadia Fawaz, S. Muthukrishnan, Aleksandar Nikolov, and Nina Taft. Private decayed predicate sums on streams. In *Proceedings of the 16th International Conference* on Database Theory, ICDT '13, page 284–295, New York, NY, USA, 2013. Association for Computing Machinery. ISBN 9781450315982. doi: 10.1145/2448496.2448530. URL https://doi.org/10.1145/2448496.2448530.
- Vladimir Braverman and Rafail Ostrovsky. Smooth histograms for sliding windows. In 48th Annual IEEE Symposium on Foundations of Computer Science (FOCS'07), pages 283–293, 2007. doi: 10.1109/FOCS.2007.55.
- Zhiqi Bu, Sivakanth Gopi, Janardhan Kulkarni, Yin Tat Lee, Judy Hanwen Shen, and Uthaipon Tantipongpipat. Fast and memory efficient differentially private-sgd via JL projections. In Marc'Aurelio Ranzato, Alina Beygelzimer, Yann N. Dauphin, Percy Liang, and Jennifer Wortman Vaughan, editors, Advances in Neural Information Processing Systems 34: Annual Conference on Neural Information Processing Systems 2021, NeurIPS 2021, December 6-14, 2021, virtual, pages 19680-19691, 2021. URL https://proceedings.neurips.cc/paper/2021/hash/a3842ed7b3d0fe3ac263bcabd2999790-Abstract.html.
- T-H Hubert Chan, Mingfei Li, Elaine Shi, and Wenchang Xu. Differentially private continual monitoring of heavy hitters from distributed streams. In *International Symposium on Privacy Enhancing Technologies Symposium*, pages 140–159. Springer, 2012.
- Moses Charikar, Kevin Chen, and Martin Farach-Colton. Finding frequent items in data streams. In *International Colloquium on Automata, Languages, and Programming*, pages 693–703. Springer, 2002.
- Seung Geol Choi, Dana Dachman-Soled, Mukul Kulkarni, and Arkady Yerukhimovich. Differentially-private multi-party sketching for large-scale statistics. Cryptology ePrint Archive, Paper 2020/029, 2020. URL https://eprint.iacr.org/2020/029. https://eprint.iacr.org/2020/029.
- Graham Cormode and Shan Muthukrishnan. An improved data stream summary: the count-min sketch and its applications. *Journal of Algorithms*, 55(1):58–75, 2005.
- Mayur Datar, Aristides Gionis, Piotr Indyk, and Rajeev Motwani. Maintaining stream statistics over sliding windows. *SIAM journal on computing*, 31(6):1794–1813, 2002.
- Marianne Durand and Philippe Flajolet. Loglog counting of large cardinalities. In European Symposium on Algorithms, pages 605–617. Springer, 2003.
- Cynthia Dwork. Differential privacy: A survey of results. In International conference on theory and applications of models of computation, pages 1–19. Springer, 2008.
- Cynthia Dwork, Moni Naor, Toniann Pitassi, and Guy N. Rothblum. Differential privacy under continual observation. In Leonard J. Schulman, editor, *Proceedings of the 42nd ACM Symposium* on Theory of Computing, STOC, pages 715–724. ACM, 2010a.
- Cynthia Dwork, Moni Naor, Toniann Pitassi, Guy N Rothblum, and Sergey Yekhanin. Pan-private streaming algorithms. In *ics*, pages 66–80, 2010b.

