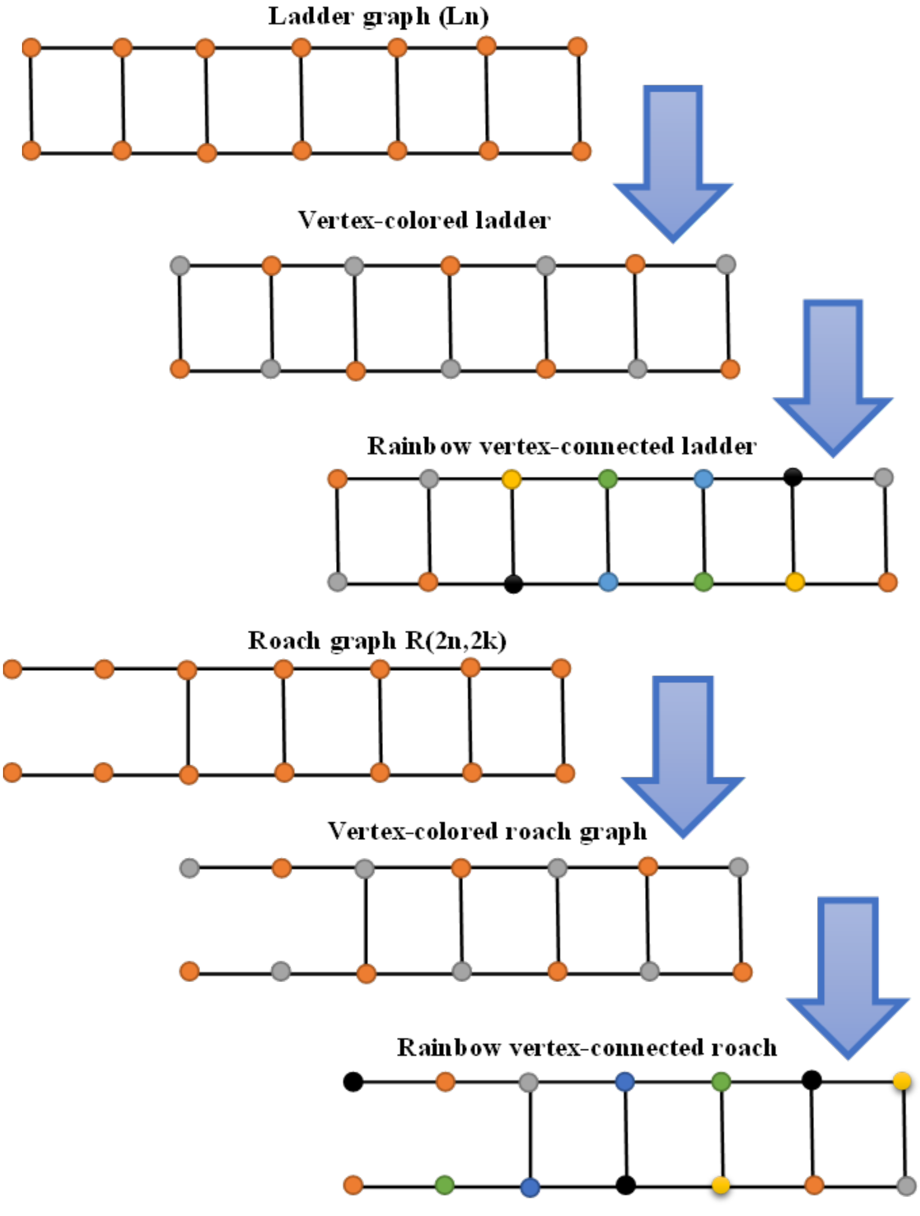
RESEARCH ARTICLE

The rainbow vertex connection number of ladder graphs and Roach graphs

W. D. D. P. Dewananda and K. K. K. R. Perera



Highlights

- Rainbow vertex connection number (rvc) of a ladder graph L_n is given by $rvc(L_n) = n - 1$, where $2n$ is the order of the graph.
- Rainbow vertex connection number of Roach graph $R(2n, 2k)$ is given by $rvc(R(2n, 2k)) = k$, when $n = 1$ and $rvc(R(2n,2k)) = \begin{cases} 2n & 2 \le k \le n+1 \\ k + (n-1) & k \ge 2+n \end{cases}$, when $n \geq 2$

