Dynamic Revenue Sharing*†

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Abstract

Many online platforms act as intermediaries between a seller and a set of buyers. Examples of such settings include online retailers (such as Ebay) selling items on behalf of sellers to buyers, or advertising exchanges (such as AdX) selling pageviews on behalf of publishers to advertisers. In such settings, *revenue sharing* is a central part of running such a marketplace for the intermediary, and fixed-percentage revenue sharing schemes are often used to split the revenue among the platform and the sellers. In particular, such revenue sharing schemes require the platform to (i) take at most a constant fraction α of the revenue from auctions and (ii) pay the seller at least the seller declared opportunity cost c for each item sold. A straightforward way to satisfy the constraints is to set a reserve price at $c/(1-\alpha)$ for each item, but it is not the optimal solution on maximizing the profit of the intermediary.

While previous studies (by Mirrokni and Gomes, and by Niazadeh et al) focused on revenue-sharing schemes in static double auctions, in this paper, we take advantage of the repeated nature of the auctions. In particular, we introduce *dynamic revenue sharing schemes* where we balance the two constraints over different auctions to achieve higher profit and seller revenue. This is directly motivated by the practice of advertising exchanges where the fixed-percentage revenue-share should be met across all auctions and not in each auction. In this paper, we characterize the optimal revenue sharing scheme that satisfies both constraints in expectation. Finally, we empirically evaluate our revenue sharing scheme on real data.

1 Introduction

The space of internet advertising can be divided in two large areas: *search ads* and *display ads*. While similar at first glance, they are different both in terms of business constraints in the market as well as algorithmic challenges. A notable difference is that in search ads the auctioneer and the seller are the same party, as the same platform owns the search page and operates the auction. Thus search ads are a one-sided market: the only agents outside the control of the auctioneer are buyers. In display ads, on the other hand, the platform operates the auction but, in most cases, it does not own the pages in which the ads are displayed, making the main problem the design of a two-sided market, referred to as *ad exchanges*.

The problem of designing an ad exchange can be decomposed in two parts: the first is to design an *auction*, which will specify how an ad impression will be allocated among different prospective buyers (advertisers) and how they will be charged from it. The second component is a *revenue sharing scheme*, which specifies how the revenue collected from buyers will be split between the seller

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(the publisher) and the platform. Traditionally the problems of designing an auction and designing a revenue sharing scheme have been merged in a single one called *double auction design*. This was the traditional approach taken by Myerson and Satterthwaite [1983], McAfee and McMillan [1987] and more recently in the algorithmic work of Gomes and Mirrokni [2014], Niazadeh et al. [2014]. The goals in those approaches have been to maximize efficiency in the market, maximize profit of the platform and to characterize when the profit maximizing policy is a simple one.

Those objectives however, do not entirely correspond to actual problem faced by advertising exchanges. Take platform-profit-maximization, for example. The ad-exchange business is a highly competitive environment. A web publisher (seller) can send their ad impressions to a dozen of different exchanges. If an exchange tries to extract all the surplus in the form of profit, web publishers will surely migrate to a less greedy platform. In order to retain their inventory, exchanges must align their incentives with the incentives of those of web publishers.

A good practical solution, which has been adopted by multiple real world platforms, is to declare a *fixed revenue sharing scheme*. The exchange promises it will keep at most an α-fraction of profits, where the constant α is typically the outcome of a business negotiation between the exchange and the web publisher. After the fraction is agreed, the objective of the seller and the exchange are aligned. The exchange maximizes profits by maximizing the seller's revenue.

If revenue sharing was the only constraint, the exchange could simply ignore sellers and run an optimal auction among buyers. In practice, however, web-publishers have outside options, typically in the form of reservation contracts, which should be taken into account by the exchange. Reservation contracts are a very traditional form of selling display ads that predates ad exchanges, where buyers and sellers make agreements offline specifying a volume of impressions to be transacted, a price per impression and a penalty for not satisfying the contract. Those agreements are entered in a system (for example Google's Doubleclick for Publishers) that manages reservations on behalf of the publisher. This reservation system determines for each arriving impression the best matching offline contract that impression could be allocated to as well as the cost of not allocating that impression. The cost of not allocating an impression takes into account the potential revenue from allocating to a contract and the probability of paying a penalty for not satisfying the contract.

From our perspective, it is irrelevant how a cost is computed by reservation systems. It is sufficient to assume that for each impression, the publisher has an opportunity cost and it is only willing to sell that particular impression in the exchange if its payout for that impression exceeds the cost. Exchanges therefore, allow the publisher to submit a cost and only sell that impression if they are able to pay the publisher at least the cost per that impression.

This allows us to design the following simple auction and revenue sharing scheme, which we call the naïve policy:

- seller sends to the exchange an ad impression with cost c.
- exchange runs a second price auction with reserve $r \geq c/(1-\alpha)$.
- if the item is sold the exchange keeps an α fraction of the revenue and sends the remaining $1-\alpha$ fraction to the seller.

This scheme is pretty simple and intuitive for each participant in the market. It guarantees that if the impression is sold, the revenue will be at least $c/(1-\alpha)$ and therefore the seller's payout will be at least c. So both the minimum payout and revenue sharing constraints are satisfied with probability 1. This scheme has also the advantage of decoupling the auction and the revenue sharing problem. The platform is free to use any auction among the buyers as long as it guarantees that whenever the impression is matched, the revenue extracted from buyers is at least $c/(1-\alpha)$.

Despite being simple, practical and allowing the exchange to experiment with the auction without worrying about revenue sharing, this mechanism is sub-optimal both in terms of platform profit and publisher payout. The exchange might be willing to accept a revenue share lower than α if this grants more freedom in optimizing the auction and extracting more revenue.

More generally, the exchange might exploit the repeated nature of the auction to improve revenue even further by adjusting the revenue share dynamically based on the bids and the cost. In this setting, we can think of the revenue share constraints to be enforced on average, i.e., over a sequence of auctions the platform is required to bound by α the ratio of the aggregate profit and the aggregate revenue collected from buyers. This allows the platform to increase the revenue share on certain queries and reduce in others.

In the repeated auctions setting, the exchange is also allowed to treat the minimum cost constraint on aggregate: the payout for the seller needs to be at least as large as the sum of costs of the impressions matched. The exchange can implement this in practice by always paying the seller at least his cost even if the revenue collected from buyers is less than the cost. This would cause the exchange to operate at a loss for some impressions. But this can be advantageous for the exchange on aggregate if it is able to offset these losses by leveraging other queries with larger profit margins.

In this paper, we attempt to characterize the optimal scheme for repeated auctions and measure on data the improvement with respect to the simple revenue sharing scheme discussed above.

Finally, while we discuss the main application of our results in the context of advertising exchanges, our model and results apply to the broad space of platforms that serve as intermediaries between buyers and sellers, and help run many repeated auctions over time. The issue of *dynamic* revenue sharing also arises when Amazon or eBay act as a platform and splits revenues from a sale with the sellers, or when ride-sharing services such as Uber or Lyft split the fare paid by the passenger between the driver and the platform. Uber for example mentions in their website[1] that: "Drivers using the partner app are charged an Uber Fee as a percentage of each trip fare. The Uber Fee varies by city and vehicle type and helps Uber cover costs such as technology, marketing and development of new features within the app."

1.1 Our Results and Techniques

We propose different designs of auctions and revenue sharing policies in exchanges and analyze them both theoretically and empirically on data from a major ad exchange. We compare against the naïve policy described above. We compare policies in terms of seller payout, exchange profit and match-rate (number of impressions sold). We note that match-rate is an important metric in practice, since it represents the volume of inventory transacted in the exchange and it is a proxy for the volume of the ad market this particular exchange is able to capture.

For the auction, we restrict our attention to second price auctions with reserve prices, since we aim at using theory as a guide to inform decisions about practical designs that can be implemented in real ad-exchanges. To be implementable in practice the designs need to follow the industry practice of running second-price auctions with reserves. This design will be automatically incentive compatible for buyers. On the seller side, instead of enforcing incentive compatibility, we will assume that impression costs are reported truthfully. Note that the revenue sharing contract guarantees, at least partially, when the constraint binds (which always happens in practice), the goals of the seller and the platform are partially aligned: maximizing profit is the same as maximizing revenue. Thus, sellers have little incentive to misreport their costs. In fact, this is one of the main reason that so many real-world platforms such as Uber adopt fixed revenue sharing contracts. In the ads market,

[1] See `https://www.uber.com/info/how-much-do-drivers-with-uber-make/`

moreover, sellers are also typically viewed as less strategic and reactive agents. Thus, we believe that the latter assumption is not too restrictive in practice.[2]

We will also assume Bayesian priors on buyer's valuations and on seller's costs. For the sake of simplicity, we will start with the assumption that seller costs are constant and then extend our results to the case where costs are sampled from a distribution.

We will focus on the exchange profit as our main objective function. While this paper will take the perspective of the exchange, the policies proposed will also improve seller's payout with respect to the naïve policy. The reason is simple: the naïve policy keeps exactly α fraction of the revenue extracted from buyers as profit. Any policy that keeps at most α and improves profit, should improve revenue extracted from buyers at least at the same rate and hence improve seller's payout.

Single Period Revenue Sharing. We first study the case where exchange is required to satisfy the revenue sharing constraint in each period, i.e., for each impression at most an α-fraction of the revenue can be retained as profit. We characterize the optimal policy. We first show that the optimal policy always sets the reserve price above the seller's cost, but not necessarily above $c/(1-\alpha)$. The exchange might voluntarily want to decrease its revenue share if this grants freedom to set lower reserve prices and extract more revenue from buyers.

