Entanglement area law for 1D gauge theories and bosonic systems

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Abstract

We prove an entanglement area law for a class of 1D quantum systems involving infinite-dimensional local Hilbert spaces. This class of quantum systems include bosonic models such as the Hubbard-Holstein model, and both U(1) and SU(2) lattice gauge theories in one spatial dimension. Our proof relies on new results concerning the robustness of the ground state and spectral gap to the truncation of Hilbert space, applied within the approximate ground state projector (AGSP) framework from previous work. In establishing this area law, we develop a system-size independent bound on the expectation value of local observables for Hamiltonians without translation symmetry, which may be of separate interest. Our result provides theoretical justification for using tensor network methods to study the ground state properties of quantum systems with infinite local degrees of freedom.

1 Introduction

It has long been conjectured that for a wide range of quantum systems described by gapped local Hamiltonians, the entanglement entropy with respect to any bipartition of the system scales as the boundary area. Such entanglement entropy scaling is known as the *entanglement area law*. An entanglement area law for the ground state of 1D quantum spin systems was first proved in the seminal paper by Hastings [18], and the scaling with respect to the spectral gap was improved by later work such as [5, 6]. Entanglement area laws have also been proved for degenerate ground state and low-lying eigenstates [7]. Limited results are also available for higher-dimensional quantum systems, especially for the case where the Hamiltonian is frustration-free [2, 4]. For 1D systems, whether a quantum state satisfies an entanglement area law is an important criterion for determining whether it can be approximated by a matrix-product state[16], which is a key component in the density-matrix renormalization group (DMRG) algorithm [34] (see also [17]). Using theoretical tools constructed for proving the 1D area law, a polynomial time algorithm for computing the ground state of 1D gapped local Hamiltonians was proposed in [24].

The aforementioned results are all proved in the setting of quantum spin systems, i.e., each lattice site is associated with a finite-dimensional local Hilbert space. However, there are many quantum systems of practical interest that involve infinite-dimensional local Hilbert spaces. Examples of such

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include bosonic systems and gauge theories. When a quantum system involves bosons, each bosonic mode corresponds to an infinite-dimensional local Hilbert space, due to the fact that its occupation number can be arbitrarily large. A similar situation arises when we consider lattice gauge theories (LGTs), with the Hamiltonians constructed according to [22]. Given a fixed lattice discretization of a gauge theory, each gauge link (an edge of the lattice) has a local Hilbert space that is spanned by all the elements of the symmetry group, which is infinite dimensional when the symmetry group contains infinitely many elements. The gauge theories that are of the greatest interests, i.e. the U(1), SU(2), and SU(3) theories, all fall into this category.

When the dimension of the local Hilbert space is infinite, the area law proof techniques developed for the finite dimensional case cannot be directly applied. By the state-of-the-art 1D area law result in [6], across a given cut in a 1D system described by a gapped local Hamiltonian, the entanglement entropy scales as $\mathcal{O}(\Delta^{-1}\log^3(d))$ where d is the local Hilbert space dimension and Δ is the spectral gap. This becomes infinity when the local Hilbert space is infinite-dimensional. Moreover, in quantum spin systems all local Hamiltonian terms can be rescaled to have operator norm at most 1, whereas in the models we consider in this work the local Hamiltonian terms can be unbounded, making it impossible to normalize them. For certain non-interacting bosonic systems, such as those involving infinite-dimensional local Hilbert spaces but are exactly solvable, the entanglement area law has been proven in Refs. [12, 13, 14] (see also [15]). However, a general methodology remains elusive for establishing area laws for quantum systems with infinite local degrees of freedom.

Main result. In this work we prove an entanglement area law for a class of 1D quantum systems that involve bosons or arise from gauge theories, using the approximate ground state projector (AGSP) framework developed in [5]. These quantum systems involve infinite-dimensional local Hilbert spaces. Examples of such quantum systems include the Hubbard-Holstein model, U(1) and SU(2) LGTs, all of which are defined on a 1D chain. For these models, we can introduce a notion of *local quantum number*, which is the occupation number in the bosonic case, the electric field value in the U(1) LGT, and the total angular momentum in the SU(2) LGT. Informally, our main result can be summarized as follows:

Theorem (Main result (informal)). *For the gapped ground state of the 1D Hubbard-Holstein model, or the 1D U(1) or SU(2) LGTs, the entanglement entropy across a cut scales like $\mathcal{O}(\mathrm{poly}(\Delta^{-1}))$, where Δ is the spectral gap, assuming that all coefficients in the Hamiltonian remain constant. In particular, this entanglement entropy scaling is independent of the system size.*

A more precise statement can be found in Theorem 20. This result generalizes to other models as long as certain assumptions about the growth of the local Hilbert space dimension and quantum numbers are satisfied. We discuss these assumptions in detail in Section 3.

Proof strategy. Our result utilizes a local quantum number tail bound recently obtained in [31]. Essentially, this tail bound tells us that for the class of quantum systems we study, a spectrally isolated energy eigenstate can be well-approximated by a truncated state with low local quantum numbers. This suggests that we may choose an effective local Hilbert space with finite dimensions and thereby apply the framework developed in [5, 6]. To implement this proof strategy, we need to apply truncation to the local Hilbert space and by extension to the Hamiltonian terms, on each lattice site close to the cut. Through this procedure we will obtain a different Hamiltonian than the one we started out with; however, we show in Theorem 12 that the truncation changes the ground state only by an exponentially small amount. This is similar to what is known as the robustness theorem [6, Theorem 6.1]. Because of this we can show that the AGSP framework developed in [6] is robust to this truncation. By the "off-the-rack" lemma with approximate target space in [1], generalized from the case [5] of exact target state, we will prove the entanglement area law for our class of unbounded quantum systems.

As an intermediate step in our proof, we develop tools to show that the mean absolute value of the local quantum number can be bounded independently of the system size for many quantum systems without translation symmetry. We state the general version of our bound in Lemmas 7 and 8 and later specialize it to the Hubbard-Holstein model (Corollary 10) and U(1) and SU(2) LGTs (Corollary 9) with open boundary conditions.

In proving area laws for unbounded quantum systems, it is tempting to plug the eigenstate tail bound from [31] directly into the area law result for spin systems [6]. However, such a naive strategy does *not* seem to work here. The main reason is that the tail bound from [31] only guarantees the proximity of the quantum states before and after truncation, but not the proximity of the corresponding entanglement entropies. In the finite-dimensional setting, one could invoke the Fannes' inequality to estimate the entanglement entropy difference [33, 35], but its explicit dependence on the dimension of the Hilbert space will ruin the area law scaling for unbounded quantum systems. In fact, the main contribution of our paper is to give a careful truncation of the local Hilbert space that does not blow up entanglement entropy of the ground state, overcoming the technical difficulties mentioned above.

Tensor network methods have been extensively applied to studying LGTs to obtain interesting numerical results [8, 9, 10, 11, 26, 27, 28, 29]. Entanglement area law is a prerequisite for the ground state to be efficiently approximable using a tensor network state, and hence a rigorous proof in the setting of LGTs is crucial to the theoretical foundation of these numerical results. Moreover, if more mathematical rigor is needed, we might want a provably accurate algorithm to compute the ground state of gauge theories in 1D, along the line of the rigorous renormalization group (RG) method proposed in Ref. [7]. The current work can be seen as a first step towards that goal, since many of the proof techniques, such as the AGSP, are also useful components of the rigorous RG method.

Organization of the paper. The rest of the paper is organized as follows: in Section 2 we will briefly introduce the class of models we consider, including the 1D gauge theories and the Hubbard-Holstein model. In Section 3 we will discuss the common structure of these models that enables a proof of area law. In Section 4 we will discuss properties of the ground state that can be used to truncate the infinite-dimensional local Hilbert space. In Section 5 we prove that the local quantum numbers have a mean absolute value that can be bounded independently of the system size. This fact removes an otherwise extra assumption in the proof of the area law. In Section 6 we will demonstrate that the ground state is robust to such truncations using properties derived in the previous section. In Section 7 we prove our main result of the entanglement area law scaling, using an "off-the-rack" lemma slightly tighter than the one from [1]. We conclude our paper in Section 8 with a brief summary of the main result and a selection of questions for future work.