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The rainbow vertex connection number of ladder graphs and Roach graphs

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Abstract: A vertex-coloured graph G is said to be rainbow vertex-connected, if every two vertices of G are connected by a path whose internal vertices have distinct colours. The rainbow vertex-connection number of a connected graph G, denoted by *rvc(G)*, is the smallest number of colours that are needed to make G, a rainbow vertex-connected. This study focuses on deriving formulas for the rainbow vertex connectivity number of a simple ladder graph and a roach graph.

***Keywords*:** Ladder Graphs; Rainbow Vertex-Connection Number; Roach Graphs

INTRODUCTION

In this paper, we assume that all graphs are simple, finite, and connected. Rainbow connectivity is viewed by some authors as a measure of the connectivity of a graph. It is also studied as a vertex connectivity as well as edge connectivity. Rainbow connection number ($rc(G)$) of a connected graph G is the minimum number of colours needed to colour its edges so that every pair of vertices is connected by at least one path in which no two edges have identical colours. A path P in a graph G with vertices k - coloring is said to be rainbow vertex-path if all internal vertices of P have distinct colors. The graph G is said to be rainbow vertex- connected, if for any two vertices u, v in G, there is a rainbow vertex-path. The rainbow vertex-connection number $rvc(G)$ of a graph G is the lowest positive integer k such that graph G is rainbow vertex connected. Finding $rvc(G)$ of a graph is benefited for different applications related to networks. When sending messages in a cellular network, each link between two vertices is assigned a separate channel and the rainbow connection number is used to find the required minimum number of separate channels.

According to the literature,(Chartrand *et al.*, 2008) discussed rainbow connection $rc(G)$ in graphs. They showed that $rc(G) = 1$, if and only if G is a complete graph, and $rc(G) = |E(G)|$, if and only if G is a tree. They also determined the rainbow connection number of cycles and wheels. Edge colouring, rainbow connectivity and rainbow connection number were explained by (Caro *et al.*, 2008). (Bau, 2018) defined proper (strong) rainbow connection on some Cayley graphs.(Li *et al.*, 2012), investigated the relationship of rainbow connection number with vertex and edge connectivity. (Li *et al.*, 2021)obtained the vertex-rainbow connection number of some graph operations and showed several upper and lower bounds in terms of radius. (Wang *et al.*, 2017)discussed the rainbow vertex-connection number of 3-connected graphs ignoring a special case. Strong rainbow connection number of line, middle and total graph of Sunlet graph was studied by (Zhao *et al.*, 2018). (Simamora & Salman, 2015) have derived a formula for rainbow vertex connection number of pencil graphs. (Bustan & Salman, 2018) determined the rainbow connectivity number of a Fan graph. (Li *et al.*, 2013) showed that, $diam(G) - 1 \leq rvc(G) \leq n - 2$, if the diameter is 1 or 2, and $rvc(G) = diam(G) - 1$, otherwise. (Kumala & Salman, 2015) determined the rainbow connection number of (C_m, K_m) flower graph and (C_3, F_n) flower graph. (Hui-min & Ya-ping, 2016) determined the rainbow vertex-connection number of Ladder and Ladder graphs. They have determined the exact value of strong rainbow vertex connection (srvc) of Ladder graphs (L_n) for even number of vertices. For odd number of vertices, upper and lower bounds of srvc(L_n) were obtained. They also gave upper and lower bounds of the srvc(L_n) of Ladder. Rainbow connection number and rainbow connectivity number for several graph classes have been considered in the literature and only a few research was conducted on graphs. Ladder graph structure a vital role in building network structures. In this paper, we derive formulas for the rainbow connectivity number of simple ladder graphs and roach graphs.