- Cynthia Dwork, Aaron Roth, et al. The algorithmic foundations of differential privacy. Foundations and Trends (R) in Theoretical Computer Science, 9(3-4):211-407, 2014.
- Cynthia Dwork, Moni Naor, Omer Reingold, and Guy N. Rothblum. Pure differential privacy for rectangle queries via private partitions. In *Proceedings, Part II, of the 21st International Conference on Advances in Cryptology — ASIACRYPT 2015 - Volume 9453*, page 735–751, Berlin, Heidelberg, 2015. Springer-Verlag. ISBN 9783662487990. doi: 10.1007/978-3-662-48800-3_30. URL https://doi.org/10.1007/978-3-662-48800-3_30.
- Hendrik Fichtenberger, Monika Henzinger, and Wolfgang Ost. Differentially private algorithms for graphs under continual observation. In Petra Mutzel, Rasmus Pagh, and Grzegorz Herman, editors, 29th Annual European Symposium on Algorithms, ESA 2021, September 6-8, 2021, Lisbon, Portugal (Virtual Conference), volume 204 of LIPIcs, pages 42:1–42:16. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021. doi: 10.4230/LIPIcs.ESA.2021.42. URL https://doi.org/10.4230/LIPIcs.ESA.2021.42.
- Philippe Flajolet and G Nigel Martin. Probabilistic counting algorithms for data base applications. Journal of computer and system sciences, 31(2):182–209, 1985.
- Philippe Flajolet, Éric Fusy, Olivier Gandouet, and Frédéric Meunier. Hyperloglog: the analysis of a near-optimal cardinality estimation algorithm. In *Discrete Mathematics and Theoretical Computer Science*, pages 137–156. Discrete Mathematics and Theoretical Computer Science, 2007.
- Sumit Ganguly. Polynomial estimators for high frequency moments. arXiv preprint arXiv:1104.4552, 2011.
- Hsiang Hsu, Natalia Martinez, Martin Bertran, Guillermo Sapiro, and Flavio P. Calmon. A survey on statistical, information, and estimation—theoretic views on privacy. *IEEE BITS the Informa*tion Theory Magazine, 1(1):45–56, 2021.
- T-H. Hubert Chan, Elaine Shi, and Dawn Song. Private and continual release of statistics. In Samson Abramsky, Cyril Gavoille, Claude Kirchner, Friedhelm Meyer auf der Heide, and Paul G. Spirakis, editors, Automata, Languages and Programming, pages 405–417, Berlin, Heidelberg, 2010. Springer Berlin Heidelberg. ISBN 978-3-642-14162-1.
- Piotr Indyk. Stable distributions, pseudorandom generators, embeddings, and data stream computation. Journal of the ACM (JACM), 53(3):307–323, 2006.
- Piotr Indyk and David Woodruff. Optimal approximations of the frequency moments of data streams. In *Proceedings of the thirty-seventh annual ACM symposium on Theory of computing*, pages 202–208, 2005.
- Palak Jain, Sofya Raskhodnikova, Satchit Sivakumar, and Adam D. Smith. The price of differential privacy under continual observation. CoRR, abs/2112.00828, 2021. URL https://arxiv.org/abs/2112.00828.
- Prateek Jain, Pravesh Kothari, and Abhradeep Thakurta. Differentially private online learning. In Shie Mannor, Nathan Srebro, and Robert C. Williamson, editors, *Proceedings* of the 25th Annual Conference on Learning Theory, volume 23 of Proceedings of Machine Learning Research, pages 24.1–24.34, Edinburgh, Scotland, 25–27 Jun 2012. PMLR. URL https://proceedings.mlr.press/v23/jain12.html.

- Hossein Jowhari, Mert Sağlam, and Gábor Tardos. Tight bounds for lp samplers, finding duplicates in streams, and related problems. In *Proceedings of the thirtieth ACM SIGMOD-SIGACT-SIGART symposium on Principles of database systems*, pages 49–58, 2011.
- Daniel M Kane, Jelani Nelson, and David P Woodruff. On the exact space complexity of sketching and streaming small norms. In Proceedings of the twenty-first annual ACM-SIAM symposium on Discrete Algorithms, pages 1161–1178. SIAM, 2010.
- Daniel M Kane, Jelani Nelson, Ely Porat, and David P Woodruff. Fast moment estimation in data streams in optimal space. In Proceedings of the forty-third annual ACM symposium on Theory of computing, pages 745–754, 2011.
- Ping Li. Estimators and tail bounds for dimension reduction in l α (0< $\alpha \leq 2$) using stable random projections. In *Proceedings of the nineteenth annual ACM-SIAM symposium on Discrete algorithms*, pages 10–19, 2008.
- Yi Li and David P Woodruff. A tight lower bound for high frequency moment estimation with small error. In Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques, pages 623–638. Springer, 2013.
- Darakhshan Mir, Shan Muthukrishnan, Aleksandar Nikolov, and Rebecca N Wright. Pan-private algorithms via statistics on sketches. In *Proceedings of the thirtieth ACM SIGMOD-SIGACT-SIGART symposium on Principles of database systems*, pages 37–48, 2011.
- Jayadev Misra and David Gries. Finding repeated elements. Science of computer programming, 2 (2):143–152, 1982.
- Robert Morris. Counting large numbers of events in small registers. *Communications of the ACM*, 21(10):840–842, 1978.
- Victor Perrier, Hassan Jameel Asghar, and Dali Kaafar. Private continual release of realvalued data streams. In 26th Annual Network and Distributed System Security Symposium, NDSS 2019, San Diego, California, USA, February 24-27, 2019. The Internet Society, 2019. URL https://www.ndss-symposium.org/ndss-paper/private-continual-release-of-real-valued-data-streams/
- Michael Saks and Xiaodong Sun. Space lower bounds for distance approximation in the data stream model. In *Proceedings of the thiry-fourth annual ACM symposium on Theory of computing*, pages 360–369, 2002.
- Or Sheffet. Differentially private ordinary least squares. In Doina Precup and Yee Whye Teh, editors, Proceedings of the 34th International Conference on Machine Learning, ICML 2017, Sydney, NSW, Australia, 6-11 August 2017, volume 70 of Proceedings of Machine Learning Research, pages 3105-3114. PMLR, 2017. URL http://proceedings.mlr.press/v70/sheffet17a.html.
- Adam Smith and Abhradeep Thakurta. (nearly) optimal algorithms for private online learning in full-information and bandit settings. In Proceedings of the 26th International Conference on Neural Information Processing Systems - Volume 2, NIPS'13, page 2733–2741, Red Hook, NY, USA, 2013. Curran Associates Inc.