2 1D gauge theories and bosonic systems

In this section we first introduce some specific quantum systems that we will study, and then extract the common structure of these systems. These quantum systems are all discussed in [31, Section I]. We first consider the 1D Hubbard-Holstein model [19], a model describing electron-phonon interactions.

The Hubbard-Holstein model. This model is defined on a 1D chain of N nodes. Each node in the lattice, indexed by x, contains two fermionic modes (spin up and down) and a bosonic mode. The

Hamiltonian is

$$H = H_f + H_{fb} + H_b, \tag{1}$$

where H_f is the Hamiltonian of the Fermi-Hubbard model [21] acting on only the fermionic modes, and

$$H_{fb} = g\sum_{x=1}^{N}(b_x^\dagger + b_x)(n_{x,\uparrow} + n_{x,\downarrow} - 1), \qquad H_b = \omega_0 \sum_{x=1}^{N} b_x^\dagger b_x, \tag{2}$$

are the boson-fermion coupling and purely bosonic parts of the Hamiltonian respectively. Here, b_x is the bosonic annihilation operator on node x, and $n_{x,\sigma}$ is the fermionic number operator for node x and spin σ.

For gauge theories, we consider the Hamiltonian formulation of the U(1) and SU(2) LGTs [22] in one dimension. The U(1) LGT is also known as the Schwinger model.

The U(1) lattice gauge theory. The system consists of a chain of N nodes with $N-1$ links between adjacent nodes. We denote each node by x, and the links by the node on its left end. The links are sometimes called *gauge links*.

On each node x we have a fermionic mode whose annihilation operator is denoted by ϕ_x. Each link consists of a planar rotor, whose configuration can be described by an angle $\theta \in [0, 2\pi]$. The local Hilbert space is the vector space of square-integrable functions on $U(1)$. An orthonormal basis of the local Hilbert space can be chosen to be the Fourier basis. More specifically we denote by $|k\rangle$ the Fourier mode $(2\pi)^{-1/2}e^{ik\theta}$, and $\{|k\rangle : k \in \mathbb{Z}\}$ form the basis we need.

We further define the operators E_x and U_x, which act on the vector space of the links, through

$$E_x |k\rangle = k |k\rangle, \qquad U_x |k\rangle = |k-1\rangle. \tag{3}$$

The Hamiltonian for $U(1)$ LGT can then be described in terms of these operators via

$$\begin{aligned} H &= H_M + H_{GM} + H_E, \\ H_M &= g_M \sum_x (-1)^x \phi_x^\dagger \phi_x, \\ H_{GM} &= g_{GM} \sum_x (\phi_x^\dagger U_x \phi_{x+1} + \phi_{x+1}^\dagger U_x^\dagger \phi_x), \\ H_E &= g_E \sum_x E_x^2. \end{aligned} \tag{4}$$

The three terms H_M, H_{GM}, H_E describe the fermionic mass (using staggered fermions [22]), the gauge-matter interaction, and the electric energy respectively.

The SU(2) lattice gauge theory. We only consider the theory using the fundamental representation of SU(2), as done in [10]. The model is still defined on a 1D chain consisting of nodes and links, similar to the U(1) case. The Hamiltonian also takes the form (4). One difference is that in the SU(2) theory, each node x now contains two fermionic modes, whose annihilation operators are denoted by ϕ_x^l, $l = 1, 2$. We write $\phi_x = (\phi_x^1, \phi_x^2)^\top$. Each gauge link consists of a rigid rotator whose configuration is described by an element of the group SU(2) [22]. For the link Hilbert space, which is the vector space of square-integrable functions with respect to the Haar measure on SU(2), we can construct an orthonormal basis that consists of the quantum states $|jmm'\rangle$, where j, m, m' are simultaneously either integers or half-integers with $-j \le m, m' \le j$. Here j is the total angular momentum of the rotator, and m, m' are the components of angular momentum along the z-axis in the body-fixed and space-fixed coordinate systems.

The Hamiltonian takes the form (4), and is invariant under SU(2) transformations acting either from the left or from the right, which correspond to rotations of the rigid rotator with respect to space-fixed or body-fixed axes respectively. The operators E_x^2 and U_x are different from the U(1) case. The operator E_x^2 is defined through

$$E_x^2 \left|jmm'\right\rangle = j(j+1) \left|jmm'\right\rangle . \tag{5}$$

Because ϕ_x has two components, where each component is a fermionic mode, U_x is a 2×2 matrix, where each of the 4 matrix entries is an operator acting on the link space

$$U_x = \begin{pmatrix} U_x^{11} & U_x^{12} \\ U_x^{21} & U_x^{22} \end{pmatrix} . \tag{6}$$

An important property that we will use later is

$$\langle j_1 m_1 m_1' | U_x^{ll'} | j_2 m_2 m_2' \rangle = 0, \text{ if } |j_1 - j_2| > 1/2, \qquad \|U_x^{ll'}\| \leq 1, \tag{7}$$

which follows from rules for the addition of angular momentum, given that U_{x,n_i} transforms as the $j = 1/2$ representation of SU(2). Here $\|O\|$ denotes the spectral norm of an operator O.

3 The abstract model

We now consider a more general class of Hamiltonians that include all the previously discussed examples. We consider a 1D system consisting of N sites indexed by $x = 1, 2, \ldots, N$. In the context of the Hubbard-Holstein model, each site x contains one fermionic mode and one bosonic mode. In the context of the LGTs, each site x contains a node x and the gauge link x to the right, except for the last site which only contains a node (unless periodic boundary conditions are assumed). As discussed in Section 2, each node contains one or two fermionic modes for the U(1) and SU(2) cases respectively. The Hamiltonian is geometrically local:

$$H = H_1 + H_2 + \cdots + H_N, \tag{8}$$

where each H_x acts on two adjacent sites x and $x + 1$.

The local vector spaces are allowed to be infinite-dimensional, but we need to keep track of which part of the space is relevant to our problem. To this end, we introduce the notion of the *local quantum number*.

Definition 1. *For any $1 \leq x \leq N$, we define an integer local quantum number λ_x and corresponding Hermitian observable O_x acting only on the local Hilbert space x, such that $O_x \left|\lambda_x\right\rangle = \lambda_x \left|\lambda_x\right\rangle$. We further define the following:*

1. *Let projector $\Pi_S^{(x)}$ be defined for any $S \subset \mathbb{R}$ such that $\Pi_S^{(x)} \left|\lambda_x\right\rangle = \left|\lambda_x\right\rangle$ if and only if $\lambda_x \in S$. Further, we define $\overline{\Pi}_S^{(x)} = I - \Pi_S^{(x)}$ to be the projector onto the complement of S.*

2. *We define for cutoff $\Lambda \in \mathbb{R}$ the* local Hilbert space dimension *of a site x to be $d(\Lambda) = \mathrm{rank}(\Pi_{[-\Lambda,\Lambda]}^{(x)})$. For simplicity we use the same cutoff Λ for all sites throughout the paper where truncation is needed.*

3. *We define for site x the norm of the local Hamiltonian H_x constrained to $[-\Lambda, \Lambda]$ to be $\mathcal{N}(\Lambda) = \|H_x \Pi^{(x)}_{[-\Lambda,\Lambda]}\|$ where $\|\cdot\|$ is the spectral norm.*

Often the local observables of interest in a physical problem may have noninteger eigenvalues. If the eigenvalues are all integer multiples of a fixed number, up to some constant shift, then simply shifting and rescaling the observable can get the local quantum numbers we want. Otherwise we can apply a non-linear transformation to the local observable to map all eigenvalues to integers. Also, henceforth we do not distinguish between O_x and λ_x and will sometimes use λ_x as an operator.

Using the notation of Definition 1, we can easily articulate a fundamental assumption for our work, namely, the norm of the local Hamiltonian as well as the local Hilbert space dimension grow modestly with the cutoff Λ. This assumption is stated formally below.