MATERIALS AND METHODS

Definition - Ladder Graph: Ladder graph is denoted by L_n, where upper path and lower path has n vertices and opposite vertices of upper and lower paths are connected.

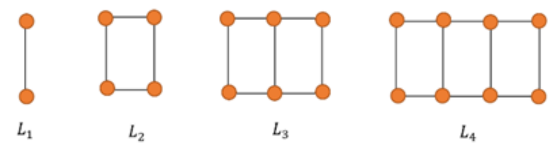

Figure 1: Ladder graphs

Definition - Roach Graph: Roach graph denoted by $R(2n, 2k)$ looks like a cockroach, where vertices have placed in head and body respectively. Upper and lower paths consist of $n + k$ vertices and all the k opposing vertices are joining as a ladder.

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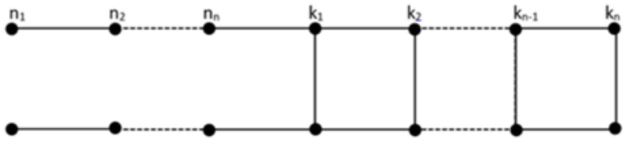


Figure 2: Roach graph with $2(n_n + k_n)$ vertices.

Theorem 1: Rainbow vertex connection number of ladder graph L_n is given by $rvc(L_n) = n - 1$, where $2n$ is the order of the graph.

Proof: We will give the proof by considering several cases.

Case I: Let $n = 2$. For any graph G, $Diameter(G) - 1 \leq rvc(G)$ and equality occurs when *diameter* = 1 or 2. According to the Figure 3, $Diam(G) = 2$. Hence the rainbow vertex connection number of this graph is 1.

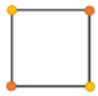


Figure 3: Ladder graph L_2

Case II: Let $n = 4$. According to the Figure 4, there exists only one path with small number of distinct vertices between vertices V_1 and V_2 which contain 3 distinct vertices with distinct colours.

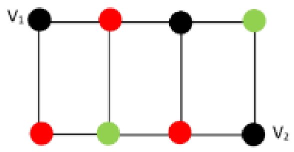


Figure 4: Ladder graph L_4

Case III: When n is an even number, and $n \geq 6$, we consider the ladder graph as in Figure 5. According to Figure 5, order of this graph is $2n$. We observed that the diameter of this graph is n. Using the equation $diam(G) - 1 \leq rvc(G)$, we have, $n - 1 \leq rvc(G)$. Next, we find a path from vertex V_1 to vertex U_n, which is equal to the diameter of the graph.

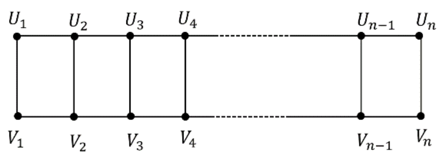


Figure 5: Ladder graph L_n.

According to the Figure 5, we can find more such paths. We choose $V_1\ U_1\ U_2 \ldots U_{n-1}\ U_n$ path. According to the definition of rainbow vertex- connected, to make this path a rainbow path, at least $(n - 1)$ number of colours need to be selected. Following the method given below, the whole graph can be converted into a rainbow vertex-connected using $n - 1$ number of colours. Accordingly, we should color the graph, $C(U_1) = C(V_2) = C(V_n) = 1$, $C(V_1) = C(U_2) = C(U_n) = 2$ where 1 and 2 are type of colours.

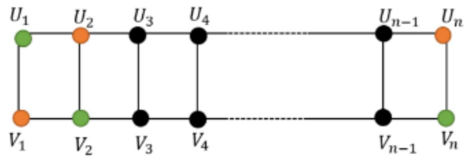


Figure 6: Ladder graph L_n with few colours assigned.

According to the Figure 6, we can see that U_1, V_2 and V_n vertices have same color and V_1, U_2 and U_n vertices have same color. Let $C(U_i) = i$, $3 \leq i \leq n - 1$, where i is the type of color. Then, upper path vertices were coloured using different colors. By using the colours used to color the (excluding above mentioned vertices) upper vertices, the lower vertices can be coloured such that no two adjacent vertices have the same colour. Now, when we consider any two vertices, the graph is rainbow vertex- connected. Therefore, when $n \geq 6$; $rvc(L_n) = n - 1$.