When the opportunity cost of the seller is low, the optimal policy for the exchange ignores the seller's cost and prices according to the optimal reserve price. When the opportunity cost is high, pricing according to $c/(1-\alpha)$ is again not optimal because demand is inelastic at that price. The exchange internalizes the opportunity cost, prices between c and $c/(1-\alpha)$, and reduces its revenue share if necessary. For intermediate values of the opportunity cost, the exchange is better off employing the naïve policy and pricing according to $c/(1-\alpha)$.

Multi Period Revenue Sharing. We then study the case where the revenue share constraint is imposed over the aggregate buyers' payments. We provide intuition on the structure of the optimal policy by first solving a Lagrangian relaxation and then constructing an *asymptotically optimal* heuristic policy (satisfying the original constraints) based on the optimal relaxation solution. In particular, we introduce a Lagrange multiplier for the revenue sharing constraint to get the optimal solution to the Lagrangian relaxation. The optimal revenue sharing policy obtained from the Lagrangian relaxation pays the publisher a convex combination between his cost c and a fraction $(1-\alpha)$ of the revenue obtained from buyers. Depending on the value of the multiplier, the reserve price could be below c, exposing the platform to the possibility of operating at a loss in some auctions.

The policy obtained from the Lagrangian relaxation, while intuitive, only satisfies the revenue sharing and cost constraints in expectation. Because this is not feasible for the platform, we discuss heuristic policies that approximate that policy in the limit, but satisfy the constraints surely in aggregate over the T periods. Then we discuss an even stronger policy that satisfies the aggregate constraints for any prefix, i.e., at any given time t, the constraints are satisfied in aggregate from time 1 to t.

Comparative Statics. We compare the structure of the single period and multi period policies. The first insight is that the optimal multi-period policy uses lower reserve prices therefore matching more queries. The key insight we obtain from the comparison is that multi-period revenue sharing policies are particularly effective when markets are thick, i.e. when a second highest bid is above a rescaled version of the cost often and cost are not too high.

[2] While in this paper we focus on the dynamic optimization of revenue sharing schemes when agents report truthfully, it is still an interesting avenue of research to study the broader market design question of designing dynamic revenue sharing schemes while taking into account agents' incentives.

Empirical Insights. To complement our theoretical results, we conduct an empirical study simulating our revenue sharing policies on real world data from a major ad exchange. Our study confirms the effectiveness of the multi period revenue sharing policies and single period revenue sharing policies over the naïve policy. The results are consistent for different values of α: the profit lifts of single period revenue sharing policies are $+1.23\% \sim +1.64\%$ and the lifts of multi period revenue sharing policies are roughly 5.5 to 7 times larger ($+8.53\% \sim +9.55\%$).

It is important to mention that while we derive policies optimizing exchange profit, there are other business constraints and real world objectives that exchanges need to balance. Most notably, exchanges need to offer attractive terms of trade so as to be competitive with respect to other exchanges by, for example, making sure publishers obtain enough revenue. Exchanges are often concerned about match rate, which measures the volume of impressions transacted in the exchange. For that reason, we also measure how those metrics are affected by our policy and show that they improve.

Our final goal is to confirm the insights obtained from theory. We prove that under concavity conditions on the profit function, the performance of multi period policies over single period ones depend on the relation between second highest bids and costs. Our empirical study confirms the existence of a sweet spot for costs: we evaluate the single period and multi period policies after rescaling costs by different factors. When costs are too low or too high, we observe similar performance for the two policies. Interestingly, the unscaled costs are in the sweet spot where multi-period policies are particularly effective. These empirical observations are consistent with our prediction for different values of α.

1.2 Related Work

Double Auctions. Revenue sharing schemes have been studied in the context of optimal double auctions. Following the seminal work of Myerson [1981], Myerson and Satterthwaite [1983] presented the optimal double auction in two-sided settings. Recently, Deng et al. [2011] studied multidimensional variants of the Bayesian optimal double auctions and present polynomial-time approximation algorithms for the problem. For the non-Bayesian (prior-free) setting, Deshmukh et al. [2002] studied revenue-maximizing double auctions when the auctioneer has no prior knowledge about bids. More recently, Gomes and Mirrokni [2014] studied revenue-maximizing double auctions in the context of advertising exchanges, and generalized the results of Myerson and Satterthwaite [1983] by studying settings in which the platform's objective function is a convex combination of the seller's profit and the platform's profit, and provided a necessary and sufficient condition under which constant sharing schemes indirectly implement the optimal mechanism. Furthermore, Niazadeh et al. [2014] developed an approximately optimal mechanism for constant revenue-sharing double auctions. To the best of our knowledge, none of the above work consider revenue sharing schemes in repeated auctions.

Exchange Design. Our work is also related to the broad question of ad exchange design. We refer to Muthukrishnan [2009] for a survey. Mansour et al. [2012] provides an overview of the auction employed by Google's ad exchange. Feldman et al. [2010], Balseiro et al. [2016] study how should the exchange design auctions when advertisers do not acquire impressions directly from the exchange, but instead contract with intermediaries to acquire impressions on their behalf. These papers, however, take a one-sided approach to the display advertising market and do not take into account the presence of revenue sharing schemes for publishers.

E-commerce Applications. The results of this paper apply to various online and offline retailers and e-commerce websites like Amazon and Ebay. More specifically, Ebay applies similar revenue-sharing auctions to the ones studied in this paper when it serves as a broker between a set of buyers and a seller. Roughly speaking, Ebay takes a 9% cut on each sale, referred to as *final value fee*, and also fixed fee for listing an item, referred to as an *insertion fee*. They also apply a convex cost function for the fixed fee as the number of listings, and a maximum of $250 for the final value fee. A recent paper by Jain and Wilkens [2012] studies EBay's double auction problem, but their setting is different from this paper as they consider multiple sellers and one buyer, and explore approximately optimal pricing schemes for this setting.

2 Preliminaries

Setting. We study a discrete-time finite horizon setting in which items arrive sequentially to an intermediary. We index the sequence of items by $t = 1, \ldots, T$. There are multiple buyers bidding in the intermediary (the exchange) and the intermediary determines the winning bidder via a second price auction. We assume that the bids from the buyers are drawn independently and identically distributed across auctions, but potentially correlated across buyers for a given auction.

We will assume that the profit function of the joint distribution of bids is quasi-concave. The expected profit function corresponds to the expected revenue of a second price auction with reserve price r and opportunity cost c:

$$\Pi(r, c) = \mathbb{E}\left[\mathbf{1}\{b^{\mathsf{f}} \geq r\}\left(\max(r, b^{\mathsf{s}}) - c\right)\right].$$

where b^{f}_t and b^{s}_t are the highest- and second-highest bid at time t. Our assumption on the bid distribution will be as follows:

Assumption 2.1. *The expected profit function $\Pi(r, c)$ is quasi-concave in r for each c.*

The previous assumption is satisfied, for example, if bids are independent and identically distributed according to a distribution with increasing hazard rates (see, e.g., Balseiro et al. [2014]).

Mechanism. The seller submitting the items sets an opportunity cost of $c \geq 0$ for the items. The profit of the intermediary is given by the difference between the revenue collected from the buyers and the payments made to the seller. The intermediary has agreed to a *revenue sharing scheme* that limits the profit of the intermediary to at most $\alpha \in (0, 1)$ of the total revenue collected from the buyers.

The intermediary implements a non-anticipative adaptive policy π that maps the history at time t to a reserve price $r^{\pi}_t \in \mathbb{R}_+$ for the second price auction and a payment function $p^{\pi}_t : \mathbb{R}_+ \to \mathbb{R}_+$ that determines the amount to be paid to the seller as a function of the buyers' payments. That is, the item is sold whenever the highest bid is above the reserve price, or equivalently $b^{\mathsf{f}}_t \geq r^{\pi}_t$. The intermediary's revenue is equal to the buyers' payments of $\max(r^{\pi}_t, b^{\mathsf{s}}_t)$ and the seller's revenue is given by $p^{\pi}_t\left(\max(r^{\pi}_t, b^{\mathsf{s}}_t)\right)$. The intermediary's profit is given by the difference of the buyers' payments and the payments to the seller, i.e., $\max(r^{\pi}_t, b^{\mathsf{s}}_t) - p^{\pi}_t\left(\max(r^{\pi}_t, b^{\mathsf{s}}_t)\right)$. From the perspective of the buyers, the mechanism implemented by the intermediary is a second price auction with (potentially dynamic) reserve price r^{π}_t. The intermediary's problem amounts to maximizing profits subject to the revenue sharing constraint. The revenue sharing constraint can be imposed at every single period or over multiple periods. We discuss each model at a time.

Naïve revenue sharing scheme. The most straightforward revenue sharing scheme is the one that sets a reserve above $c/(1 - \alpha)$ and pay the sellers a $(1 - \alpha)$-fraction of the revenue:

$$r^{\pi}_t \geq \frac{c}{1 - \alpha}, \quad p^{\pi}_t(x) = (1 - \alpha)x. \tag{1}$$

Since the revenue sharing is fixed, the intermediary's profit is given by $\alpha \max(r_t^\pi, b_t^{\mathsf{s}})$. Thus, the intermediary optimizes profits by optimizing revenues, and the optimal reserve price is given by:

$$r^* = \arg \max_{r \geq c/(1-\alpha)} \Pi(r, 0)\,.$$

The naïve revenue sharing scheme sets a reserve above $c/(1-\alpha)$ and pays the seller $(1-\alpha)$ of the buyers' payments. This guarantees that the payment to the seller is always no less than c, by construction, because the payment of the buyers is at least the reserve price. Since the intermediary's profit is a fraction α of the buyers' payment, the seller's cost does not appear in the objective, and the objective of the seller is $\alpha\Pi(r, 0)$. Note, however, that the seller's cost does appear as a constraint in the intermediary's optimization problem: the reserve price should be at least $c/(1-\alpha)$.