Assumption 2 (Local quantum numbers)**.** *The scaling of the maximum local dimension $d(\Lambda)$ and the norm of the local Hamiltonian H_x constrained to the local quantum number range $[-\Lambda, \Lambda]$ obey*

$$d(\Lambda),\ \mathcal{N}(\Lambda) = \mathcal{O}(\mathrm{poly}(\Lambda)). \tag{9}$$

More conditions are needed for us to handle the infinite dimensional local Hilbert space. Following [31], we assume that:

Assumption 3 (Growth of local quantum numbers)**.** *There exist non-negative real-valued constants χ and r such that, for any x, the Hamiltonian can be decomposed as*

$$H = H_W^{(x)} + H_R^{(x)}, \tag{10}$$

where $H_W^{(x)}$ and $H_R^{(x)}$ satisfy (for $\Pi^{(x)}_\lambda \equiv \Pi^{(x)}_{\{\lambda\}}$)

$$\Pi^{(x)}_\lambda H_W^{(x)} \Pi^{(x)}_{\lambda'} = 0,\ \text{if } |\lambda - \lambda'| > 1, \tag{11a}$$

$$\|H_W^{(x)} \Pi^{(x)}_{[-\Lambda,\Lambda]}\| \le \chi(\Lambda + 1)^r, \tag{11b}$$

$$[H_R^{(x)}, \Pi^{(x)}_\lambda] = 0 \text{ for all } \lambda, \tag{11c}$$

for all $x = 1, 2, \ldots, N$.

We note that the $H = H_W^{(x)} + H_R^{(x)}$ is a decomposition of the entire Hamiltonian for each site x, and neither $H_W^{(x)}$ nor $H_R^{(x)}$ is necessarily local.

These assumptions are satisfied by many physical models of interest. For example, for the Hubbard-Holstein model we can take our local quantum number to be the bosonic occupation number of each of the sites. Furthermore, we can choose for any x [31]

$$H_W^{(x)} = g(b_x^\dagger + b_x)(n_{x,\uparrow} + n_{x,\downarrow} - 1), \qquad H_R^{(x)} = \sum_{x'} \omega_0 b_{x'}^\dagger b_{x'} + \sum_{x' \neq x} g(b_x^\dagger + b_x)(n_{x,\uparrow} + n_{x,\downarrow} - 1) + H_f. \tag{12}$$

If we take our local quantum number, λ_x, to be the phonon number at a particular site with $O_x = b_x^\dagger b_x$ we then see that (11a) holds because the bosonic ladder operators only couple between adjacent occupancy states. Using the commutation properties of b_x and $b_x^\dagger$ it is easy to see that (11b) holds with $r = 1/2$ and $\chi \le 2g$. Since bosonic ladder operators commute across different sites, we have that (11c) also holds for this model. A similar decomposition can be shown to exist for U(1) and SU(2) lattice gauge theories in one spatial dimension.

While such decompositions exist for many cases of interest, they are obviously not necessarily unique. In fact, the choice of decomposition may potentially impact the value of r and χ that appears in (11b). But for our purpose of proving area law, it suffices to take any such decomposition of the Hamiltonian.

4 Truncating the local Hilbert space

Although we consider the setting where the local Hilbert spaces are infinite dimensional, we can approximate spectrally isolated eigenstates with states containing low local quantum numbers. This fact is made rigorous in [31, Theorem 12] which we restate here in a slightly modified way:

Theorem 4 (Quantum number distribution tail bound)**.** *Let H be a Hamiltonian satisfying Assumption 3. Let $|\Psi\rangle$ be an eigenstate of H corresponding to an eigenvalue ε with multiplicity 1, and ε be separated from the rest of the spectrum of H by a spectral gap Δ. Moreover, for a fixed site x we assume local quantum number $|\lambda_x|$ has a finite expectation value:*

$$\sum_{\lambda} |\lambda| \langle\Psi|\Pi_{\lambda}^{(x)}|\Psi\rangle = \bar{\lambda}_x < \infty,$$

we then have that

$$\|(I - \Pi^{(x)}_{[-\Lambda,\Lambda]})\,|\Psi\rangle\,\| \leq e^{-\Omega\left(\sqrt{\chi^{-1}\Delta(\Lambda^{1-r}-(2\bar{\lambda}_x)^{1-r})}\right)}.$$

We will further need to apply this result to bound the error that emerges from applying the truncation bound to many sites. The following corollary provides such a result in a convenient form.

Corollary 5. *Let $\Pi' = \prod_{x=\ell+1}^{\ell+s} \Pi^{(x)}_{[-\Lambda,\Lambda]}$, then under the same assumption as in Theorem 4, we have*

$$\|(I - \Pi')\,|\Psi\rangle\,\| \leq \sqrt{s}e^{-\Omega\left(\sqrt{\chi^{-1}\Delta(\Lambda^{1-r}-(2\bar{\lambda})^{1-r})}\right)},$$

where $\bar{\lambda} = \max_{\ell<x\leq\ell+s} \bar{\lambda}_x$.

Proof. Proof follows from Theorem 4 via a straightforward application of the triangle inequality and the sub-multiplicative property of the spectral norm:

$$\begin{aligned}
\|(I - \Pi')\,|\Psi\rangle\,\| &= \sqrt{\langle\Psi|\,(I-\Pi')^2\,|\Psi\rangle} = \sqrt{\langle\Psi|\,(I-\Pi')\,|\Psi\rangle} \\
&= \sqrt{\sum_{x=\ell+1}^{\ell+s} \langle\Psi| \prod_{x'=\ell+1}^{x-1} \Pi^{(x')}_{[-\Lambda,\Lambda]}(I - \Pi^{(x)}_{[-\Lambda,\Lambda]})\,|\Psi\rangle} \\
&\leq \sqrt{\sum_{x=\ell+1}^{\ell+s} \|(I - \Pi^{(x)}_{[-\Lambda,\Lambda]})\,|\Psi\rangle\,\|} \leq \sqrt{s}e^{-\Omega\left(\sqrt{\chi^{-1}\Delta(\Lambda^{1-r}-(2\bar{\lambda})^{1-r})}\right)}.
\end{aligned} \tag{13}$$

□

A final important consequence of Theorem 4 is a bound on the quantity $\|\Pi' H(I-\Pi')\,|\Psi\rangle\,\|$ where Π' is defined as in Corollary 5. We use the above corollary to derive such a bound below.

Corollary 6. *Let $\Pi' = \prod_{x=\ell+1}^{\ell+s} \Pi^{(x)}_{[-\Lambda,\Lambda]}$, then under the same assumption as in Theorem 4, we have*

$$\|\Pi' H(I - \Pi')\,|\Psi\rangle\,\| \leq s^{3/2}\mathcal{N}(\Lambda)e^{-\Omega\left(\sqrt{\chi^{-1}\Delta(\Lambda^{1-r}-(2\bar{\lambda})^{1-r})}\right)},$$

where $\bar{\lambda} = \max_{\ell<x\leq\ell+s} \bar{\lambda}_x$.

Proof. We only need to use Corollary 5 along with a bound for $\|\Pi' H(I-\Pi')\|$. We have

$$\Pi' H(I-\Pi') = \sum_{x=1}^{N} \Pi' H_x (I-\Pi') = \sum_{x=\ell+1}^{\ell+s} \Pi' H_x (I-\Pi'). \tag{14}$$

The second equality is because $\Pi' H_x(I-\Pi') = \Pi'(I-\Pi')H_x = 0$ if $x \notin \{\ell+1,\ldots,\ell+s\}$. Therefore

$$\|\Pi' H(I-\Pi')\| \leq \sum_{x=\ell+1}^{\ell+s} \|\Pi' H_x(I-\Pi')\| \leq \sum_{x=\ell+1}^{\ell+s} \|\Pi^{(x)}_{[-\Lambda,\Lambda]} H_x\| \leq s\mathcal{N}(\Lambda). \tag{15}$$

□

We remark that the bound in the above corollary does not depend on the system size N. However, there is a dependence on $\bar{\lambda}$, an upper bound on the mean absolute value of the local quantum numbers on sites $\ell+1, \ell+2, \ldots, \ell+s$. One might worry that $\bar{\lambda}$ will show up as an independent parameter in our expression of the entanglement entropy. However, we will show in the next section that for the ground states of U(1) and SU(2) LGTs as well as the Hubbard-Holstein model, $\bar{\lambda}$ depends only on the coefficients in the Hamiltonian, and is thus independent of the system size.