Case IV: When $n = 3$

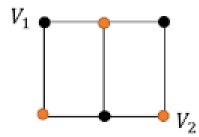


Figure 7: Rainbow colored Ladder graph L_3.

There exists only one path with small number of distinct vertices between vertices V_1 and V_2 which contain 2 distinct vertices. Therefore $rvc(L_3) = 2$.

Case V: Let $n \geq 5$ and n is an odd number. According to the Figure 8, we should color the graph such that $C(U_1) = C(V_2) = C(V_n) = 1$, $C(V_1) = C(U_2) = C(U_n) = 2$, where 1 and 2 are types of colours.

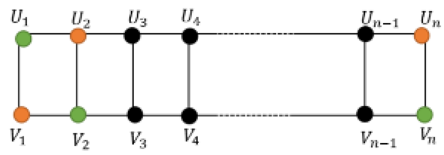


Figure 8: Ladder graph L_n, $n \geq 5$, and n odd with few colours.

According to the Figure 8, we can see U_1, V_2 and V_n vertices have the same color and V_1, U_2 and U_n vertices have same color. Then, using the method below we should complete this graph coloring. Let $C(U_i) = i$, $3 \leq i \leq n - 1$, where i is the type of color and $C(V_i) = n + 2 - i$. Now, when any two vertices are considered, the graph is rainbow vertex-connected.

Theorem 2: For a Roach graph $R\ (2n, 2k)$, rainbow vertex connectivity number can be given by the following formula:

$$rvc\big(R(2n, 2k)\big) = \begin{cases} k, & n = 1 \\ 2n, & 2 \leq k \leq n + 1,\ n \geq 2 \\ k + (n - 1), & k \geq 2 + n,\ n \geq 2 \end{cases}$$

Proof: We will give the proof by considering two parts of the graph.

Part 1: Let $n = 1$, $k = N$.

First, consider the ladder part of the graph as given in Figure 9. Rainbow vertex connection number of ladder graph L_n is given by $rvc(L_n) = n - 1$, where $2n$ is the order of the graph.

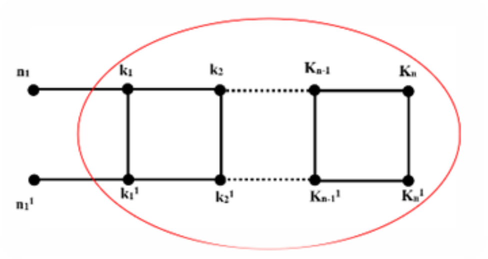


Figure 9: Roach graph $R(2, 2k)$.

Graph in Figure 9 has order $2N$. Therefore, rainbow vertex connection number of this graph is $N - 1$. Then, consider the other part of this graph. We can see that the diameter of the graph is increasing (Consider $n_1\ k_1 k_2 \ldots k_n\ k_n^1$ path). Therefore, we can add another one color for this graph. Now, the rainbow vertex connection number of this graph is $N - 1 + 1 = N$. When any two vertices are considered, the graph is rainbow vertex- connected.

Part 2: Case 1: Let $n \geq 2$, $k = 2$: $2 + (n - 1)$

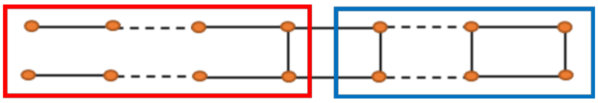


Figure 10: Labeled Roach graph $R(2n, 2k)$ with $n \geq 2$, $2 \leq k \leq n + 1$.

First, we consider the left side of Figure 10. Order of the above subgraph is $2n + 2$ and the diameter is $2n + 1$. Since $rvc(G) = dima(G) - 1$, to make the path rainbow connected, at least $2n$ number of colours are needed. Next, we consider the right side of Figure 10. Using the same $2n$ colours, we can colour the right side of the graph such that the whole graph is rainbow vertex connected. Figure 11 shows the rainbow connected graph using $2n$ colours.