This is the baseline that we will use to compare the proposed policies with in the experiment section. This policy is suboptimal for various reasons. Consider for example the extreme case where the buyers alway bid more than c and less than $c/(1-\alpha)$. In this case, the profit from the naïve revenue sharing scheme is zero. However, the intermediary can still obtain a non-zero profit by setting the reserve somewhere between c and $c/(1-\alpha)$, which results in a revenue share less than α. If the revenue sharing constraint is imposed over multiple periods instead of each single period, we are able to dynamically balance out the deficit and surplus of the revenue sharing constraint over time.

3 Single Period Revenue Sharing Scheme

In this case the revenue sharing scheme imposes that in every single period the profit of the intermediary is at most α of the buyers' payment. We start by formulating the profit maximization problem faced by the intermediary as a mathematical program with optimal value J^S.

$$J^S \triangleq \max_{\pi} \sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\{b_t^{\mathsf{f}} \geq r_t^\pi\}\left(\max(r_t^\pi, b_t^{\mathsf{s}}) - p_t^\pi\left(\max(r_t^\pi, b_t^{\mathsf{s}})\right)\right)\right] \tag{2a}$$

$$\text{s.t. } p_t^\pi(x) \geq (1-\alpha)x\,, \quad \forall x \tag{2b}$$

$$p_t^\pi(x) \geq c\,, \quad \forall x\,. \tag{2c}$$

The objective (2a) gives the profit of the intermediary as the difference between the payments collected from the buyers and the payments made to the seller. The revenue sharing constraint (2b) imposes that intermediary's profit is at most a fraction α of the total revenue, or equivalently $(x - p_t^\pi(x))/x \leq \alpha$ where x is the payment from the buyers. The floor constraint (2c) imposes that the seller is paid at least c. These constraints are imposed at every auction.

We next characterize the optimal decisions of the seller in the single period model. Some definitions are in order. Let $r^*(c)$ be an optimal reserve price in the second price auction if the seller's cost is c:

$$r^*(c) = \arg\max_{r\geq 0} \mathit{\Pi}(r, c).$$

To avoid trivialities we assume that the optimal reserve price is unique. Because the profit function $\mathit{\Pi}(r, c)$ has increasing differences in (r, c) then the optimal reserve price is non-decreasing with the cost, that is, $r^*(c) \geq r^*(c')$ for $c \geq c'$.

Our main result in this section characterizes the optimal decision of the intermediary in this model. All the proofs in this paper can be found in the appendix.

Theorem 3.1. *The optimal decision of the intermediary is to set* $p_t^\pi(x) = \max(c, (1-\alpha)x)$ *and* $r_t^\pi = \max\{\min\{\bar{c}, r^*(c)\}, r^*(0)\}$ *where* $\bar{c} = c/(1-\alpha)$.

The reserve price $\bar{c} = c/(1-\alpha)$ in the statement of the theorem is the naïve reserve price that satisfies the revenue sharing scheme by inflating the opportunity cost by $1/(1-\alpha)$. When the opportunity cost c is very low ($\bar{c} \le r^*(0)$), pricing according to $\bar{c}$ is not optimal because demand is elastic at $\bar{c}$ and the intermediary can improve profits by increasing the reserve price. Here the intermediary ignores the opportunity cost, prices optimally according to the optimal reserve price $r_t^\pi = r^*(0)$ and pays the seller according to $p_t^\pi(x) = (1-\alpha)x$, that is, the seller is paid a constant fraction $(1-\alpha)$ of the buyers' payments. When the opportunity cost c is very high ($\bar{c} \ge r^*(c)$), pricing according to $\bar{c}$ is again not optimal because demand is inelastic at $\bar{c}$ and the intermediary can improve profits by decreasing the reserve price. Here the intermediary internalizes the opportunity cost, prices optimally according to $r_t^\pi = r^*(c)$, the optimal reserve price with cost c, and pays the seller according to $p_t^\pi(x) = \max(c, (1-\alpha)x)$.

4 Multi Period Revenue Sharing Scheme

In this case the revenue sharing scheme imposes that the aggregate profit of the intermediary is at most α of the buyers' aggregate payment. Additionally, in this model the opportunity costs are satisfied on an aggregate fashion over all actions, that is, the payments to the seller need to be at least the floor price times the number of items sold. The intermediary decision's problem can be characterized by the following mathematical program with optimal value J^M.

$$J^M \triangleq \max_\pi \sum_{t=1}^T \mathbb{E}\left[\mathbf{1}\{b_t^{\mathsf{f}} \ge r_t^\pi\}\left(x_t^\pi - p_t^\pi\left(x_t^\pi\right)\right)\right] \tag{3a}$$

$$\text{s.t.} \sum_{t=1}^T \mathbf{1}\{b_t^{\mathsf{f}} \ge r_t^\pi\}\left(p_t^\pi\left(x_t^\pi\right) - (1-\alpha)x_t^\pi\right) \ge 0\,, \tag{3b}$$

$$\sum_{t=1}^T \mathbf{1}\{b_t^{\mathsf{f}} \ge r_t^\pi\}\left(p_t^\pi\left(x_t^\pi\right) - c\right) \ge 0\,, \tag{3c}$$

$$\text{where } x_t^\pi = \max(r_t^\pi, b_t^{\mathsf{s}})\,. \tag{3d}$$

The objective (3a) gives the profit of the intermediary as the difference between the payments collected from the buyers and the payments made to the seller. The revenue sharing constraint (3b) imposes that intermediary's profit is at most a fraction α of the total revenue. The floor constraint (3c) imposes that the seller is paid at least c. These constraints are imposed over the whole horizon.

The stochastic decision problem (3) can be solved via Dynamic Programming. To provide some intuition of the structure of the optimal solution we solve a Lagrangian relaxation of the problem where we introduce a dual variable $\lambda \ge 0$ for the floor constraint (3c) and a dual variable $\mu \ge 0$ for the revenue sharing constraint (3b). Lagrangian relaxations provide upper bounds on the optimal objective value and introduce heuristic policies of provably good performance in many settings (e.g., see Talluri and van Ryzin [1998]). Moreover, we shall see the optimal policy derived from the Lagrangian relaxation is optimal for problem (3) if constraints (3c) and (3b) are imposed in expectation instead of almost surely.

4.1 Lagrangian Relaxation

The dual function $\phi(\mu, \lambda)$ is given by supremum of the Lagrangian over the feasible set:

$$\begin{aligned}
\phi(\mu, \lambda) &\triangleq \sup_{\pi} \sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\{b_t^{\mathsf{f}} \geq r_t^{\pi}\}\left(x_t^{\pi} - p_t^{\pi}(x_t^{\pi}) + \lambda\big(p_t^{\pi}(x_t^{\pi}) - c\big) + \mu\big(p_t^{\pi}(x_t^{\pi}) - (1-\alpha)x_t^{\pi}\big)\right)\right] \\
&= \sup_{\pi} \sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\{b_t^{\mathsf{f}} \geq r_t^{\pi}\}\left(\big(1-\mu(1-\alpha)\big)x_t^{\pi} - \big(1-\lambda-\mu\big)p_t^{\pi}(x_t^{\pi}) - \lambda c\right)\right] \\
&= T \sup_{r} \mathbb{E}\left[\mathbf{1}\{b^{\mathsf{f}} \geq r\}\left(\big(1-\mu(1-\alpha)\big)\max(r, b_t^{\mathsf{s}}) - \lambda c\right)\right] + \mathcal{X}_{\{\lambda+\mu=1\}} \\
&= T\big(1-\mu(1-\alpha)\big) \sup_{r} \mathit{\Pi}\left(r, \frac{\lambda c}{1-\mu(1-\alpha)}\right) + \mathcal{X}_{\{\lambda+\mu=1\}}
\end{aligned}$$

where the third equality follows because bids are i.i.d. and thus the problem is separable, and from optimizing over the payment function p_t^{π} and noting that the objective is unbounded unless $\lambda + \mu = 1$ and denoting by $\mathcal{X}_A$ the characteristic function which is zero if A is true and ∞ otherwise.

Because the Lagrangian is unbounded unless $\lambda + \mu = 1$, then the optimal dual objective is given by

$$\inf_{\mu \geq 0, \lambda \geq 0 : \mu+\lambda=1} \phi(\mu, \lambda) = \inf_{0 \leq \mu \leq 1} \phi(\mu, 1-\mu) = \inf_{0 \leq \mu \leq 1} \hat{\phi}(\mu)$$

where

$$\begin{aligned}
\hat{\phi}(\mu) \triangleq \phi(\mu, 1-\mu) &= T\big(1-\mu(1-\alpha)\big) \sup_{r} \mathit{\Pi}\left(r, \frac{(1-\mu)c}{1-\mu(1-\alpha)}\right) \\
&= T\big(1-\mu(1-\alpha)\big) \mathit{\Pi}\left(r^*(c(\mu)), c(\mu)\right) . \qquad (4)
\end{aligned}$$

and $c(\mu) = \frac{1-\mu}{1-\mu(1-\alpha)} \cdot c \leq c$. Because $\hat{\phi}(\mu)$ is the pointwise maximum of linear functions, then $\hat{\phi}(\mu)$ is convex μ and thus the dual problem is convex. Because the feasible set is compact then the dual problem admits a solution. Furthermore, from weak duality we have that

$$J^M \leq \inf_{0 \leq \mu \leq 1} \hat{\phi}(\mu).$$

This suggests us to choose μ minimizing $\phi(\mu)$ and use it to infer the optimal policy from the dual relaxation. The next result proposes a stationary policy for the intermediary that satisfies the floor and revenue sharing constraints in expectation. The proof is deferred to the appendix.