5 Bounding the mean absolute value of the local quantum number

In this section we prove bounds on the mean absolute value $\bar{\lambda}_x = \langle|\lambda_x|\rangle$ of the local quantum numbers in the ground states of U(1) and SU(2) LGTs, as well as the Hubbard-Holstein model. λ_x here is the local quantum number of site x and $\langle\cdot\rangle$ denotes the ground state expectation value. We will drop the subscript x in this section because we will focus on only a single bosonic mode or gauge link. These bounds are independent of the system size and only depend on the coefficients in the Hamiltonians.

In the following discussion, we view our lattice models as a bipartite system, where a subsystem A is a gauge link or bosonic mode, and a subsystem B is the rest of the system. We then bound $\langle|\lambda|\rangle$, where λ is the local quantum number for A, using the variational principle.

Lemma 7. *Consider a bipartite system with Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A$ and $\mathcal{H}_B$ are the Hilbert spaces for subsystems A and B respectively. Let $H = H_A + H_{AB} + H_B$ be the Hamiltonian, where H_A acts non-trivially only on A, and H_B on B. Let $|\Psi\rangle$ be the ground state of H. Furthermore, we assume that there exists an operator K_A acting non-trivially only on A such that $|H_{AB}| \preceq K_A$. Then we have*

$$\langle\Psi| (H_A - K_A) |\Psi\rangle \leq \langle\Psi_A| (H_A + K_A) |\Psi_A\rangle, \tag{16}$$

for any $|\Psi_A\rangle \in \mathcal{H}_A$. Here $\preceq$ denotes the partial order induced by the convex cone of positive semi-definite operators and $|O| = \sqrt{O^\dagger O}$ for operator O.

Proof. Let $|\Psi_P\rangle = |\Psi_A\rangle |\Psi_B\rangle$ be a product state, where $|\Psi_B\rangle \in \mathcal{H}_B$ is the ground state of H_B, and $|\Psi_A\rangle \in \mathcal{H}_A$ is chosen arbitrarily. Then because $|\Psi\rangle$ is the ground state of H we have

$$\langle\Psi|H|\Psi\rangle \leq \langle\Psi_P|H|\Psi_P\rangle. \tag{17}$$

For $\langle\Psi|H|\Psi\rangle$ we have

$$\begin{aligned}\langle\Psi|H|\Psi\rangle &= \langle\Psi|H_A|\Psi\rangle + \langle\Psi|H_{AB}|\Psi\rangle + \langle\Psi|H_B|\Psi\rangle \\ &\geq \langle\Psi|H_A|\Psi\rangle - \langle\Psi|K_A|\Psi\rangle + \langle\Psi|H_B|\Psi\rangle \\ &\geq \langle\Psi|\left(H_A - K_A\right)|\Psi\rangle + \langle\Psi_B|H_B|\Psi_B\rangle\,,\end{aligned} \tag{18}$$

where in the first inequality we have used $|H_{AB}| \preceq K_A$ and in the second inequality we have used $\langle\Psi|H_B|\Psi\rangle \geq \langle\Psi_B|H_B|\Psi_B\rangle$, which is true because $|\Psi_B\rangle$ is chosen to be the ground state of H_B.

For $\langle\Psi_P|H|\Psi_P\rangle$ we have

$$\begin{aligned}\langle\Psi_P|H|\Psi_P\rangle &= \langle\Psi_P|H_A|\Psi_P\rangle + \langle\Psi_P|H_{AB}|\Psi_P\rangle + \langle\Psi_B|H_B|\Psi_B\rangle \\ &\leq \langle\Psi_P|\left(H_A + K_A\right)|\Psi_P\rangle + \langle\Psi_B|H_B|\Psi_B\rangle \\ &= \langle\Psi_A|\left(H_A + K_A\right)|\Psi_A\rangle + \langle\Psi_B|H_B|\Psi_B\rangle\,.\end{aligned} \tag{19}$$

Combining (17), (18), (19), we have (16).

□

Lemma 8. *Under the same assumptions as in Lemma 7, we further let Ξ be a Hermitian operator on A. Assume that $H_A - K_A \succeq L(|\Xi|)$ where $L(x)$ is a convex non-decreasing function for $x \in \mathbb{R}^+$ satisfying $L(x) \to +\infty$ when $x \to +\infty$. Then we have*

$$\langle\Psi||\Xi||\Psi\rangle \leq L^{-1}\left(\langle\Psi|\left(H_A - K_A\right)|\Psi\rangle\right) \leq L^{-1}\left(\min_{|\Psi_A\rangle\in\mathcal{H}_A}\langle\Psi_A|\left(H_A + K_A\right)|\Psi_A\rangle\right). \tag{20}$$

Proof. By the assumption that $H_A - K_A \succeq L(|\Xi|)$ and (16), we have

$$\langle\Psi|L(|\Xi|)|\Psi\rangle \leq \langle\Psi_A|\left(H_A + K_A\right)|\Psi_A\rangle\,.$$

Because L is convex, by Jensen's inequality $L(\langle\Psi||\Xi||\Psi\rangle) \leq \langle\Psi|L(|\Xi|)|\Psi\rangle$. Because L is non-decreasing and $|\Psi_A\rangle$ can be arbitrarily chosen, we have (20). □

Note that the right-hand side of (20) is independent of the subsystem B. This allows us to bound the mean absolute value of the local quantum number in the lattice models in a way that is independent of the system size. We will now apply this lemma to the case of gauge theories and the Hubbard-Holstein model.

First, for U(1) and SU(2) gauge theories with the Hamiltonian defined in (4), we take A to be a gauge link x. Then we let

$$H_A = g_E E_x^2, \qquad H_{AB} = g_{GM}(\phi_x^\dagger U_x \phi_{x+1} + \phi_{x+1}^\dagger U_x^\dagger \phi_x), \qquad H_B = H - H_A - H_B.$$

A nice feature about this Hamiltonian is that H_{AB} is bounded: $\|H_{AB}\| \leq 2|g_{GM}|$. Therefore we can simply choose $K_A = 2|g_{GM}|$. The local quantum number λ_x in this case is the electric field value in the U(1) case and the total angular momentum in the SU(2) case. We recall that $E_x^2 = \lambda_x^2$ for U(1) LGT and $E_x^2 = \lambda_x(\lambda_x + 1) \succeq \lambda_x^2$ for SU(2) LGT (for the latter see (5), also $\lambda_x \succeq 0$ for SU(2) LGT). Therefore we have

$$H_A - K_A \succeq g_E\lambda_x^2 - 2|g_{GM}|$$

for both cases. For the right-hand side of (20) we have

$$\min_{|\Psi_A\rangle\in\mathcal{H}_A}\langle\Psi_A|\left(H_A + K_A\right)|\Psi_A\rangle = 2|g_{GM}|,$$

where the minimum is attained by $|\Psi_A\rangle = |0\rangle$. Combining the above facts, we apply Lemma 8 to get:

Corollary 9. *For the U(1) and SU(2) LGTs with the Hamiltonian defined in* (4)*, let λ_x be the local quantum number on gauge link x, then $\langle|\lambda_x|\rangle \leq 2\sqrt{|g_{GM}|/g_E}$, where $\langle\cdot\rangle$ denotes the ground state expectation value.*

Then let us consider the Hubbard-Holstein model with the Hamiltonian described in (1). We let A be a bosonic mode x, and let B be the rest of the system. We have

$$H_A = \omega_0 b_x^\dagger b_x, \qquad H_{AB} = g(b_x^\dagger + b_x)(n_{x,\uparrow} + n_{x,\downarrow} - 1), \qquad H_B = H - H_A - H_{AB}.$$

Here H_{AB} is no longer bounded, but we can still construct a K_A such that $|H_{AB}| \preceq K_A$. To simplify the discussion we introduce the position and momentum operators X and P:

$$X = \frac{1}{\sqrt{2}}(b_x^\dagger + b_x), \qquad P = \frac{i}{\sqrt{2}}(b_x^\dagger - b_x).$$