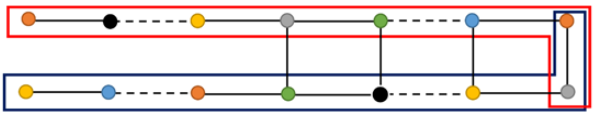


Figure 11: Rainbow vertex connected graph.

Case 2: When, $n \geq 2$, $k \geq 2 + n$

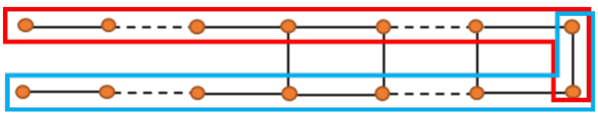


Figure 12: Labeled Roach graph $R(2n, 2k)$

Consider the upper path coloured in red as shown in Figure 12. Order of the subgraph is $n + k + 1$ and the diameter is $n + k$. To make this path a rainbow connected, we need to use at least $n + k - 1$ colours. Similarly, if we consider the lower path as in Figure 12, we can use $n + k - 1$ colours alternatively.

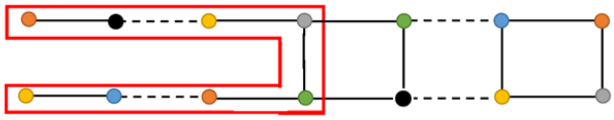


Figure 13: Rainbow connected path.

After colouring the Figure 12 using above colours, if we consider any two vertices, it is rainbow vertex connected. Figure 13 shows the rainbow path from upper left corner vertex to lower left corner vertex.

RESULTS AND DISCUSSION

Rainbow vertex connection number of ladder graph L_n is given by $rvc(L_n) = n - 1$, where $2n$ is the order of the graph.

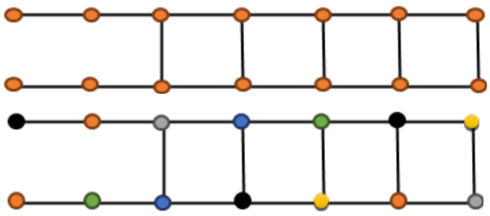
Rainbow vertex connection number of Roach graph $R(2n, 2k)$ is given by $rvc(R(2n, 2k)) = k$, when $n = 1$ and

$$rvc(R(2n, 2k)) = \begin{cases} 2n & 2 \leq k \leq n + 1 \\ k + (n - 1) & k \geq 2 + n \end{cases}, \text{ when}$$

$n \geq 2$

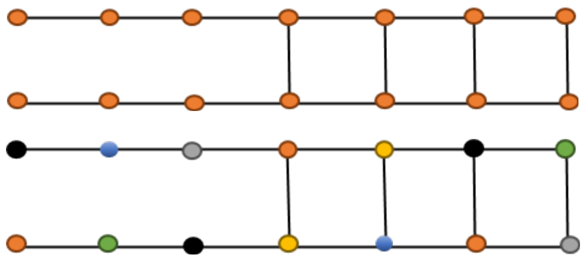
We will give some counter examples for the above cases.

Example 1: When, , $n = 2$ and $k = 5$ then $rvc(R(2n, 2k)) = k + (n - 1)$



$rvc(R(4, 10)) = 6$

Example 2: When, $n = 3$ and $k = 4$. This is similar to the case $(2 \leq k \leq n + 1)$



$rvc(R(6, 8)) = 6$

CONCLUSION

Rainbow connectivity of a graph is a measurement of the connectivity. It has been studied as a vertex connectivity as well as edge connectivity. The rainbow vertex-connection number $rvc(G)$ of a graph G is the smallest positive integer k such that graph G is rainbow vertex connected. In this research, we have obtained formulae for rainbow connectivity number of a ladder graph and a roach graph.

DECLARATION OF CONFLICTS OF INTERESTS

The authors declare no competing interests.

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