Theorem 4.1. *Let $\mu^* \in \arg\min_{0 \leq \mu \leq 1} \hat{\phi}(\mu)$. The policy $p_t^{\pi}(x) = (1-\mu^*)c + \mu^*(1-\alpha)x$ and $r_t^{\pi} = r^*(c(\mu^*))$ is optimal for problem* (3) *when constraints* (3c) *and* (3b) *are imposed in expectation instead of almost surely.*

Remark 4.2. *Although the multi period policy we just proposed is not a solution to the original program* (3)*, we emphasize that it naturally induces heuristic policies (e.g., see Algorithm 1) that are asymptotically optimal solutions to the original multi period problem* (3) *without relaxation (see Theorem 6.1).*

4.2 Random Opportunity Costs

In this section, we extend our characterization result for the multi stage case (Theorem 4.1) to the case where the opportunity cost c for the seller is not fixed but drawn from some distribution for each item t. This setting describes the scenario where the sequentially arriving items are heterogeneous and randomly drawn from a population. In practice, impressions are sometimes heterogeneous and the publisher-declared opportunity costs can vary with the attributes of the impressions.

Denote by c_t the opportunity cost at time t. The optimization program of the intermediary is as in (3) with the exception that the floor constraint (3c) is now

$$\sum_{t=1}^{T} \mathbf{1}\{b_t^{\mathsf{f}} \geq r_t^{\pi}\} \left(p_t^{\pi}\left(x_t^{\pi}\right) - c_t\right) \geq 0 \,, \tag{5}$$

and in the objective the expectations are taken over the buyers' bids and the opportunity costs c_t.

Because the intermediary observes the opportunity cost declared by the publisher, it can adjust the reserve price for the auction depending on the opportunity cost. Thus, in the Lagrangian relaxation the intermediary can optimize the reserve price point-wise for each opportunity cost and we obtain the dual function:

$$\hat{\varphi}(\mu) = T\big(1 - \mu(1-\alpha)\big)\mathbb{E}_c\left[\Pi\left(r^*(c(\mu)), c(\mu)\right)\right] \,,$$

where $c(\mu) = \frac{1-\mu}{1-\mu(1-\alpha)} \cdot c$ as before.

Theorem 4.3. *Let $\mu^* \in \arg\min_{0 \leq \mu \leq 1} \hat{\varphi}(\mu)$. The policy $p_t^{\pi}(x) = (1-\mu^*)c_t + \mu^*(1-\alpha)x$ and $r_t^{\pi} = r^*(c_t(\mu^*))$ with $c_t(\mu) = \frac{1-\mu}{1-\mu(1-\alpha)} \cdot c_t$ is optimal for problem* (3) *when constraints* (5) *and* (3b) *are imposed in expectation instead of surely.*

5 Comparative Analysis

We first compare the optimal reserve price of the single period and multi period model (when opportunity costs are deterministic).

Proposition 5.1. *Let $r^S \triangleq \max\{\min\{\bar{c}, r^*(c)\}, r^*(0)\}$ be the optimal reserve price of the single period constrained model and $r^M \triangleq r^*(c(\mu^*))$ be the optimal reserve price of the multi period constrained model. Then $r^S \geq r^M$.*

The previous result shows that the reserve price of the single-period constrained model is larger or equal than the one of the multi-period constrained model. As a consequence, in the multi-period constrained model items are allocated more frequently and the social welfare is larger.

We next compare the intermediary's optimal profit under the single period and multi period model. This result quantifies the benefits of dynamic revenue sharing and provides insight into when dynamic revenue sharing is profitable for the intermediary.

Proposition 5.2. *Let $\mu^S \in [0,1]$ be such that $r^*(c(\mu^S)) = r^S$. Then*

$$J^S \leq J^M \leq J^S + (1-\mu^S)T\mathbb{E}\left[(1-\alpha)b^{\mathsf{s}} - c\right]^+ .$$

The previous result shows that the benefit of dynamic revenue sharing is driven, to a large extent, by the second-highest bid and the opportunity cost c. If the market is thin and the second-highest bid b^{s} is low, then the truncated expectation $E \triangleq \mathbb{E}\left[(1-\alpha)b^{\mathsf{s}} - c\right]^+$ is low and the benefit from dynamic revenue sharing is small, that is, $J^S \sim J^M$. If the market is thick and the second-highest bid b^{s} is high, then the benefit of dynamic revenue sharing depends on the opportunity cost c. If the floor price c is very low, then $r^S = r^*(0)$ and $\mu^S = 1$, implying that the coefficient in front of E is zero, and there is no benefit of dynamic revenue sharing $J^S = J^M$. If the floor price c is very high, then $r^S = r^*(c)$ and $\mu^S = 0$, implying that the coefficient in front of E is 1. However, in this case the truncated expectation E is small and again there is little benefit of dynamic revenue sharing, that is, $J^S \sim J^M$. *Thus the sweet spot for dynamic revenue sharing is when the second-highest bid is high and the opportunity cost is neither too high nor too low.*

6 Heuristic Revenue Sharing Schemes

So far we focused on the theory of revenue sharing schemes. We now switch our focus to applying insights derived from theory to the practical implementation of revenue sharing schemes. First we note that while the policies in the statement of Theorem 4.1 and Theorem 4.3 are only guaranteed to satisfy constraints in expectations, a feasible policy of the stochastic decision problems should satisfy the constraints in an almost sure sense.

We start then by providing two transformations that convert a given policy satisfying constraints in expectation to another policy satisfying the constraints in every sample path.

6.1 Multi-period Refund Policy

Our first transformation will keep track of how much each constraint is violated and will issue a refund to the seller in the last period (see Algorithm 1).

ALGORITHM 1: Heuristic Refund Policy from Lagrangian Relaxation

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Determine the optimal dual variable $\mu^* \in \arg\min_{0 \le \mu \le 1} \hat{\phi}(\mu)$
for $t = 1, \ldots, T$ **do**
  Set the reserve price $r^\pi_t = r^*(c(\mu^*))$
  if item is sold, that is, $b^{\mathsf{f}}_t \ge r^\pi_t$ **then**
    Collect the buyers' payment $x^\pi_t = \max(r^\pi_t, b^{\mathsf{s}}_t)$
    Pay the seller $p^\pi_t(x^\pi_t) = (1-\mu^*)c + \mu^*(1-\alpha)x^\pi_t$
  end if
end for
Let $D^F = \sum_{t=1}^T \mathbf{1}\{b^{\mathsf{f}}_t \ge r^\pi_t\}\left(p^\pi_t(x^\pi_t) - c\right)$ be the floor deficit.
Let $D^R = \sum_{t=1}^T \mathbf{1}\{b^{\mathsf{f}}_t \ge r^\pi_t\}\left(p^\pi_t(x^\pi_t) - (1-\alpha)x^\pi_t\right)$ be the revenue sharing deficit.
Pay the seller $-\min\{D^F, D^R, 0\}$

The following result analyzes the performance of the heuristic policy. We omit the proof as this is a standard result in the revenue management literature.

Theorem 6.1 (Theorem 1, Talluri and van Ryzin [1998])**.** *Let J^H be the expected performance of the heuristic policy. Then*

$$J^H \le J^M \le J^H + O(\sqrt{T}).$$

The previous result shows that the heuristic policy given by Algorithm 1 is asymptotically optimal for the multi-period constrained model, that is, it implies that $J^H/J^M \to 1$ as $T \to \infty$. When the number of auctions is large, by the Law of Large Numbers, stochastic quantities tend to concentrate around their means. So the floor and revenue sharing deficits incurred by violations of the respective constraints are small relative to the platform's profit and the policy becomes asymptotically optimal.

6.2 Prefix Revenue Sharing Policy

Now we look at how to transform a policy to satisfy even more stringent business constraints. A business constraint that arises in practice is that revenue sharing constraints can be satisfied in aggregate over all past auctions *at every point in time.* Formally, it means that for every prefix of the sequence, we should have:

$$\forall t \geq 1, \sum_{\tau=1}^{t} \mathbf{1}\{b^{\mathsf{f}}_{\tau} \geq r^{\pi}_{\tau}\} \left(p^{\pi}_{\tau}(x^{\pi}_{\tau}) - (1-\alpha)x^{\pi}_{\tau}\right) \geq 0. \tag{6}$$

Another important business constraint in practice is that we should pay the seller at least his cost c_t for each impression matched, i.e.:

$$\forall t \geq 1, \mathbf{1}\{b^{\mathsf{f}}_{t} \geq r^{\pi}_{t}\} \left(p^{\pi}_{t}(x^{\pi}_{t}) - c\right) \geq 0. \tag{7}$$

For any given revenue sharing scheme π, we can construct another $\hat{\pi}$ that satisfies the two constraints above and only differs with π on the payment rule.

The construction is based on the following ideas: The revenue share constraints (6) are imposed on each prefix of the sequence of the auctions, hence we need to increase the payment to the seller whenever we are about to violate the constraint by simply following the given revenue sharing scheme π. To do this, we use a "bank account" B to keep the track of the left-hand-side of (6), and make sure that for each period, $B_t = B_{t-1} + p^{\hat{\pi}}_t(x^{\pi}_t) - (1-\alpha)x^{\pi}_t \geq 0$.

The opportunity cost constraints (7) are imposed on each single period, hence we need to increase the payment to the seller if $p^{\pi}_t(x^{\pi}_t) < c_t$, which could be done by making $p^{\hat{\pi}}_t(x^{\pi}_t) \geq \max\{c_t, p^{\pi}_t(x^{\pi}_t)\}$.