Then $H_{AB} = \sqrt{2}gX(n_{x,\uparrow} + n_{x,\downarrow} - 1)$. Because $\|n_{x,\uparrow} + n_{x,\downarrow} - 1\| \leq 1$, we can define

$$K_A = \sqrt{2}|g||X|,$$

which satisfies $|H_{AB}| \preceq K_A$. With this choice of K_A we have

$$\begin{aligned}
H_A - K_A &= \frac{\omega_0}{2}(X^2 + P^2 - 1) - \sqrt{2}|g||X| \\
&\succeq \frac{\omega_0}{2}(X^2 + P^2 - 1) - \frac{\omega_0}{4}X^2 - \frac{2g^2}{\omega_0} \\
&\succeq \frac{\omega_0}{4}(X^2 + P^2 - 1) - \frac{\omega_0}{4} - \frac{2g^2}{\omega_0} \\
&= \frac{\omega_0}{2}b_x^\dagger b_x - \frac{\omega_0}{4} - \frac{2g^2}{\omega_0}.
\end{aligned}$$

The local quantum number here is the bosonic occupation number, which has to be non-negative. Therefore

$$H_A - K_A \succeq \frac{\omega_0}{2}|\lambda_x| - \frac{\omega_0}{4} - \frac{2g^2}{\omega_0}.$$

For the right-hand side of (20), we have

$$\min_{|\Psi_A\rangle \in \mathcal{H}_A} \langle\Psi_A| (H_A + K_A) |\Psi_A\rangle \leq \langle 0| \left(\omega_0 b_x^\dagger b_x + 2|g||X|\right) |0\rangle = \frac{2|g|}{\sqrt{\pi}},$$

where we have used the analytic solution of the ground state of the harmonic oscillator in deriving the equality. Combining these results with Lemma 8 we have:

Corollary 10. *For the Hubbard-Holstein model with the Hamiltonian defined in* (1)*, let λ_x be the local quantum number on site x, then*

$$\langle|\lambda_x|\rangle \leq \frac{1}{2} + \frac{4|g|}{\omega_0\sqrt{\pi}} + \frac{4g^2}{\omega_0^2},$$

where $\langle\cdot\rangle$ denotes the ground state expectation value.

6 Robustness of the ground state to truncation

In this section, we show that the ground state, the ground state energy, and the spectral gap are all robust to the truncation of the Hamiltonian, in a way that we will specify later. Following [6] we focus on the s sites from $\ell+1$ to $\ell+s$. For convenience we relabel the sites so that the original site x is now labelled $x-\ell$. The Hamiltonian can be rewritten as

$$H = H_L + H_1 + \cdots + H_s + H_R, \tag{21}$$

where $H_L = \sum_{x\le 0} H_x$ and $H_R = \sum_{x\ge s+1} H_x$. We need to shift each Hamiltonian term (H_L, H_1, $H_2, \ldots, H_s$, H_R) by a constant to ensure that they are all positive semi-definite. As in [6] we look at the entanglement entropy across a cut between sites $s/2$ and $s/2+1$. If $\|H_x\| \le 1$, the local Hilbert space dimension is d and the spectral gap is Δ, then it is known that the entanglement entropy scales as $\mathcal{O}(\log^3(d)/\Delta)$ [6, Theorem 6.2].

Now we want to consider the case where the local Hilbert space dimension is infinite. This compels us to truncate the local Hilbert space, and consequently the local Hamiltonian terms H_i as well. We denote the truncation threshold, defined according to the local quantum number introduced in Section 3, by Λ, and correspondingly the truncated Hamiltonian term by H'_x. The truncated Hilbert space dimension is $d(\Lambda)$ and the truncated local term has a norm that is upper bounded by $\mathcal{N}(\Lambda)$.

6.1 The two truncations

We first clarify in more detail what we mean by truncating the local Hilbert space. The original Hilbert space is $\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_1 \otimes \cdots \mathcal{H}_s \otimes \mathcal{H}_R$. We consider a subspace $\mathcal{H}' = \mathcal{H}_L \otimes \mathcal{H}'_1 \otimes \cdots \mathcal{H}'_s \otimes \mathcal{H}_R \subseteq \mathcal{H}$, where each $\mathcal{H}_x$ has dimension $d(\Lambda)$. We denote by Π'_x the projection operator onto $\mathcal{H}'_x$, and define

$$\Pi' = I_L \otimes \Pi'_1 \otimes \cdots \Pi'_s \otimes I_R,$$

which is the projection operator onto $\mathcal{H}'$. I_L and I_R are the identity operators on $\mathcal{H}_L$ and $\mathcal{H}_R$ respectively. The truncated Hamiltonian is a Hermitian operator H' defined to be the *restriction* of $\Pi' H \Pi'$ to the subspace $\mathcal{H}'$, by which we mean that H' maps elements from $\mathcal{H}'$ to $\mathcal{H}'$. We do not directly define H' to be $\Pi' H \Pi'$ because that would introduce an artificial eigenvalue 0 corresponding to the part of the Hilbert space that is truncated out. We can write H' out as

$$H' = H'_L + H'_1 + \cdots + H'_s + H'_R, \tag{22}$$

where each H'_x is the restriction of $\Pi' H_x \Pi'$ to the subspace $\mathcal{H}'$, and the same is true for H'_L and H'_R. Locality is preserved in this truncation as H'_x still acts non-trivially on sites x and $x+1$, H'_L on sites to the left of and including site 1, and H'_R on sites to the right of site s.

The second truncation we consider comes from Ref. [6]. We adopt the definition in [6], for a Hermitian operator A,

$$A^{\le t} = AP_t + \|AP_t\|(I - P_t), \tag{23}$$

where P_t is the projection operator onto the subspace spanned by eigenvectors of A with eigenvalues at most t. Then the second truncation yields the Hamiltonian

$$H'' = (H'_L + H'_1)^{\le t} + H'_2 + \cdots + H'_{s-1} + (H'_s + H'_R)^{\le t}. \tag{24}$$

The goal here is to show that these two truncations (i) preserve the spectral gap up to a constant factor, and (ii) preserve the ground state up to an error exponentially small in Λ and t.

6.2 Truncation robustness of the ground state and energy

Definition 11. *For a self-adjoint operator A bounded from below, define the sequence $\epsilon_0(A) \leq \epsilon_1(A) \leq \ldots$ as follows. Let $\sigma_{\text{ess}}(A)$ be its (closed) essential spectrum and let $K \in \mathbb{N}_0 \cup \{\infty\}$ be the number of eigenvalues in $[-\infty, \min \sigma_{\text{ess}}(A))$, including multiplicity. For $k < K$, let $\epsilon_k(A)$ be the k-th eigenvalue with multiplicity (K eigenvalues as we start from $k = 0$), and for $k \geq K$ let $\epsilon_k(A) = \min \sigma_{\text{ess}}(A)$.*

Each $\epsilon_k(A)$ is in the spectrum[1] of A but is not necessarily an eigenvalue when $\epsilon_k(A) = \min \sigma_{\text{ess}}(A)$. The min-max principle [30, Theorem 4.10] states that for $k \in \mathbb{N}_0$,

$$\epsilon_k(A) = \inf_{\phi_0,\ldots,\phi_k \in \mathcal{D}(A)} \sup \left\{ \langle\psi|A|\psi\rangle \;\middle|\; |\psi\rangle \in \text{span}\{\phi_0, \ldots, \phi_k\}, \| \, |\psi\rangle \, \| = 1 \right\}, \tag{25}$$

where $\mathcal{D}$ is the domain of A.

Writing $\epsilon_k = \epsilon_k(H)$, recall that we assume that the Hamiltonian H has a non-degenerate lowest eigenvalue ϵ_0 (the ground state energy), with a unique ground state $|\psi_0\rangle$. We also assume that ϵ_0 is separated from the rest of the spectrum by a gap $\Delta = \epsilon_1 - \epsilon_0$.

For truncated Hamiltonians H' and H'', we will prove that there exist unique ground states $|\psi_0'\rangle$ and $|\psi_0''\rangle$, corresponding to non-degenerate lowest eigenvalues ϵ_0' and ϵ_0'', for the two truncated Hamiltonians respectively. Write $\epsilon_k = \epsilon_k(H)$, $\epsilon_k' = \epsilon_k(H')$, and $\epsilon_k'' = \epsilon_k(H'')$ and $\Delta' = \epsilon_1' - \epsilon_0'$, $\Delta'' = \epsilon_1'' - \epsilon_0''$.