- Adam D. Smith, Shuang Song, and Abhradeep Thakurta. The flajolet-martin sketch itself preserves differential privacy: Private counting with minimal space. In Hugo Larochelle, Marc'Aurelio Ranzato, Raia Hadsell, Maria-Florina Balcan, and Hsuan-Tien Lin, editors, Advances in Neural Information Processing Systems 33: Annual Conference on Neural Information Processing Systems 2020, NeurIPS 2020, December 6-12, 2020, virtual, 2020. URL https://proceedings.neurips.cc/paper/2020/hash/e3019767b1b23f82883c9850356b71d6-Abstract.html.
- Shuang Song, Susan Little, Sanjay Mehta, Staal Vinterbo, and Kamalika Chaudhuri. Differentially private continual release of graph statistics, 2018. URL https://arxiv.org/abs/1809.02575.
- Robert H. Morris Sr. Counting large numbers of events in small registers. *Commun. ACM*, 21(10): 840–842, 1978.
- Mikkel Thorup and Yin Zhang. Tabulation based 4-universal hashing with applications to second moment estimation. In SODA, volume 4, pages 615–624, 2004.
- Jalaj Upadhyay. Sublinear space private algorithms under the sliding window model. In Kamalika Chaudhuri and Ruslan Salakhutdinov, editors, *Proceedings of the 36th International Conference* on Machine Learning, volume 97 of Proceedings of Machine Learning Research, pages 6363–6372. PMLR, 09–15 Jun 2019. URL https://proceedings.mlr.press/v97/upadhyay19a.html.
- Lun Wang, Iosif Pinelis, and Dawn Song. Differentially private fractional frequency moments estimation with polylogarithmic space. In *The Tenth International Conference on Learning Representations, ICLR 2022, Virtual Event, April 25-29, 2022.* OpenReview.net, 2022. URL https://openreview.net/forum?id=718LPkcx8V.

A Missing Details of Section 3

A.1 Proof of Lemma 3.1

Proof. We first show that the output groups G_1, G_2, \dots, G_m is ε_0 -DP. Let O_1, O_2, \dots, O_m be any fixed grouping. Let c'_1, c'_2, \dots, c'_T be any neighboring stream, i.e., $\exists q \in [T]$ such that $|c_q - c'_q| \leq 1$ and $\forall j \neq q, c_j = c'_j$. Let $G'_1, G'_2, \dots, G'_{m'}$ be the output groups of the neighboring stream. Suppose $q \in O_r$ for some $r \in [m]$. Let us consider $\Pr[(G_1, G_2, \dots, G_m) = (O_1, O_2, \dots, O_m)]$ and $\Pr[(G'_1, G'_2, \dots, G'_{m'}) = (O_1, O_2, \dots, O_m)]$ (in this case m' = m). We have:

$$\begin{aligned} &\frac{\Pr\left[(G_1, G_2, \cdots, G_m) = (O_1, O_2, \cdots, O_m)\right]}{\Pr\left[(G_1', G_2', \cdots, G_{m'}) = (O_1, O_2, \cdots, O_{m'})\right]} \\ &= \frac{\Pr\left[(G_1, G_2, \cdots, G_{r-1}) = (O_1, O_2, \cdots, O_{r-1})\right]}{\Pr\left[(G_1', G_2', \cdots, G_{r-1}') = (O_1, O_2, \cdots, O_{r-1})\right]} \cdot \frac{\Pr\left[G_r = O_r \mid (G_1, G_2, \cdots, G_{r-1}) = (O_1, O_2, \cdots, O_{r-1})\right]}{\Pr\left[G_r' = O_r \mid (G_1', G_2', \cdots, G_{r-1}') = (O_1, O_2, \cdots, O_{r-1})\right]} \\ &\quad \cdot \frac{\Pr\left[(G_{r+1}, G_{r+2}, \cdots, G_m) = (O_{r+1}, O_{r+2}, \cdots, O_m) \mid (G_1, G_2, \cdots, G_r) = (O_1, O_2, \cdots, O_r)\right]}{\Pr\left[(G_{r+1}', G_{r+2}', \cdots, G_m') = (O_{r+1}, O_{r+2}, \cdots, O_m) \mid (G_1', G_2', \cdots, G_r') = (O_1, O_2, \cdots, O_r)\right]} \end{aligned}$$

Since $\forall j \in O_1 \cup O_2 \cup \cdots \cup O_{r-1}, c_j = c'_j$, the behavior of running Algorithm 1 on the prefix $O_1 \cup O_2 \cup \cdots \cup O_{r-1}$ of c_1, c_2, \cdots, c_T is the same as the behavior of running it on the prefix $O_1 \cup O_2 \cup \cdots \cup O_{r-1}$ of c'_1, c'_2, \cdots, c'_T . Therefore, we have $\frac{\Pr[(G_1, G_2, \cdots, G_{r-1}) = (O_1, O_2, \cdots, O_{r-1})]}{\Pr[(G'_1, G'_2, \cdots, G'_{r-1}) = (O_1, O_2, \cdots, O_{r-1})]} = 1$. Similarly, since $\forall j \in O_1 \cup O_2 \cup \cdots \cup O_r$.

 $O_{r+1} \cup O_{r+2} \cup \cdots \cup O_m, c_j = c'_j, \text{ when } G_r = G'_r = O_r, \text{ the behavior of running Algorithm 1 on the suffix } O_{r+1} \cup O_{r+2} \cup \cdots \cup O_m \text{ of } c_1, c_2, \cdots, c_T \text{ is the same as the behavior of running it on the same suffix of } c'_1, c'_2, \cdots, c'_T. \text{ Therefore, we have } \frac{\Pr[(G_{r+1}, G_{r+2}, \cdots, G_m) = (O_{r+1}, O_{r+2}, \cdots, O_m) | (G_1, G_2, \cdots, G_r) = (O_1, O_2, \cdots, O_r)]}{\Pr[(G'_{r+1}, G'_{r+2}, \cdots, G'_m) = (O_{r+1}, O_{r+2}, \cdots, O_m) | (G'_1, G'_2, \cdots, G'_r) = (O_1, O_2, \cdots, O_r)]}.$ Thus, we have:

$$\frac{\Pr\left[(G_1, G_2, \cdots, G_m) = (O_1, O_2, \cdots, O_m)\right]}{\Pr\left[(G'_1, G'_2, \cdots, G'_m) = (O_1, O_2, \cdots, O_m)\right]} = \frac{\Pr\left[G_r = O_r \mid (G_1, G_2, \cdots, G_{r-1}) = (O_1, O_2, \cdots, O_{r-1})\right]}{\Pr\left[G'_r = O_r \mid (G'_1, G'_2, \cdots, G'_{r-1}) = (O_1, O_2, \cdots, O_{r-1})\right]}.$$

Suppose $O_r = \{x + 1, \dots, x + k\}.$

Consider the first case where $r \neq m$. In this case, we have

$$\Pr \left[G_r = O_r \mid (G_1, G_2, \cdots, G_{r-1}) = (O_1, O_2, \cdots, O_{r-1}) \right]$$