Formally, the policy is given as follows:

ALGORITHM 2: Converting any revenue sharing scheme to the one that obeys constraints (6) and (7).

For any give revenue sharing scheme $\langle r^{\pi}_t, p^{\pi}_t \rangle$.
Let $B \leftarrow 0$ be the bank account of the seller.
for $t = 1, \ldots, T$ **do**
  Set the reserve price $r^{\hat{\pi}}_t = r^{\pi}_t$
  if item is sold, that is, $b^{\mathsf{f}}_t \geq r^{\hat{\pi}}_t$ **then**
    Collect the buyers' payment $x^{\hat{\pi}}_t = x^{\pi}_t$
    Pay the seller $p^{\hat{\pi}}_t(x^{\hat{\pi}}_t) = \max\left\{c_t, (1-\alpha)x^{\hat{\pi}}_t - B, p^{\pi}_t(x^{\hat{\pi}}_t)\right\}$
    Update the bank account $B \leftarrow B + p^{\hat{\pi}}_t(x^{\hat{\pi}}_t) - (1-\alpha)x^{\hat{\pi}}_t$
  end if
end for

Theorem 6.2. *For any revenue sharing scheme π, the corresponding revenue sharing scheme $\hat{\pi}$ defined by Algorithm 2 satisfies constraints* (6) *and* (7) *in every sample path.*

6.3 Hybrid Revenue Sharing Policy

In the next section we will evaluate the policies discussed on data from a major ad exchange. One conclusion will be that while the prefix policy satisfies more stringent business constraints, it had poor performance in terms of exchange profit compared, for example, with the refund policy. Our goal in this section is to combine the insights on optimal formats of reserve prices and revenue sharing policies from the theory in Sections 3 and 4 together with empirical observations from experiments. Thus motivated, we design a hybrid policy that has profit performance compared with the refund policy but satisfies the stringent constraints in the previous section.

ALGORITHM 3: Hybrid multi period prefix policy.

Let $B \leftarrow 0$ be the bank account of the seller.
Determine the optimal dual variable $\mu^* \in \arg\min_{0 \leq \mu \leq 1} \hat{\phi}(\mu)$
for $t = 1, \ldots, T$ **do**
  Set the reserve price $r_t^\pi = \max\{\min\{\bar{c}_t, r^*(c_t(\mu^*))\}, r^*(0)\}$
  if item is sold, that is, $b_t^{\mathsf{f}} \geq r_t^\pi$ **then**
    Collect the buyers' payment $x_t^\pi = \max(r_t^\pi, b_t^{\mathsf{s}})$
    Pay the seller $p_t^\pi(x_t^\pi) = \max\left\{c_t, (1-\alpha)x_t^\pi - B\right\}$
    Update the bank account $B \leftarrow B + p_t^\pi(x_t^\pi) - (1-\alpha)x_t^\pi$
  end if
end for

We call it a hybrid policy since the reserve price r_t^π is a hybrid of the reserve prices computed in Sections 3 and 4. The payment to sellers is the least payment required to satisfy the prefix revenue share constraint (6) and the per-period opportunity cost constraint (7).

7 Empirical Evaluation

In this section, we use anonymized real bid data from a major ad exchange to evaluate the policies we discussed in previous sections. Our goal will be to validate our insights on data. In the theoretical part of this paper we made simplifying assumptions, that not necessarily hold on data. For example, we assume quasi-concavity of the expected profit function $\Pi(r, c)$. Even though this function is not concave, we can still estimate it from data and optimize using linear search. Our theoretical results also assume we have access to distributions of buyers' bids. We build such distributions from past data. Finally, in our real data set bids are not necessarily stationary and identically distributed over time. Even though there might be inaccuracies from bids changing from one day to another, our revenue sharing policies are also robust to such non-stationarity.

7.1 Data Sets

The data set is a collection of auction records, where each record corresponds to a real time auction for an impression and consists of:

- a seller (publisher) id,
- the seller declared opportunity cost,
- a set of bid records. Each bid record corresponds to a buyer id and the value of the bid submitted by that buyers to the auction.

The maximum revenue share α that the intermediary could take is set to be a constant. To show that our results do not rely on the selection of this constant, we run the simulation for different values of α ($\alpha = 0.15, 0.2, 0.25$).

Our data set will consist of a random sample of auctions from 20 large publishers over the period of 2 days. We will partition the data set in a *training set* consisting of data for the first day and a *testing set* consisting of data for the second day.

7.2 Preprocessing Steps

Before running the simulation, we need to do some preprocessing of the data set. The goal of the preprocessing is to learn the parameters required by the policies we introduced for each seller, in particular, the optimal reserve function r^* and the optimal Lagrange multiplier μ^*. We will do this estimation using the training set, i.e., the data from the first day.

The first problem is to estimate $\Pi(r, c)$ and $r^*(c)$. In order to estimate $\Pi(r, c)$ for a given impression we look at all impressions in the training set with the same seller and obtain a list of $(b^{\mathsf{f}}, b^{\mathsf{s}})$ pairs. We build the empirical distribution where each of those pairs is picked with equal probability. This allows us to evaluate and optimize $\Pi(r, c)$ with a single pass over the data using the technique described in Paes Leme et al. [2016].

For each seller, to estimate μ^*, we enumerate different μ's from the discretization of $[0, 1]$ (denoted by D) and evaluate the profits of these policies on the training set. Then the estimation ($\hat{\mu}^*$) of μ^* is the μ that yields the maximum profit on the training set, i.e.,

$$\hat{\mu}^* \triangleq \arg\max_{\mu \in D} \hat{\mathsf{profit}}(\mu)$$

7.3 Evaluating Revenue Sharing Policies

We will evaluate the different policies discussed in the paper on testing set (day 2 of the data set) using the parameters $\hat{r}^*(c)$ and $\hat{\mu}^*$ learned from the training set during preprocessing. For each revenue sharing policy we evaluate, we will be concerned with the following metrics: *profit* of the exchange, *payout* to the sellers, *match rate* which corresponds the number of impressions allocated, *revenue* extracted from buyers and *buyers values* which is the sum of highest bids over allocated impressions (here we assume that buyers report their values truthfully in the second-price auction run by the exchange). In addition, the average *intermediary's revenue share* will be calculated.

The policies evaluated will be the following:

- `NAIVE`: naïve policy (Section 2),
- `SINGLE`: single period policy (Section 3),
- `REFUND`: multi period refund policy (Algorithm 1 in Section 6.1),
- `PREFIX`: multi period prefix policy (Algorithm 2 in Section 6.2),
- `HYBRID`: multi period hybrid policy (Algorithm 3 in Section 6.3).

In Table 1, we report the results of the policies for different values of α $(0.15, 0.2, 0.25)$. The metrics are reported with respect to the `NAIVE` policy. In other words, the cell in the table corresponding to revenue of policy `P` is the revenue lift of `P` with respect to `NAIVE`:

$$\text{revenue lift}(\texttt{P}) = \frac{\text{revenue}(\texttt{P})}{\text{revenue}(\texttt{NAIVE})} - 1$$

The only metric that is not reported as a percentage lift is the revenue share in the last column which corresponds to:

$$\text{rev share}(\texttt{P}) = \frac{\text{profit}(\texttt{P})}{\text{revenue}(\texttt{P})}$$

7.4 Interpreting Simulation Results

What conclusions can we draw from the lift numbers? The first conclusion is that even though the theoretical model deviates from practice in a number of different ways (concavity of $\Pi(r, c)$, precise distribution estimates, stationarity of bids), we are still able to improve over the naïve policy. Notice

(a) $\alpha = 0.15$

policy	profit	payout	match rate	revenue	buyers values	rev. share
NAIVE	0.00%	0.00%	0.00%	0.00%	0.00%	15.00%
SINGLE	+1.23%	+1.74%	+0.83%	+1.66%	+0.83%	14.94%
REFUND	+8.53%	+8.53%	+3.71%	+8.53%	+7.81%	15.00%
PREFIX	−3.60%	−1.97%	−23.96%	−2.22%	−6.40%	14.79%
HYBRID	+3.34%	+5.38%	+4.09%	+5.08%	+3.31%	14.75%

(b) $\alpha = 0.20$

policy	profit	payout	match rate	revenue	buyers values	rev. share
NAIVE	0.00%	0.00%	0.00%	0.00%	0.00%	20.00%
SINGLE	+1.29%	+2.33%	+0.86%	+2.12%	+1.11%	19.84%
REFUND	+9.37%	+9.37%	+8.30%	+9.37%	+9.09%	20.00%
PREFIX	−2.17%	+0.69%	−21.87%	+0.12%	−4.41%	19.54%
HYBRID	+3.81%	+5.93%	+5.26%	+5.51%	+3.78%	19.68%

(c) $\alpha = 0.25$

policy	profit	payout	match rate	revenue	buyers values	rev. share
NAIVE	0.00%	0.00%	0.00%	0.00%	0.00%	25.00%
SINGLE	+1.64%	+2.97%	+1.07%	+2.64%	+1.39%	24.76%
REFUND	+9.55%	+9.57%	+10.71%	+9.56%	+9.64%	25.00%
PREFIX	−1.00%	+2.16%	−18.51%	+1.37%	−2.90%	24.41%
HYBRID	+4.61%	+6.90%	+6.74%	+6.33%	+4.55%	24.60%

Table 1: Performance of the policies for different α's.

that the naïve policy implements the optimal reserve price subject to a fixed revenue sharing policy. So all the gains from reserve price optimization are already accounted for in our baseline.

