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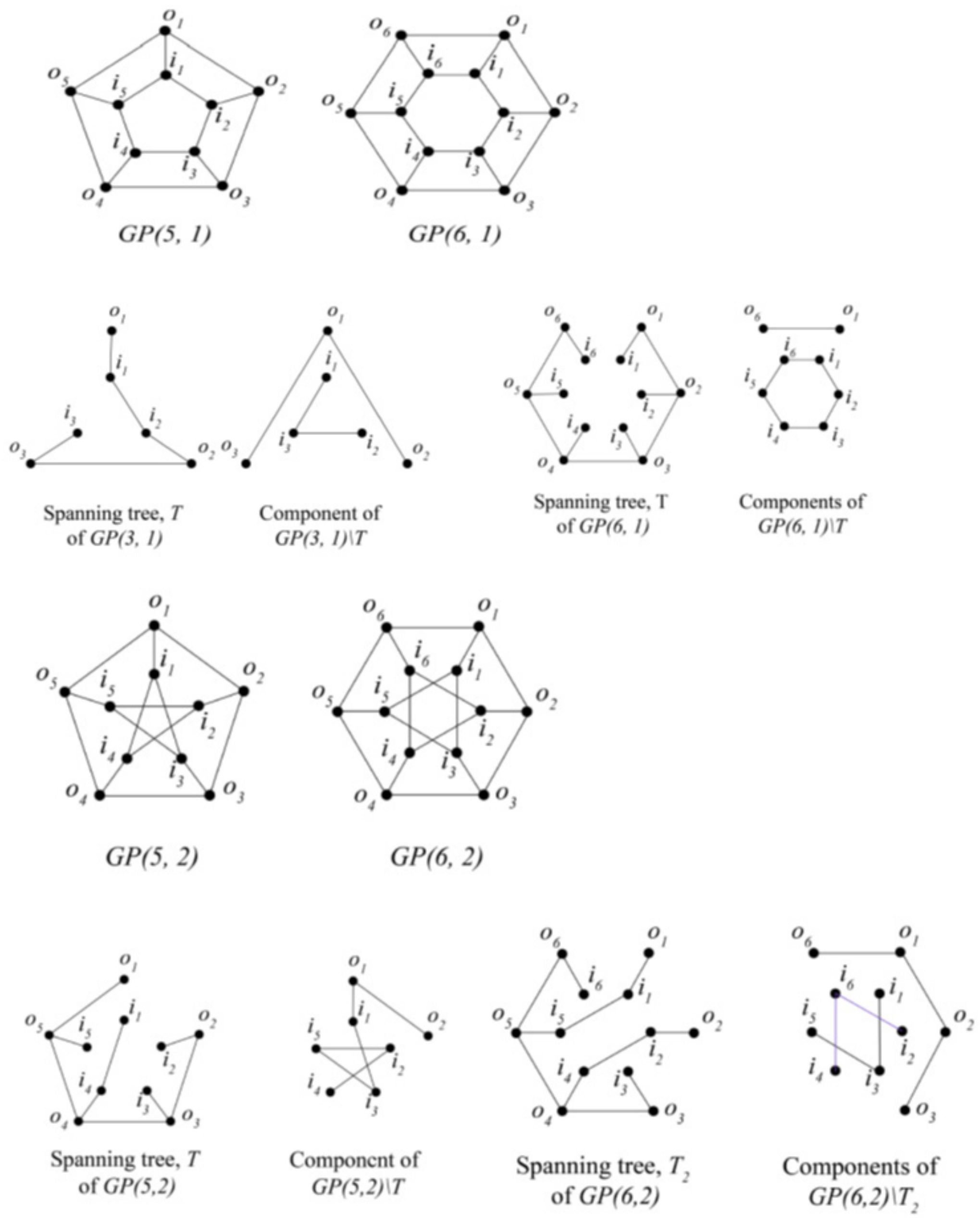
RESEARCH ARTICLE

The maximum genus of the generalized Petersen Graph, *GP* (*n, k*) for the cases *k* = 1, 2

P.A.D.S.P. Caldera, S.V.A. Almeida* and G. S. Wijesiri



Highlights

- The generalized Petersen graphs for the case *k* = 1, *GP* (*n*, 1) where n ≥ 3 is upper embeddable.
- *GP* (*n*, 2), the generalized Petersen graphs for the case *k* = 2 are upper embeddable for n ≥ 5.
- The maximum genus of generalized Petersen graphs, *GP* (*n*, *k*) for the cases *k* = 1, 2 is given by, $\gamma_M\big(GP\ (n,k)\big) = \left\lfloor\frac{n+1}{2}\right\rfloor$.