=
$$\Pr \left[G_r = O_r \mid G_{r-1} = O_{r-1} \right]$$

=
$$\Pr \left[\left(\forall j \in [k-1], \nu_{x+j} + \sum_{b=1}^j c_{x+b} < \tau_r \right) \bigwedge \left(\nu_{x+k} + \sum_{b=1}^k c_{x+b} \ge \tau_r \right) \right]$$

Now, let us fix $\nu_{x+1}, \nu_{x+2}, \dots, \nu_{x+k-1}$ and let $g = \max_{j \in [k-1]} \nu_{x+j} + \sum_{b=1}^{j} c_{x+b}$. Then,

$$\Pr_{\nu_{x+k},\tau_{r}} [G_{r} = O_{r} \mid G_{r-1} = O_{r-1}]$$

$$= \Pr_{\nu_{x+k},\tau_{r}} \left[\tau_{r} \in (g, \nu_{x+k} + \sum_{b=1}^{k} c_{x+b}] \right]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{\nu_{x+k}}(v) \cdot p_{\tau_{r}}(\tau) \cdot \mathbf{1} \left(\tau \in (g, v + \sum_{b=1}^{k} c_{x+b}) \right) dv d\tau$$
(3)

where $p_{\nu_{x+k}}(\cdot)$ and $p_{\tau_r}(\cdot)$ are density functions of ν_{x+k} and τ_r respectively. Let $g' = \max_{j \in [k-1]} \nu_{x+j} + \sum_{b=1}^{j} c'_{x+b}$. Let $v' = v + g - g' + \sum_{b=1}^{k} c'_{x+b} - \sum_{b=1}^{k} c_{x+b}$. Let $\tau' = \tau + g - g'$. Since $|c_q - c'_q| \leq 1$, it is easy to see that $|v' - v| \leq 2$ and $|\tau - \tau'| \leq 1$. Note that dv' = dv and $d\tau' = d\tau$. Therefore, Equation (3) is equal to the following:

$$\begin{split} &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{\nu_{x+k}}(v') \cdot p_{\tau_r}(\tau') \cdot \mathbf{1} \left(\tau + g - g' \in \left(g, v + g - g' + \sum_{b=1}^{k} c'_{x+b} \right) \right) \mathrm{d}v \mathrm{d}\tau \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{\nu_{x+k}}(v') \cdot p_{\tau_r}(\tau') \cdot \mathbf{1} \left(\tau \in \left(g', v + \sum_{b=1}^{k} c'_{x+b} \right) \right) \mathrm{d}v \mathrm{d}\tau \\ &\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(\varepsilon_0/2) \cdot p_{\nu_{x+k}}(v) \cdot \exp(\varepsilon_0/2) \cdot p_{\tau_r}(\tau) \cdot \mathbf{1} \left(\tau \in \left(g', v + \sum_{b=1}^{k} c'_{x+b} \right) \right) \mathrm{d}v \mathrm{d}\tau \\ &= \exp(\varepsilon_0) \cdot \Pr_{\nu_{x+k},\tau_r} \left[\tau_r \in (g', \nu_{x+k} + \sum_{b=1}^{k} c'_{x+b}] \right] \\ &= \exp(\varepsilon_0) \cdot \Pr_{\nu_{x+k},\tau_r} \left[G'_r = O_r \mid G'_{r-1} = O_{r-1} \right] \end{split}$$

$$= \exp(\varepsilon_0) \cdot \Pr_{\nu_{x+k}, \tau_r} \left[G'_r = O_r \mid (G'_1, G'_2, \cdots, G'_{r-1}) = (O_1, O_2, \cdots, O_{r-1}) \right].$$

Next, consider the second case where r = m. In this case, we have

$$\Pr \left[G_r = O_r \mid (G_1, G_2, \cdots, G_{r-1}) = (O_1, O_2, \cdots, O_{r-1}) \right]$$

=
$$\Pr \left[G_r = O_r \mid G_{r-1} = O_{r-1} \right]$$

=
$$\Pr \left[\forall j \in [k], \nu_{x+j} + \sum_{b=1}^j c_{x+b} < \tau_r \right].$$

Now, let us fix $\nu_{x+1}, \nu_{x+2}, \dots, \nu_{x+k}$ and let $g = \max_{j \in [k]} \nu_{x+j} + \sum_{b=1}^{j} c_{x+b}$. Let $g' = \max_{j \in [k]} \nu_{x+j} + \sum_{b=1}^{j} c'_{x+b}$. Since $|c_q - c'_q| \leq 1$, we have $|g - g'| \leq 1$. Then,

$$\Pr_{\tau_r} [\tau_r > g] \le \exp(\varepsilon_0) \cdot \Pr_{\tau_r} [\tau_r > g']$$