We start by observing that even for `SINGLE`, which is a simple policy, we are able to considerably improve over `NAIVE` across all performance metrics. This highlights that the observation that "profit and revenue can be improved by reducing the share taken by the exchange" is not only a theoretical possibility, but a reality on real-world data.

Next we compare the lifts of `SINGLE`, which enforces revenue sharing constraints per impression, versus `REFUND`, which enforces constraints in aggregate. We can see that the lift is 5.5 to 7 times larger for `REFUND` compared to `SINGLE`. For $\alpha = 0.15$, the lift[3] for `SINGLE` is +1.23% while `REFUND` is +8.53%. This shows the importance of optimizing revenue shares across all auctions instead of per auction. Additionally, we observe that the match rate and buyers values of `REFUND` are higher than those of `SINGLE`. This is in agreement with Proposition 5.1: because the reserve price of the single-period constrained model is typically larger than the one of the multi-period constrained model, we expect `REFUND` to clear more auctions, which in turns leads to higher buyer values.

Next we analyze the performance of `PREFIX` and `HYBRID` policies. While `PREFIX` is able to raise payout and revenue in some cases, it fails to have a positive impact on profit in all experiments.

[3] The reader might ask how to interpret lift numbers. The annual revenue of display advertising exchanges is on the order of billions of dollars. At that scale, each 1% lift corresponds to tens of millions of dollars in incremental annual revenue. We emphasize that this lift is in addition to that obtained by reserve price optimization, since `NAIVE` already captures the gains from setting reserve prices optimally given a simple revenue sharing policy.

In PREFIX the exchange ends up sacrificing too much of its revenue share. At first glance, such result seems to be counterintuitive. However, it is not surprising because there is no theoretical guarantee on the profit of policy PREFIX at all. In particular, PREFIX is subject to tighter constraints than REFUND, and the reserve prices of policy SINGLE and policy NAIVE are not achievable by policy PREFIX with $\mu^* \in [0, 1]$ in general.

This is our motivation for policy HYBRID. We address the shortcomings of policy PREFIX by granting the intermediary more freedom in picking reserve prices. When $\mu^* = 0$, for example, $r^{\texttt{HYBRID}} = r^{\texttt{SINGLE}}$. As a result, we obtain a policy that is consistently better than SINGLE. Even though HYBRID is not as good REFUND in terms of revenue lift, it satisfied the more stringent constraints defined in Section 6.2, which are not necessarily satisfied by REFUND.

One other interesting observation is that the larger the revenue share α, the larger the improvement. So the higher revenue share the exchange can negotiate with sellers, the more important it is to invest in sophisticated revenue sharing policies.

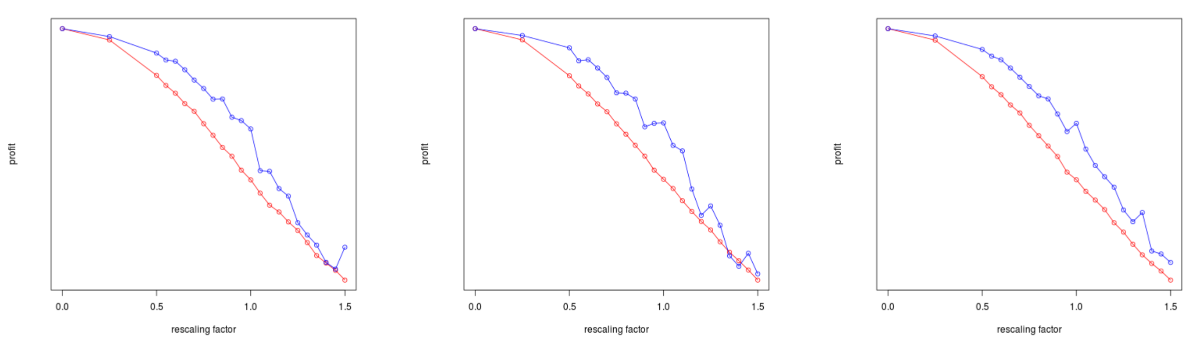
To sum up, the policies can be ranked as follows in terms of performance:

$$\texttt{REFUND} \succ \texttt{HYBRID} \succ \texttt{SINGLE} \succ \texttt{NAIVE} \sim \texttt{PREFIX}.$$

7.5 Effectiveness of Multi-period Policies

In Proposition 5.2 we provide a theoretical comparison of single-period and multi-period revenue sharing policies and concludes that there are two effects at play: the first is the effect of the Lagrange multiplier $1 - \mu^S$ which increases as the cost grows and the second is the expected expected lift provided by second bids over a rescaled version of the cost $\mathbb{E}[b^{\mathsf{s}} - \bar{c}]^+$ for $\bar{c} = c/(1-\alpha)$. This effect decreases with c. For very low values of c there is not a significant difference between policies due to μ^S being close to 1. For large values of c, there is again no significant difference since the second bid is rarely above the cost. Our theorems indicate that there is a sweet spot for costs values which makes multi-period policies particularly effective with respect to single-period policies.

To verify this hypothesis experimentally on data we perform the following experiment: we choose a rescaling factor between 0 and 1.5 and evaluate the profit obtained by both SINGLE and REFUND when all the costs are rescaled by that factor. We obtain the result in Figure 1. First we



(a) $\alpha = 0.15$ (b) $\alpha = 0.20$ (c) $\alpha = 0.25$

Figure 1: We compare profit(SINGLE) in red and profit(REFUND) in blue for different cost rescaling parameters. The absolute values in the y-axis are removed for privacy reasons.

observe that the total revenue is decreasing in the cost scaling, since the larger the cost the more constrained the optimizaton problem is. But more interestingly, observe that for very small costs (cost scaling close to zero) there is little difference between SINGLE and REFUND . The gap grows as the costs increase but as costs become large, the gap again closes and the two policies again produce similar revenue. Interestingly, the actual unscaled costs (rescaling factor equal to 1) are in the sweet spot where REFUND is particularly more effective than SINGLE.

We next provide some intuition for these results. When the opportunity cost is very low, the revenue constraint binds ($\mu = 1$). Both policies ignore the seller's opportunity cost, price according to the Myerson optimal reserve $r^*(0)$, and pay the seller $(1 - \alpha)$ of the buyers' payments. When the opportunity cost is very high, the floor constraint binds ($\mu = 0$). Both policies internalize the seller's opportunity cost, price according to $r^*(c)$ and pay the seller his opportunity cost. Thus, both policies coincide when the opportunity cost is too low or too high. In the intermediate regime, both constraints are binding. Here, REFUND can take advantage of the repeated nature of the auctions to accept a revenue share lower than α or operate at a loss for some impressions. This grants REFUND more freedom in optimizing the auction and extracting more revenue.

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A Proof of Results

A.1 Proof of Theorem 3.1

Proof. Constraints (2b) and (2c) readily imply that $p_t^\pi(x) \geq \max(c, (1-\alpha)x)$. Because the exchange is maximizing profits he would like to set the payment to the seller as small as possible, which implies that $p_t^\pi(x) = \max(c, (1-\alpha)x)$. Furthermore, because bids are stationary we can simplify the problem to

$$J^S = T \max_r \underbrace{\mathbb{E}\left[\mathbf{1}\{b^{\mathsf{f}} \geq r\}\left(\max(r, b^{\mathsf{s}}) - \max\left(c, (1-\alpha)\max(r, b^{\mathsf{s}})\right)\right)\right]}_{\triangleq \Pi^S(r)}, \tag{8}$$

where we denote by $\Pi^S(r)$ the objective in (8).

Let $\bar{c} = c/(1-\alpha)$. We next show that the objective $\Pi^S(r)$ can be written as

$$\Pi^S(r) = \begin{cases} \bar{\Pi}^S(r) \triangleq \alpha \Pi(r, 0) & \text{if } r \geq \bar{c}\,, \\ \underline{\Pi}^S(r) \triangleq \Pi(r, c) - \mathbb{E}\left[(1-\alpha)b^{\mathsf{s}} - c\right]^+ & \text{if } r < \bar{c}\,. \end{cases} \tag{9}$$

When $r \geq \bar{c}$ the payment to the seller when the item is sold is $(1-\alpha)\max(r, b^{\mathsf{s}})$, which implies that $\Pi^S(r) = \alpha \mathbb{E}\left[\mathbf{1}\{b^{\mathsf{f}} \geq r\}\max(r, b^{\mathsf{s}})\right] = \alpha \Pi(r, 0)$. When $r \leq \bar{c}$ we can write the objective as follows by adding and subtracting the expected cost $c\mathbb{P}\left\{b^{\mathsf{f}} \geq r\right\}$:

$$\begin{aligned} \Pi^S(r) &= \Pi(r, c) - \mathbb{E}\left[\mathbf{1}\{b^{\mathsf{f}} \geq r\}\left(\max\left(c, (1-\alpha)\max(r, b^{\mathsf{s}})\right) - c\right)\right] \\ &= \Pi(r, c) - \mathbb{E}\left[\mathbf{1}\{b^{\mathsf{f}} \geq r\}\left((1-\alpha)\max(r, b^{\mathsf{s}}) - c\right)^+\right] \\ &= \Pi(r, c) - \mathbb{E}\left[\mathbf{1}\{b^{\mathsf{f}} \geq r, b^{\mathsf{s}} \geq \bar{c}\}\left((1-\alpha)\max(r, b^{\mathsf{s}}) - c\right)^+\right] \\ &= \Pi(r, c) - \mathbb{E}\left[\left((1-\alpha)b^{\mathsf{s}} - c\right)^+\right] \end{aligned}$$

where the third equality follows because the second term is non-zero when $\max(r, b^{\mathsf{s}}) \geq \bar{c}$ which is equivalent to $b^{\mathsf{s}} \geq \bar{c}$ because $r \leq \bar{c}$, and the last equation follows because $b^{\mathsf{s}} \geq \bar{c}$ implies that (i) $b^{\mathsf{s}} \geq r$ because $r \leq \bar{c}$ and (ii) $b^{\mathsf{f}} \geq r$ since $b^{\mathsf{f}} \geq b^{\mathsf{s}}$.