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The maximum genus of the generalized Petersen Graph, *GP* (*n*, *k*) for the cases *k* = 1, 2

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Abstract: In Topological graph theory, the maximum genus of graphs has been a fascinating subject. For a simple connected graph G, the maximum genus $\gamma_M(G)$ is the largest genus of an orientable surface on which G has a 2-cell embedding. $\gamma_M(G)$ has the upper bound $\gamma_M(G) \leq \left\lfloor \frac{\beta(G)}{2} \right\rfloor$, where $\beta(G)$ denotes the Betti number and G is said to be upper embeddable if the equality holds. In this study, the maximum genus of $GP(n, k)$ is established as $\gamma_M(GP(n,k)) = \left\lfloor \frac{n+1}{2} \right\rfloor$ for $k = 1$ and $k = 2$ by proving the upper embeddability of generalized Petersen graph, $GP(n, k)$ for the cases $k = 1$ and $k = 2$. The proof is done by obtaining spanning trees T and examining the components in the edge complements $GP(n, k)\backslash T$ for the cases $k = 1$ and $k = 2$ of $GP(n, k)$.

Keywords: 2-cell embedding; Generalized Petersen graph; Maximum genus; Upper embeddability; Spanning tree

INTRODUCTION

An embedding of a connected graph G on a surface S is a drawing of G on S without edge crossings and the embedding is referred to as a 2-cell embedding or simply cellular if the interior of each region called the faces of the embedding is homeomorphic to an open disk. The genus of the 2-cell graph embedding of G, is the genus g of the surface S, the minimum genus (known as the genus of G) is denoted by $\gamma(G)$ and the maximum genus is denoted by $\gamma_M(G)$. With the results of previous research (Duke, 1996; Nordhaus et al., 1971; Youngs, 1963) it is established that G has a 2-cell embedding on an orientable surface S of genus g if and only if $\gamma(G) \leq g \leq \gamma_M(G)$. The connection between maximum genus and other graph invariants has received significant interest (Han & Gang, 2010). In this paper we study and establish results for $\gamma_M(G)$ where $G = GP(n, k)$ for the cases $k = 1$ and $k = 2$.

The maximum genus $\gamma_M(G)$ of a connected graph G was determined using 2-cell embeddings by Nordhaus et al. (1971); in particular, they have derived an upper bound for the maximum genus of a connected graph G, $\gamma_M(G)$ in terms of the Betti number $\beta(G)$. The maximum genus of graphs has been an essential factor in graph embeddings since then and it has led to many advanced findings (Jungerman, 1978; Xuong, 1979) on how to determine the maximum genus.

Upper embeddability is related to the maximal embeddings of graphs. Many families of graphs are shown to be upper-embeddable; in particular, 4-edge connected graphs are upper-embeddable (Jungerman, 1978), and finite connected vertex-transitive graphs (meaning that the graph has symmetries that take any vertex to any other vertex) are upper-embeddable whenever its valency or girth is at least 4 (Skoviera & Nedela, 1989).

Jungerman (1978) gave a characterization for the upper embeddability of a graph in terms of splitting trees for the graph. A tree is a connected graph that has no cycles. A Spanning tree T of a graph G is a subgraph of G that is a tree that includes all vertices of the graph. We say T is a splitting tree for G if the edge complement $G\backslash T$ contains at most one component with an odd number of edges.

The connected cubic graph denoted as $GP(n, k)$ is called a generalized Petersen graph with parameters n and k, where $n \geq 3$ and $1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. These graphs were introduced and named by Coxeter (1950) and Watkins (1969), respectively. The name for $GP(n, k)$ is derived from the well-known graph, the Petersen graph $GP(5, 2)$. $GP(n, k)$ is formed of an inner star polygon, and an outer regular polygon with parameters $\{n, k\}$, and n, respectively. Furthermore, the equivalent vertices in the inner and outer polygons are connected by edges (some authors refer to as spokes).