Theorem 12 (Robustness to truncations)**.** *Let Π' be the projection operator onto $\mathcal{H}'$. Let $\delta_1 = \|(I - \Pi') |\psi_0\rangle \|$, $\delta_2 = \|\Pi' H (I - \Pi') |\psi_0\rangle \|$, and*

$$\frac{\delta_2}{1 - \delta_1^2} \leq \frac{\Delta}{18}, \tag{26}$$

then for every cutoff of the local quantum number $\Lambda > 0$ there exists

$$T = \mathcal{O}\left(\frac{\mathcal{N}(\Lambda)^2}{\Delta} \left(\frac{\epsilon_0 + \delta_2/(1 - \delta_1^2)}{\Delta} + 1 \right) \right), \tag{27}$$

such that for all $t \geq T$,

(i) For H'', there exists a non-degenerate ground state $|\psi_0''\rangle$ corresponding to the lowest eigenvalue ϵ_0''.

(ii) $\Delta'' = \Omega(\Delta)$;

(iii) The trace distance between $|\psi_0\rangle$ and $|\psi_0''\rangle$ can be bounded as follows:

$$D(|\psi_0''\rangle, |\psi_0\rangle) \leq \sqrt{\frac{2\delta_2}{\Delta(1 - \delta_1^2)}} + e^{-\Omega(t/\mathcal{N}(\Lambda))}. \tag{28}$$

Before we proceed with the proof we establish the following lemma, which follows from a similar reasoning as in [6, Lemma 6.4] except we correct a minor mistake in their bound due to not taking the global phase into account.

[1] I.e., the set of ϵ such that $A - \epsilon I$ has no bounded inverse.

Lemma 13 (Markov)**.** *Let H be a Hamiltonian with the lowest eigenvalue ϵ_0 and all other eigenvalues at least $\epsilon_1 > \epsilon_0$, and assume $|\psi_0\rangle$ is its unique ground state. Given a quantum state $|\phi\rangle$ with expectation value $\langle\phi|H|\phi\rangle = E$, we have*

$$|\langle\phi|\psi_0\rangle|^2 \geq \frac{\epsilon_1 - E}{\epsilon_1 - \epsilon_0}. \tag{29}$$

Proof. Since the eigenstate $|\psi_0\rangle$ is the unique ground state, the expectation value of H in any other eigenstate must be at least ϵ_1 by assumption. Using the fact that $\epsilon_1 - \epsilon_0 > 0$, we have from Markov's inequality

$$E = \langle\phi| H |\phi\rangle \geq \epsilon_0 |\langle\phi|\psi_0\rangle|^2 + \epsilon_1(1 - |\langle\phi|\psi_0\rangle|^2). \tag{30}$$

The result then follows by re-arranging the above expression. □

The proof of Theorem 12 proceeds as follows: we first show that when we go from H to H', the spectral gap is preserved and the ground state is changed by a small amount, and then show the same is true when we go from H' to H''. In the first step we obtain:

Lemma 14. *Let Π' be the projection operator onto $\mathcal{H}'$. Let $\delta_1 = \|(I - \Pi')|\psi_0\rangle\|$, $\delta_2 = \|\Pi' H(I - \Pi')|\psi_0\rangle\|$. Then if $\delta_1 < 1$ and*

$$\frac{\delta_2}{1 - \delta_1^2} \leq \frac{\Delta}{4}, \tag{31}$$

we have the following

(i) For H', there exists a non-degenerate ground state $|\psi'_0\rangle$ corresponding to the lowest eigenvalue ϵ'_0.

(ii) $\Delta' = \Omega(\Delta)$.

(iii) The trace distance between $|\psi_0\rangle$ and $|\psi'_0\rangle$ can be bounded as

$$D(|\psi'_0\rangle, |\psi_0\rangle) \leq \sqrt{\frac{2\delta_2}{\Delta(1 - \delta_1^2)}}. \tag{32}$$

(iv) $\epsilon_0 \leq \epsilon'_0 \leq \epsilon_0 + \frac{2\delta_2}{1-\delta_1^2}$.

Here $D(\cdot, \cdot)$ denotes the trace distance.

Proof. By the min-max theorem (Equation 25), we have for $k \in \mathbb{N}_0$

$$\begin{aligned}\epsilon_k &= \inf_{\phi_0,\ldots,\phi_k \in \mathcal{H}} \sup \left\{ \langle\psi|H|\psi\rangle \,\middle|\, |\psi\rangle \in \operatorname{span}\{\phi_0, \ldots, \phi_k\}, \| |\psi\rangle \| = 1 \right\} \\ &\leq \inf_{\phi_0,\ldots,\phi_k \in \mathcal{H}'} \sup \left\{ \langle\psi|H|\psi\rangle \,\middle|\, |\psi\rangle \in \operatorname{span}\{\phi_0, \ldots, \phi_k\}, \| |\psi\rangle \| = 1 \right\} = \epsilon'_k \leq \min \sigma_{\mathrm{ess}}(H').\end{aligned} \tag{33}$$

In particular we have $\epsilon_1 \leq \epsilon'_1 \leq \min \sigma_{\mathrm{ess}}(H')$. To establish a gap Δ' we need an upper bound on ϵ'_0:

$$\begin{aligned}\epsilon_0 &= \langle\psi_0|H|\psi_0\rangle \\ &= \langle\psi_0|\Pi' H \Pi'|\psi_0\rangle + \langle\psi_0|(I - \Pi')H\Pi'|\psi_0\rangle \\ &\quad + \langle\psi_0|\Pi' H(I - \Pi')|\psi_0\rangle + \langle\psi_0|(I - \Pi')H(I - \Pi')|\psi_0\rangle \\ &\geq \epsilon'_0 \|\Pi'|\psi_0\rangle\|^2 - 2\|\Pi' H(I - \Pi')|\psi_0\rangle\| + \epsilon_0 \|(I - \Pi')|\psi_0\rangle\|^2 \\ &= \epsilon'_0(1 - \delta_1^2) - 2\delta_2 + \epsilon_0 \delta_1^2.\end{aligned} \tag{34}$$

As a result,

$$\epsilon_0' \le \epsilon_0 + \frac{2\delta_2}{1-\delta_1^2}. \tag{35}$$

The bound $\epsilon_0 \le \epsilon_0'$ is immediate from the variational principle (or (33)), so (iv) is established. The assumption (31) then yields

$$\epsilon_0' \le \epsilon_0 + \Delta/2 \le \epsilon_1 - \Delta/2 \le \epsilon_1' - \Delta/2,$$

which implies $\Delta' \ge \Delta/2$, hence (ii). In particular it implies (i), that ϵ_0' is a simple eigenvalue with eigenvector $|\psi_0'\rangle$.

To establish closeness between $|\psi_0'\rangle$ and $|\psi_0\rangle$ we apply Lemma 13,

$$\langle\psi_0'|\psi_0\rangle^2 \ge \frac{\epsilon_1 - \langle\psi_0'|H|\psi_0'\rangle}{\epsilon_1-\epsilon_0} = \frac{\epsilon_1-\epsilon_0'}{\Delta} = 1 - \frac{\epsilon_0'-\epsilon_0}{\Delta} \ge 1 - \frac{1}{\Delta}\frac{2\delta_2}{1-\delta_1^2},$$

where the last inequality follows from (35). Claim (iii) follows since $D(|\psi_1'\rangle, |\psi_0\rangle) \le \sqrt{1-|\langle\psi_1'|\psi_0\rangle|^2}$. □

We are then ready to prove the main result.