Note that $r^*(0)$ is the maximizer of $\bar{\Pi}^S(r)$ and $r^*(c)$ is the maximizer of $\underline{\Pi}^S(r)$. We prove the result by considering three cases, that is, whether $\bar{c}$ falls above, within or below the interval $[r^*(0), r^*(c)]$.

Case 1 ($\bar{c} \geq r^*(c)$). In this case the optimal reserve price will be shown to be $r^*(c)$. For $r < \bar{c}$ we have that $\Pi^S(r) = \underline{\Pi}^S(r)$ for which the maximizer is $r^*(c)$. This solution is feasible because $r^*(c) \leq \bar{c}$. We need to show that the profit of all reserves $r \geq \bar{c}$ are dominated by that of $r^*(c)$. For any $r \geq \bar{c}$ we have that

$$\Pi^S(r) = \bar{\Pi}^S(r) \leq \bar{\Pi}^S(\bar{c}) = \underline{\Pi}^S(\bar{c}) \leq \underline{\Pi}^S(r^*(c)) = \Pi^S(r^*(c))\,, \tag{10}$$

where the first equality follows because $r \geq \bar{c}$, the first inequality because $\bar{\Pi}^S(r) = \alpha \Pi(r, 0)$ is quasi-concave in r and thus $\bar{\Pi}^S(r)$ is non-increasing when $r \geq r^*(0)$ (to the right of the maximizer) together with the fact that $r \geq \bar{c} \geq r^*(0)$, the second equality because of continuity of the objective function at $\bar{c}$, the second inequality because $r^*(c)$ is the maximizer of $\underline{\Pi}^S(r)$, and the last equality because $r^*(c) \leq \bar{c}$. Therefore $r^*(c)$ is the optimal reserve.

Case 2 ($r^*(0) \le \bar{c} \le r^*(c)$). In this case the optimal reserve price will be shown to be $\bar{c}$. We first show that for all $r \ge \bar{c}$ the profit is dominated by that of $\bar{c}$. For any $r \ge \bar{c}$ we have that

$$\Pi^S(r) = \bar{\Pi}^S(r) \le \bar{\Pi}^S(\bar{c}) = \Pi^S(\bar{c}),$$

where the first equality follows because $r \ge \bar{c}$ and the first inequality because $\bar{\Pi}^S(r) = \alpha\Pi(r,0)$ is quasi-concave in r and thus $\bar{\Pi}^S(r)$ is non-increasing when $r \ge r^*(0)$ (to the right of the maximizer) together with the fact that $r \ge \bar{c} \ge r^*(0)$. We next show that for all $r \le \bar{c}$ the profit is dominated by that of $\bar{c}$. For any $r \le \bar{c}$ we have that

$$\Pi^S(r) = \underline{\Pi}^S(r) \le \underline{\Pi}^S(\bar{c}) = \Pi^S(\bar{c}),$$

where the first equality follows because $r \le \bar{c}$ and the first inequality because $\underline{\Pi}^S(r) = \Pi(r,c) - \mathbb{E}\left[(1-\alpha)b^{\mathsf{s}} - c\right]^+$ is quasi-concave in r and thus $\underline{\Pi}^S(r)$ is non-decreasing when $r \le r^*(c)$ (to the left of the maximizer) together with the fact that $r \le \bar{c} \le r^*(c)$.

Case 3 ($\bar{c} \le r^*(0)$). In this case the optimal reserve price will be shown to be $r^*(0)$. This case follows similarly to case 1 and the proof is omitted. □

A.2 Proof of Theorem 4.1

Proof. Consider the following relaxed version of problem (3) when constraints (3c) and (3b) are imposed in expectation instead of surely.

$$\begin{aligned}
\bar{J}^M \triangleq \max_{\pi} \; & \sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\{b_t^{\mathsf{f}} \ge r_t^{\pi}\}\left(x_t^{\pi} - p_t^{\pi}(x_t^{\pi})\right)\right] && \text{(11a)}\\
\text{s.t.} \; & \sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\{b_t^{\mathsf{f}} \ge r_t^{\pi}\}\left(p_t^{\pi}(x_t^{\pi}) - (1-\alpha)x_t^{\pi}\right)\right] \ge 0\,, && \text{(11b)}\\
& \sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\{b_t^{\mathsf{f}} \ge r_t^{\pi}\}\left(p_t^{\pi}(x_t^{\pi}) - c\right)\right] \ge 0\,, && \text{(11c)}\\
& \text{where } x_t^{\pi} = \max(r_t^{\pi}, b_t^{\mathsf{s}})\,. && \text{(11d)}
\end{aligned}$$

Notice that the Lagrange relaxation of problem (11) is equivalent to that of problem (3), which implies that $\bar{J}^M \le \inf_{0 \le \mu \le 1} \hat{\phi}(\mu)$. We prove the result by showing that (i) the proposed policy attains the dual objective and (ii) the proposed policy is primal feasible in (11).

Step 1 (primal objective). Let J^{π} be the expected performance of policy π. The expected performance of the current policy is

$$\begin{aligned}
J^{\pi} &= \sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\{b_t^{\mathsf{f}} \ge r_t^{\pi}\}\left(x_t^{\pi} - p_t^{\pi}(x_t^{\pi})\right)\right]\\
&= \sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\{b_t^{\mathsf{f}} \ge r_t^{\pi}\}\left((1-\mu^*(1-\alpha))x_t^{\pi} - (1-\mu^*)c\right)\right]\\
&= T\mathbb{E}\left[\mathbf{1}\{b^{\mathsf{f}} \ge r^*(c(\mu^*))\}\left((1-\mu^*(1-\alpha))\max(r^*(c(\mu^*)), b^{\mathsf{s}}) - (1-\mu^*)c\right)\right]\\
&= T(1-\mu^*(1-\alpha))\Pi\left(r^*(c(\mu^*)), c(\mu^*)\right) = \hat{\phi}(\mu^*)\,,
\end{aligned}$$

where the third equation follows because the policy is stationary and bids are i.i.d.

Step 1 (primal feasibility). Let $\hat{\phi}'(\mu)$ be the derivative of the dual objective, which is given by

$$\hat{\phi}'(\mu) = -T\mathbb{E}\left[\mathbf{1}\{b^{\mathsf{f}} \geq r^*(c(\mu))\}\left((1-\alpha)\max(r^*(c(\mu)), b^{\mathsf{s}}) - c\right)\right]. \tag{12}$$

The first-order conditions of μ^* for the dual problem imply that

1. if $\mu^* = 0$ then $\hat{\phi}'(\mu^*) \geq 0$
2. if $\mu^* \in (0,1)$ then $\hat{\phi}'(\mu^*) = 0$
3. if $\mu^* = 1$ then $\hat{\phi}'(\mu^*) \leq 0$

Let $\text{LHS}(11c)^\pi$ be the expectation on the left hand side of the floor constraint (11c) under policy π. We have that

$$\begin{aligned}
\text{LHS}(11c)^\pi &= \sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\{b_t^{\mathsf{f}} \geq r_t^\pi\}\left(p_t^\pi\left(x_t^\pi\right) - c\right)\right] \\
&= \mu^* T\mathbb{E}\left[\mathbf{1}\{b^{\mathsf{f}} \geq r^*(c(\mu^*))\}\left((1-\alpha)\max(r^*(c(\mu^*)), b_t^{\mathsf{s}}) - c\right)\right] \\
&= -\mu^*\hat{\phi}'(\mu^*) \geq 0
\end{aligned}$$

where the second equality follows because $p_t^\pi(x) = (1-\mu^*)c + \mu^*(1-\alpha)x$, third equality follows from the formula for the derivative of the dual objective in (12) and the last inequality from the first-order conditions of μ^* for the dual problem. Let $\text{LHS}(11b)^\pi$ be the expectation on the left hand side of the floor constraint (11b) under policy π. We have that

$$\begin{aligned}
\text{LHS}(11b)^\pi &= \sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\{b_t^{\mathsf{f}} \geq r_t^\pi\}\left(p_t^\pi\left(x_t^\pi\right) - (1-\alpha)x_t^\pi\right)\right] \\
&= -(1-\mu^*)T\mathbb{E}\left[\mathbf{1}\{b^{\mathsf{f}} \geq r^*(c(\mu^*))\}\left((1-\alpha)\max(r^*(c(\mu^*)), b_t^{\mathsf{s}}) - c\right)\right] \\
&= (1-\mu^*)\hat{\phi}'(\mu^*) \geq 0
\end{aligned}$$

where the second equality follows because $p_t^\pi(x) = (1-\mu^*)c + \mu^*(1-\alpha)x$, the third equality follows from the formula for the derivative of the dual objective and the last inequality from the first-order conditions of μ^* for the dual problem. □