Thus, we have

$$\Pr \left[G_r = O_r \mid (G_1, G_2, \cdots, G_{r-1}) = (O_1, O_2, \cdots, O_{r-1}) \right]$$

$$\leq \exp(\varepsilon_0) \cdot \Pr \left[G'_r = O_r \mid (G'_1, G'_2, \cdots, G'_{r-1}) = (O_1, O_2, \cdots, O_{r-1}) \right].$$

Therefore, we can conclude that (G_1, G_2, \cdots, G_m) is always ε_0 DP.

Notice that, given any fixed (O_1, O_2, \dots, O_m) and condition on $(G_1, G_2, \dots, G_m) = (O_1, O_2, \dots, O_m)$, $\hat{c}_1, \hat{c}_2, \dots, \hat{c}_T$ is ε_0 -DP by Laplace mechanism. Thus, by composition theorem, the output stream $\hat{c}_1, \hat{c}_2, \dots, \hat{c}_T$ is ε -DP.

A.2 Proof of Lemma 3.2

Let G_1, G_2, \dots, G_m be the groups produced during Algorithm 1. Let $\tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_m$ be that $\forall i \in [m], \tilde{c}_i = \hat{c}_{\max(G_i)}, \text{ i.e., } \tilde{c}_i$ is the noisy count of the group G_i .

Lemma A.1. With probability at least $1-\xi$, the output stream of Algorithm 1 satisfies the following properties:

 $1. \ \forall i \in [m-1], \sum_{j \in G_i \setminus \{\max_{j' \in G_i} j'\}} c_j \leq \frac{7}{\eta \varepsilon_0} \cdot \ln(3 \cdot T/\xi) + \frac{13}{\varepsilon_0} \cdot \ln(3 \cdot T/\xi).$ $2. \ \sum_{j \in G_m} c_j \leq \frac{7}{\eta \varepsilon_0} \cdot \ln(3 \cdot T/\xi) + \frac{13}{\varepsilon_0} \cdot \ln(3 \cdot T/\xi).$ $3. \ \forall i \in [m-1], (1-\eta) \sum_{j \in G_i} c_j \leq \tilde{c}_i \leq (1+\eta) \sum_{j \in G_i} c_j.$

Proof. Let \mathcal{E} denote the event that

1.
$$\forall i \in [m], \left| \tau_i - \left(\frac{1}{\eta} + 1\right) \cdot \frac{7}{\varepsilon_0} \cdot \ln\left(3 \cdot T/\xi\right) \right| \leq \frac{2}{\varepsilon_0} \cdot \ln\left(3 \cdot T/\xi\right).$$

2. $\forall t \in [T], |\nu_t| \leq \frac{4}{\varepsilon_0} \cdot \ln\left(3 \cdot T/\xi\right).$
3. $\forall i \in [m-1], |\tilde{c}_i - \sum_{j \in G_i} c_j| \leq \frac{1}{\varepsilon_0} \cdot \ln(3 \cdot T/\xi)$

According to the CDF of Laplace noise, it is easy to show that \mathcal{E} happens with probability at least $1-\xi$ by a union bound over all $i \in [m], t \in [T]$. In the remaining of the proof we condition on the event \mathcal{E} .

Consider property 1. Consider any $i \in [m-1]$. Let $t = \max_{j \in G_i} j - 1$. Then we have $\nu_t + \sum_{j \in G_i, j \leq t} c_j \leq \tau_i$. Since $|\nu_t| \leq \frac{4}{\varepsilon_0} \cdot \ln(3 \cdot T/\xi)$ and $\tau_i \leq \frac{7}{\eta \cdot \varepsilon_0} \cdot \ln(3 \cdot T/\xi) + \frac{9}{\varepsilon_0} \cdot \ln(3 \cdot T/\xi)$, we have $\sum_{j \in G_i, j \leq t} c_j \leq \frac{7}{\eta \varepsilon_0} \cdot \ln(3 \cdot T/\xi) + \frac{13}{\varepsilon_0} \cdot \ln(3 \cdot T/\xi)$. The proof of property 2 is the same as the proof of property 1.