Proof of Theorem 12. The existence and uniqueness of the lowest eigenvalue ϵ_0'' and the ground state $|\psi_0''\rangle$ in (i) can be proved similarly to the proof of Lemma 14 (i). We rescale the Hamiltonian H' by a factor $\mathcal{N}(\Lambda)$ and then apply [6, Theorem 6.1]. That theorem tells us that $\Delta'' = \Omega(\Delta')$. Combining this fact with Lemma 14 (ii), we have (ii). [6, Theorem 6.1] also tells us that there exists

$$T = \mathcal{O}\left(\frac{\mathcal{N}(\Lambda)^2}{\Delta'}\left(\frac{\epsilon_0'}{\Delta'}+1\right)\right) = \mathcal{O}\left(\frac{\mathcal{N}(\Lambda)^2}{\Delta}\left(\frac{\epsilon_0+\delta_2/(1-\delta_1^2)}{\Delta}+1\right)\right), \tag{36}$$

such that $D(|\psi_0'\rangle, |\psi_0''\rangle) = e^{-\Omega(t/\mathcal{N}(\Lambda))}$ for $t \ge T$, where $\mathcal{N}(\Lambda)$ comes from the rescaling. Here we have used Lemma 14 (iv). Combining with Lemma 14 (iii) and the triangle inequality, we have proved (iii). □

Now we can use Corollaries 5 and 6 to bound δ_1 and δ_2 in Theorem 12, which leads to the following robustness result:

Corollary 15 (Robustness to truncations)**.** *Assume that the Hamiltonian satisfies Assumptions 2 and 3, and let $\bar{\lambda} = \max_{1\le x\le s}\langle|\lambda_x|\rangle$. Then the truncated Hamiltonian H'' has a lowest eigenvalue ϵ_0'' corresponding to a non-degenerate ground state $|\psi_0''\rangle$. And there exist constants C_1 and C_2 such that for any Λ and t satisfying*

$$\Lambda^{1-r} \ge (2\bar{\lambda})^{1-r} + C_1\Delta^{-1}\mathrm{polylog}(s,\Delta^{-1}), \qquad t \ge \frac{C_2\mathcal{N}(\Lambda)^2}{\Delta^2}, \tag{37}$$

we have

(i) $\Delta'' = \Omega(\Delta)$;

(ii) The trace distance between $|\psi_0\rangle$ and $|\psi_0''\rangle$ can be bounded as follows:

$$D(|\psi_0''\rangle, |\psi_0\rangle) = \mathrm{poly}(s,\Delta^{-1},\Lambda)e^{-\Omega\left(\sqrt{\Delta(\Lambda^{1-r}-(2\bar{\lambda})^{1-r})}\right)} + e^{-\Omega(t/\mathcal{N}(\Lambda))}. \tag{38}$$

We recall that r is the exponent involved in (11b), and that $r = 0$ for LGTs, $r = 1/2$ for the Hubbard-Holstein model. Δ here is the spectral gap, and s is the size of the region around the cut that we want to pay special attention to in (21).

If we assume $\bar{\lambda} = O(1)$, which we proved for the Hubbard-Holstien model and U(1) and SU(2) LGTs in Section 5, then to achieve $D(|\psi_0''\rangle, |\psi\rangle)^2 \leq \delta$, it suffices to require, for some constant C,

$$\sqrt{\Delta(\Lambda^{1-r} - C)} \geq C \log \frac{s\Lambda}{\delta\Delta} \quad \text{and} \quad t \geq C\mathcal{N}(\Lambda)\log(1/\delta) \vee \frac{C\mathcal{N}(\Lambda)^2}{\Delta^2},$$

where $\vee$ denotes the maximum, which can be satisfied by choosing

$$\Lambda = \mathrm{poly}(\Delta^{-1})\mathrm{polylog}(s/\delta) \quad \text{and} \quad t = \Theta(\mathcal{N}(\Lambda)\log(1/\delta) \vee \mathcal{N}(\Lambda)^2/\Delta^2). \tag{39}$$

7 Area law

In this section, we establish our main result of an entanglement area law for unbounded quantum systems. We first recall the notion of AGSP from [6, Definition 2.1].

Definition 16. *K is a (σ, R)-AGSP of a Hamiltonian H on a bipartite system consisting of two parts A and B that has a non-degenerate ground state, if*

1. *$K|\Psi\rangle = |\Psi\rangle$, where $|\Psi\rangle$ is the ground state of H.*
2. *$\|K|\Phi\rangle\| \leq \sigma$ for any $|\Phi\rangle$ such that $\langle\Phi|\Psi\rangle = 0$.*
3. *There exist operators K_j^A and K_j^B, acting on A and B respectively, $j = 1, 2, \ldots, R$, such that $K = \sum_{j=1}^R K_j^A \otimes K_j^B$.*

An AGSP preserves the ground state, suppresses the excited states, and increases the entanglement by a finite amount. The existence of an AGSP onto the exact ground state is known to imply a bound on the entanglement of ground state. More precisely, Corollary III.4 of [5] states that if $\sigma R \leq 1/2$ where σ is the shrinking factor and R is the entanglement rank of the AGSP, then the entanglement entropy of the ground state satisfies a bound of order $\log R$. For frustrated systems the target space of the AGSP becomes perturbed away from the exact ground state(s) as its construction involves spectral truncations. Analyses of such a situation are undertaken in [6] and [7]. These tools were simplified in [1], where an "off-the-rack" lemma was stated which generalizes the one of [5] to perturbed and degenerate target spaces.

Here we make a further improvement to [1] to obtain the cleaner and slightly stronger statement of Lemma 17 below. For two subspaces $\mathcal{Y}, \mathcal{Z} \subset \mathcal{H}$, we say $\mathcal{Y}$ is δ-viable for $\mathcal{Z}$ if $\langle z|P_{\mathcal{Y}^\perp}|z\rangle \leq \delta$ for all unit vectors $|z\rangle \in \mathcal{Z}$, where $P_{\mathcal{Y}^\perp}$ is the projection onto the orthogonal complement of subspace $\mathcal{Y}$. We write $\mathcal{Y} \approx_\delta \mathcal{Z}$ if $\langle z|P_{\mathcal{Y}^\perp}|z\rangle \leq \delta$ and $\langle y|P_{\mathcal{Z}^\perp}|y\rangle \leq \delta$ for unit vectors $|y\rangle \in \mathcal{Y}$ and $|z\rangle \in \mathcal{Z}$ respectively.

Lemma 17. *Let $\mathcal{Z}$ be a subspace of bipartite space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Let $\tilde{\mathcal{Z}}_n$ be a sequence of subspaces of $\mathcal{H}$ such that $\tilde{\mathcal{Z}}_n \approx_{\delta_n} \mathcal{Z}$ where δ_n is a sequence such that $\sum_{n=0}^\infty n\delta_n = O(1)$.*

Suppose there exist $K_1, K_2, \ldots$ such that K_n is an (σ^n, R^n)-AGSP with target space $\tilde{\mathcal{Z}}_n$ where $\sigma = \frac{1}{2R}$. Then the entanglement entropy of any state in $\mathcal{Z}$ is $O(\log R + \log\dim\mathcal{Z})$.

Proof. This follows from the proof of [1, Lemma 4.7] and the following strengthening of [1, Lemma 4.6]. $\square$

Lemma 18. *Let $\mathcal{Z}$ be a subspace of $\mathcal{H}_1 \otimes \mathcal{H}_2$ and suppose there exists a subspace $\mathcal{V} \subset \mathcal{H}_1$ with* $\dim(\mathcal{V}) = V$ *which is δ-viable for $\mathcal{Z}$. Pick any normalized state $|\psi\rangle \in \mathcal{Z}$ and write the Schmidt decomposition $\sum_i \sqrt{\lambda_i}\,|x_i\rangle\,|y_i\rangle$ with nonincreasing Schmidt coefficients. Then we have the tail bound $\sum_{i>V} \lambda_i \leq \delta$.*

Proof. By the definition of δ-viability, we have

$$\langle z'|(P_{\mathcal{V}^\perp} \otimes I)|z'\rangle \leq \delta \tag{40}$$

for all normalized state $|z'\rangle$ in $\mathcal{Z}$. In particular, this implies

$$\langle z|(P_{\mathcal{V}^\perp} \otimes I)|z\rangle = \sum_i \lambda_i \langle x_i|P_{\mathcal{V}^\perp}|x_i\rangle = \operatorname{tr}\left(P_{\mathcal{V}^\perp} \sum_i \lambda_i |x_i\rangle\langle x_i|\right) \leq \delta. \tag{41}$$