A.3 Proof of Theorem 4.3

Proof. We will again apply the Lagrangian relaxation technique and derive from it an optimal policy for the problem where the constraints (5) and (3b) are imposed in expectation instead of almost surely. We rewrite the dual function $\phi(\mu,\lambda)$ for the random opportunity cost case as follows,

$$\begin{aligned}
\varphi(\mu,\lambda) &\triangleq \sup_{\pi} \sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\{b_t^{\mathsf{f}} \geq r_t^\pi\}\left(x_t^\pi - p_t^\pi\left(x_t^\pi\right) + \lambda\big(p_t^\pi\left(x_t^\pi\right) - c_t\big) + \mu\big(p_t^\pi\left(x_t^\pi\right) - (1-\alpha)x_t^\pi\big)\right)\right] \\
&= T\big(1-\mu(1-\alpha)\big)\sup_{r(c)} \mathbb{E}_c\left[\mathit{\Pi}\left(r(c), c(\mu)\right)\right] + \mathcal{X}_{\{\lambda+\mu=1\}}.
\end{aligned}$$

Again, to prevent the last term $\mathcal{X}_{\{\lambda+\mu=1\}}$ being unbounded, $\lambda+\mu=1$ and then the optimal dual objective is given by

$$\inf_{\mu\geq 0, \lambda\geq 0: \mu+\lambda=1} \varphi(\mu,\lambda) = \inf_{0\leq\mu\leq 1} \varphi(\mu, 1-\mu) = \inf_{0\leq\mu\leq 1} \hat{\varphi}(\mu),$$

where

$$\hat{\varphi}(\mu) \triangleq \phi(\mu, 1-\mu) = T\big(1-\mu(1-\alpha)\big) \sup_{r(c)} \mathbb{E}_c \left[\Pi\left(r, c(\mu)\right)\right] .$$

Because the reserve price can be adjusted depending on the cost (i.e., the reserve price is measurable w.r.t. the publisher's opportunity cost), we can interchange the order of the supreme sup and expectation $\mathbb{E}$ to obtain that

$$\hat{\varphi}(\mu) = T\big(1-\mu(1-\alpha)\big)\mathbb{E}_c \left[\sup_r \Pi\left(r, c(\mu)\right)\right] = T\big(1-\mu(1-\alpha)\big)\mathbb{E}_c \left[\Pi\left(r^*(c(\mu)), c(\mu)\right)\right] .$$

We omit the rest of proof as it follows the same steps as in the proof of Theorem 4.1 except that all the expectations are now taken over c as well. □

A.4 Proof of Proposition 5.1

Proof. We prove the result by considering three cases, that is, whether $\bar{c}$ falls above, within or below the interval $[r^*(0), r^*(c)]$.

Case 1 ($\bar{c} \geq r^*(c)$). In this case $r^S = r^*(c)$. The result follows because

$$r^M = r^*(c(\mu^*)) \leq r^*(c(0)) = r^*(c) = r^S ,$$

where the inequality follows because $\mu^* \geq 0$, $r^*(\cdot)$ is non-decreasing and $c(\cdot)$ is non-increasing.

Case 2 ($r^*(0) \leq \bar{c} \leq r^*(c)$). In this case $r^S = \bar{c}$. First note that $c(\mu) = \frac{(1-\mu)c}{1-\mu(1-\alpha)}$ is non-increasing in μ, $c(0) = c$ and $c(1) = 0$. Thus there exist $\mu^S \in [0, 1]$ such that $r^*(c(\mu^S)) = r^S$. We claim that $\hat{\phi}'(\mu^S) \leq 0$, which implies that $\mu^* \geq \mu^S$ and as a result $r^M = r^*(c(\mu^*)) \leq r^*(c(\mu^S)) = r^S$ because $r^*(\cdot)$ is non-decreasing and $c(\cdot)$ is non-increasing.

We prove the claim that $\hat{\phi}'(\mu^S) \leq 0$. Because $r^*(c(\mu^S)) = r^S$ and using the formula for $\hat{\phi}'(\cdot)$ in (12) we have that

$$\begin{aligned}
\hat{\phi}'(\mu^S) &= -T\mathbb{E}\left[\mathbf{1}\{b^{\mathsf{f}} \geq r^S\}\left((1-\alpha)\max(r^S, b^{\mathsf{s}}) - c\right)\right] \\
&= -T\mathbb{E}\left[\mathbf{1}\{b^{\mathsf{f}} \geq r^S\}\left(\max(r^S, b^{\mathsf{s}}) - c\right)\right] + \alpha T\mathbb{E}\left[\mathbf{1}\{b^{\mathsf{f}} \geq r^S\}\max(r^S, b^{\mathsf{s}})\right] \\
&= \alpha T\Pi(r^S, 0) - T\Pi(r^S, c) = T\bar{\Pi}^S(r^S) - T\underline{\Pi}^S(r^S) - T\mathbb{E}\left[(1-\alpha)b^{\mathsf{s}} - c\right]^+ , \qquad (13)
\end{aligned}$$

where the last equation follows from the fourth equation from (9). Therefore because $r^S = \bar{c}$

$$\hat{\phi}'(\mu^S) = T\bar{\Pi}^S(\bar{c}) - T\underline{\Pi}^S(\bar{c}) - T\mathbb{E}\left[(1-\alpha)b^{\mathsf{s}} - c\right]^+ = -T\mathbb{E}\left[(1-\alpha)b^{\mathsf{s}} - c\right]^+ \leq 0$$

where the last equation follows because $\bar{\Pi}^S(\bar{c}) = \underline{\Pi}^S(\bar{c})$ since the objective of the single period constrained model is continuous at $\bar{c}$.

Case 3 ($\bar{c} \leq r^*(0)$). In this case $r^S = r^*(0)$. From case 3 of the proof of Proposition 5.2 we have that $J^M = J^S$ which implies that r^S is an optimal solution for the multi-period constrained model and $r^S = r^M$. □

A.5 Proof of Proposition 5.2

Proof. The first inequality $J^S \le J^M$ is trivial because every policy of the single period constrained problem is feasible in the multi period constrained problem. For the second bound, we prove the second result by comparing the optimal objective value of the single period constrained problem to the objective value of the Lagrange relaxation.

First note that $c(\mu) = \frac{(1-\mu)c}{1-\mu(1-\alpha)}$ is non-increasing in μ, $c(0) = c$ and $c(1) = 0$. Thus there exist $\mu^S \in [0,1]$ such that $r^*(c(\mu^S)) = r^S$. As a result:

$$\begin{aligned}
J^M &\le \inf_{0\le\mu\le 1} \hat{\phi}(\mu) \le \hat{\phi}(\mu^S) = T\big(1-\mu^S(1-\alpha)\big)\Pi\left(r^*(c(\mu^S)), c(\mu^S)\right) \\
&= T\mathbb{E}\left[\mathbf{1}\{b^{\mathsf{f}} \ge r^S\}\left(\big(1-\mu^S(1-\alpha)\big)\max(r^S, b_t^{\mathsf{s}}) - (1-\mu^S)c\right)\right] \\
&= \mu^S \alpha T \Pi(r^S, 0) + (1-\mu^S) T\Pi(r^S, c) \\
&= \mu^S T\bar{\Pi}^S(r^S) + (1-\mu^S)T\underline{\Pi}^S(r^S) + (1-\mu^S)T\mathbb{E}\left[(1-\alpha)b^{\mathsf{s}} - c\right]^+ ,
\end{aligned} \tag{14}$$

where the first inequality follows from weak duality; the second inequality because $\mu^S \in [0,1]$ is dual feasible; the first equality follows from (4); the second equality follows because $r^*(c(\mu^S)) = r^S$ together with $c(\mu) = \frac{(1-\mu)c}{1-\mu(1-\alpha)}$; and the fourth equation from (9). We conclude the proof by considering three cases, that is, whether $\bar{c}$ falls above, within or below the interval $[r^*(0), r^*(c)]$.

Case 1 ($\bar{c} \ge r^*(c)$). In this case $r^S = r^*(c)$, which implies that $\mu^S = 0$ because $c(0) = c$ where $c(\mu) = \frac{(1-\mu)c}{1-\mu(1-\alpha)}$. Here (14) gives

$$J^M \le T\underline{\Pi}^S(r^*(c)) + T\mathbb{E}\left[(1-\alpha)b^{\mathsf{s}} - c\right]^+ = J^S + \mathbb{E}\left[(1-\alpha)b^{\mathsf{s}} - c\right]^+ ,$$

where the last equation follows because $T\underline{\Pi}^S(r^*(c)) = J^S$ since the optimal reserve price in the single period constrained model is $r^*(c)$ and $r^*(c) \le \bar{c}$.

Case 2 ($r^*(0) \le \bar{c} \le r^*(c)$). In this case $r^S = \bar{c}$ and $\mu^S \in [0,1]$. Here (14) gives

$$\begin{aligned}
J^M &\le \mu^S T\bar{\Pi}^S(\bar{c}) + (1-\mu^S)T\underline{\Pi}^S(\bar{c}) + (1-\mu^S)T\mathbb{E}\left[(1-\alpha)b^{\mathsf{s}} - c\right]^+ \\
&= J^S + (1-\mu^S)T\mathbb{E}\left[(1-\alpha)b^{\mathsf{s}} - c\right]^+ ,
\end{aligned}$$

where the last equation follows because $T\bar{\Pi}^S(\bar{c}) = T\underline{\Pi}^S(\bar{c}) = J^S$ since the objective of the single period constrained model is continuous at $\bar{c}$ together with the fact the optimal reserve price in the single period constrained model is $\bar{c}$.

Case 3 ($\bar{c} \le r^*(0)$). In this case $r^S = r^*(0)$, which implies that $\mu^S = 1$ because $c(1) = 0$ where $c(\mu) = \frac{(1-\mu)c}{1-\mu(1-\alpha)}$. Here (14) gives

$$J^M \le \alpha T\Pi\left(r^*(0), 0\right) = \bar{\Pi}^S(r^*(0)) = J^S ,$$

where the last equation follows because the optimal reserve price in the single period constrained model is $r^*(0)$ and $r^*(0) \ge \bar{c}$. □