Consider property 3. Consider any $i \in [m-1]$. Let $t = \max_{j \in G_i} j$. Then we have $\nu_t + \sum_{j \in G_i} c_j \geq \tau_i$. Since $|\nu_t| \leq \frac{4}{\varepsilon_0} \cdot \ln(3 \cdot T/\xi)$ and $\tau_i \geq \frac{7}{\eta \cdot \varepsilon_0} \cdot \ln(3 \cdot T/\xi) + \frac{5}{\varepsilon_0} \cdot \ln(3 \cdot T/\xi)$, we have $\sum_{j \in G_i} c_j \geq \frac{7}{\eta \varepsilon_0} \cdot \ln(3 \cdot T/\xi)$. Note that $|\tilde{c}_i - \sum_{j \in G_i} c_j| \leq \frac{1}{\varepsilon_0} \cdot \ln(3 \cdot T/\xi)$. Thus, we know that $\sum_{j \in G_i} c_j \geq \frac{7}{\eta \varepsilon_0} \cdot \ln(3 \cdot T/\xi)$. Thus, $\forall i \in [m-1]$ we have $(1-\eta) \sum_{j \in G_i} c_j \leq \tilde{c}_i \leq (1+\eta) \sum_{j \in G_i} c_j$.

Now, we are able to prove Lemma 3.2.

Proof of Lemma 3.2. With probability at least $1-\xi$, the properties listed in Lemma A.1 hold.

Let $l' \in [m]$ be the smallest index such that $\max_{j \in G_{l'}} j \ge l$ and $r' \in [m]$ be the largest index such that $\max_{j \in G_{r'}} j \leq r$.

We have:

$$\sum_{j=l}^{r} \hat{c}_{j} = \sum_{j'=l'}^{r'} \tilde{c}_{j'}$$

$$\geq (1-\eta) \sum_{j'=l'}^{r'} \sum_{j\in G_{j'}} c_{j} - \left(\frac{7}{\eta\varepsilon_{0}} \cdot \ln\left(3 \cdot T/\xi\right) + \frac{13}{\varepsilon_{0}} \cdot \ln(3 \cdot T/\xi)\right)$$

$$\geq (1-\eta) \sum_{j=l}^{\max_{b\in G_{r'}} b} c_{j} - \left(\frac{7}{\eta\varepsilon_{0}} \cdot \ln\left(3 \cdot T/\xi\right) + \frac{13}{\varepsilon_{0}} \cdot \ln(3 \cdot T/\xi)\right)$$

$$\geq (1-\eta) \sum_{j=l}^{r} c_{j} - \frac{7}{\eta\varepsilon_{0}} \cdot \ln\left(3 \cdot T/\xi\right) - \frac{26}{\varepsilon_{0}} \cdot \ln\left(3 \cdot T/\xi\right),$$

where the first inequality follows from property 3 and property 2 of Lemma A.1, the second inequality follows from the choice of l', and the third inequality follows from property 1 of Lemma A.1.

Similarly, we can show

$$\sum_{j=l}^{r} \hat{c}_{j} = \sum_{j'=l'}^{r'} \tilde{c}_{j'}$$

$$\leq (1+\eta) \sum_{j'=l'}^{r'} \sum_{j\in G_{j'}} c_{j} + \left(\frac{7}{\eta\varepsilon_{0}} \cdot \ln\left(3 \cdot T/\xi\right) + \frac{13}{\varepsilon_{0}} \cdot \ln\left(3 \cdot T/\xi\right)\right)$$

$$\leq (1+\eta) \sum_{j=\min_{b\in G_{l'}} b}^{r} c_{j} + \left(\frac{7}{\eta\varepsilon_{0}} \cdot \ln\left(3 \cdot T/\xi\right) + \frac{13}{\varepsilon_{0}} \cdot \ln\left(3 \cdot T/\xi\right)\right)$$

$$\leq (1+\eta) \sum_{j=l}^{r} c_{j} + \frac{7}{\eta\varepsilon_{0}} \cdot \ln\left(3 \cdot T/\xi\right) + \frac{26}{\varepsilon_{0}} \cdot \ln\left(3 \cdot T/\xi\right),$$

where the first inequality follows from property 3 and property 2 of Lemma A.1, the second inequality follows from the choice of r', and the third inequality follows from property 1 of Lemma A.1.