Here, $P_{\mathcal{V}^\perp}$ projects onto $\mathcal{V}^\perp$ with $\dim(\mathcal{V}^\perp) = \dim(\mathcal{H}_1) - V$. Now use Poincaré's inequalities [20, Corollary 4.3.39] to conclude that

$$\operatorname{tr}\left(P_{\mathcal{V}^\perp} \sum_i \lambda_i |x_i\rangle\langle x_i|\right) \geq \sum_{i=\dim(\mathcal{H}_1)-\dim(\mathcal{V}^\perp)+1}^{\dim(\mathcal{H}_1)} \lambda_i = \sum_{i=V+1}^{\dim(\mathcal{H}_1)} \lambda_i. \tag{42}$$

This establishes the claimed bound. ∎

We apply Lemma 17 to the case of a simple ground state. Since $\operatorname{span}|\psi_1\rangle \approx_\delta \operatorname{span}|\psi_2\rangle$ where $\delta = D(|\psi_1\rangle, |\psi_2\rangle)^2$ we obtain

Corollary 19. *Suppose there exist $K_1, K_2, \ldots$ such that K_n is an (σ^n, R^n)-AGSP with target state $|\psi_n\rangle$ where $\sigma = \frac{1}{2R}$. If $\sum_{n=0}^\infty nD(|\psi_n\rangle, |\psi\rangle)^2 = O(1)$ then the entanglement entropy of $|\psi\rangle$ is $O(\log R)$.*

The Hamiltonian H'' of (24) has spectral norm $O(s\mathcal{N}(\Lambda) + T(\Lambda))$. Under the conditions of Corollary 15, H'' has a spectral gap $\Delta'' = \Omega(\Delta)$, so we have

$$\Delta''/\|H''\| = \Omega\Big(\frac{\Delta}{s\mathcal{N}(\Lambda) + T(\Lambda)}\Big).$$

The Chebyshev polynomial of degree ℓ is bounded by 1 on the unit interval but satisfies $T_\ell(1+\Delta) \geq \frac{1}{2}(1+\sqrt{2\Delta})^\ell = \frac{1}{2}\exp(\Omega(\ell\sqrt{\Delta}))$, so composing it with linear transformations yields a polynomial f with $f_\ell(0) = 1$ and $|f_\ell(\lambda)| \leq 2\exp(-\Omega(\ell\sqrt{\Delta/M}))$ for $\lambda \in [\Delta, M]$. Picking $M = \|H''\|$ we get that $K = f_\ell(H'')$ is an AGSP with shrinking factor

$$\sigma \lesssim \exp\Big(-\Omega(\ell\sqrt{\Delta''/\|H''\|})\Big) = \exp\Big[-\Omega\Big(\frac{\ell\sqrt{\Delta}}{\sqrt{sN + T}}\Big)\Big]. \tag{43}$$

Assume $\bar{\lambda} = O(1)$. We may pick parameters as in Eq. (39) so that $D(|\psi_0''\rangle, |\psi_0\rangle)^2 \leq \delta$. Since $\mathcal{N}(\Lambda) = \operatorname{poly}(\Lambda)$, Eq. (39) implies

$$\Lambda, \mathcal{N}(\Lambda), T(\Lambda) = \operatorname{poly}(\Delta^{-1})\operatorname{polylog}(s/\delta). \tag{44}$$

Eq. (43) then becomes

$$\sigma(\delta) \lesssim \exp\Big[-\Omega\Big(\ell \cdot \frac{\Delta^{O(1)}}{\sqrt{s}\cdot\operatorname{polylog}(s/\delta)}\Big)\Big].$$

H'' is a spin chain with a length-s segment such that each qudit on the segment has local dimension $d(\Lambda) = \Lambda^{O(1)}$. The amortization bound from the proof of [6, Lemma 4.2] yields that K has entanglement rank:

$$R(\delta) = (\ell d)^{O(\ell/s+s)} \le \exp(C(\ell/s+s)\log(\ell\Lambda)) \le \exp[C(\ell/s+s)\big(\log(\ell/\Delta) + \log\log(s/\delta)\big)].$$

Let $\ell = s^2$:

$$R(\delta) = \exp\Big[Cs\big(\log s + \log(\Delta^{-1}) + \log\log(1/\delta)\big)\Big], \qquad \sigma(\delta) = \exp\Big[-cs^{3/2}\cdot(\tfrac{\Delta}{\log(s/\delta)})^{O(1)}\Big]. \quad (45)$$

Theorem 20 (Area law)**.** *Under Assumptions 2 and 3 with $\bar\lambda = O(1)$, the ground state of H satisfies an area law with entanglement entropy bounded by* $\mathrm{poly}(\Delta^{-1})$.

We remark that $\bar\lambda = O(1)$ is proved for the U(1) and SU(2) LGTs (Corollary 9), as well as for the Hubbard-Holstein model (Corollary 10), in Section 5.

Proof. For $k = 1, 2, \ldots$ consider $s = rk/\log(rk)$ and $\delta_k = k^{-3}$, chosen so that $\sum k\delta_k < \infty$. Then (45) yields entanglement bounds and shrinking factors

$$R_k = \exp[C_0 rk\log(\Delta^{-1})], \qquad \sigma_k = \exp[-c(rk)^{3/2}\cdot(\tfrac{\Delta}{\log(rk)})^{O(1)}].$$

In particular we have $R_k = R^k$ where $R = R_1$. To apply Corollary 19 it remains to ensure that $\sigma_k \le (2R)^{-k}$ for all $k \in \mathbb{N}$

$$(2R)^k\sigma_k \le \exp\Big[k + C_0 rk\log(\Delta^{-1}) - c(rk)^{3/2}\cdot(\tfrac{\Delta}{\log(rk)})^{O(1)}\Big].$$

So we just need the expression in the square brackets to be negative. Dividing the exponent by rk we see that it suffices to pick r such that for all k,

$$1/r + C_0\log(\Delta^{-1}) \le c\sqrt{rk}\cdot\Big(\frac{\Delta}{\log(rk)}\Big)^{C_1}.$$

Pick $r = \tilde\Theta(\Delta^{-2C_1})$ to achieve this for all $k \ge 1$. By Lemma 17 the ground state entanglement entropy is bounded by:

$$\log R = C_0 r\log(\Delta^{-1}) = \mathrm{poly}(\Delta^{-1}).$$

This completes our proof. $\square$

8 Conclusions and discussion

In this work we have rigorously established an entanglement area law for the gapped ground state of 1D quantum systems with infinite-dimensional local Hilbert spaces under natural assumptions 2 and 3 on the local quantum numbers. In particular, our entanglement area law applies to U(1) and SU(2) LGTs and the Hubbard-Holstein model in 1D, and the result may be adapted to handle other bosonic and gauge theory models of interest. As an intermediate result, we develop a system-size independent bound on the ground state expectation value of local observables for Hamiltonians without translation symmetry, which may be useful in other contexts beyond our area law proof.

Proving the entanglement area law is an important step toward justifying the use of tensor network methods in classical simulation of quantum systems. The proof techniques also provide many useful tools for designing a rigorous RG algorithm for these quantum systems, which should be investigated by future work. It is also worth considering how the result in this work can be generalized to degenerate ground state or low-energy states as in Ref. [7].

When studying the entanglement area law scaling in LGTs, we considered the limit where the lattice size grows to infinity while the lattice spacing remains constant, which is needed to ensure that the coefficients in the Hamiltonian remain constant. This corresponds to the thermodynamic limit rather than the continuum limit, where the lattice spacing should also decrease to 0. It remains an open question whether the entanglement entropy can be bounded in the continuum limit as well. It would also be interesting to investigate the scaling of other entropic quantities beyond the standard von Neumann entropy considered here.

Our result relies on the local quantum number tail bound proved in Ref. [31], which in turn follows from technical tools for analyzing the dynamical simulation of unbounded Hamiltonians on digital quantum computers. Along this line, previous work such as [3, 23, 25, 32] have found other applications of quantum simulation techniques, to solving problems in quantum many-body physics beyond quantum computing. We consider finding further applications of this kind to be very interesting, as they demonstrate the immediate utility of studying quantum algorithms, while the building of scalable quantum computers remains a long-term goal.

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