These graphs are typically utilized in interconnection networks. The Petersen graph, $GP(5, 2)$, is highly sought-after in graph theory as it has served as a counterexample to several unresolved issues and hypotheses. All generalized Petersen graphs have $2n$ vertices and $3n$ edges and are unit-distance graphs (Zitnik et al., 2012). These graphs are 3-edge connected, meaning that it remains connected whenever fewer than 3 edges are removed, and are vertex-transitive if and only if $(n, k) = (10, 2)$ or $k^2 \equiv \pm 1 \ (mod\ n)$.

There have been several studies carried out regarding the generalized Petersen graphs, but none focused on the orientable genus of $GP(n, k)$. In our study, the results related to upper embeddability and the maximum genus of $GP(n, k)$ for the cases $k = 1, 2$ are established, which has not yet been determined in the previous literature.

MATERIALS AND METHODS

Throughout this work, the notation $G(V, E)$ is used to describe graphs, where V and E represents the set of vertices and the set of edges, respectively. The following theorems

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are used in proving the results in this paper.

Theorem 1: (Nordhaus et al., 1971) The maximum genus of a connected graph $G(V, E)$ has the upper bound,

$$\gamma_M(G) \leq \left\lfloor \frac{\beta(G)}{2} \right\rfloor, \quad (1)$$

where $\beta(G)$ signifies the Betti number,

$$\beta(G) = 1 + |E| - |V|. \quad (2)$$

A graph G is called upper-embeddable if $\gamma_M(G) = \left\lfloor \frac{\beta(G)}{2} \right\rfloor$.

Theorem 2: (Jungerman, 1978) A graph G is upper-embeddable if and only if G has a splitting tree.

The following labeling is introduced for the vertices of $GP(n, k)$. Take j ∈ {1, 2, …, n}. Let $i_1, i_2, \ldots, i_n$ denote the vertices of the inner star-polygon and $o_1, o_2, \ldots, o_n$ denote the vertices of the outer regular polygon such that o_j and o_{j+1} are adjacent with $o_{n+1} = o_1$, and i_j and o_j are adjacent for each j. For $GP(n, 1)$ with $n \geq 3$, i_j and i_{j+1} are adjacent with $i_{n+1} = i_1$, for $GP(n, 2)$ with $n \geq 5$, i_j is adjacent to i_{j+1} and i_{n-1} for each j. The following Figure 1 depicts the labeling of graphs $GP(5, 1)$ and $GP(5, 2)$.

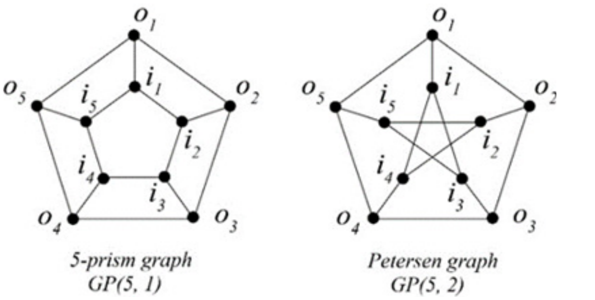


Figure 1: The labeling of graphs $GP(5, 1)$ and $GP(5, 2)$.

In the next section, we prove the upper embeddability of $GP(n, k)$ for $k = 1, 2$ and its maximum genus using theorem 2.

RESULTS AND DISCUSSION

Theorem 3: $GP(n, k)$ is upper embeddable for n ≥ 3 with $k = 1$ and for $n \geq 5$ with $k = 2$.

Proof:

This theorem will be proved independently for $k = 1$ ($n \geq 3$) and $k = 2$ ($n \geq 5$). For each k, two cases are considered separately based of n being even or odd.

<u>Case I:</u> Let $k = 1$.

Let n be even.

Consider the subgraph obtained by removing all edges in the inner star-polygon and any one edge, say, o_1o_2 in the outer regular polygon. It is easy to see that this subgraph is a spanning tree T, for $GP(n, 1)$. To see this, observe that the subgraph contains all vertices of $GP(n, 1)$ and is connected. The removal of the edges in the inner star-polygon results in a possible single cycle, $o_1, o_2, \ldots, o_n, o_1$ for the subgraph. The subgraph then becomes a graph with no cycles with the removal of o_1o_2.

For instance, the corresponding spanning tree, T for the graph $GP(6, 1)$ and the components of the $GP(6, 1) \setminus T$ are shown in Figure 2.

Observe that $GP(n, 1) \setminus T$ has the following components:

The inner star-polygon, which has an even number of edges (n is even)

The component containing the vertices o_1, o_2 and the corresponding adjacent edge o_1o_2, which has an odd number of edges.

Hence, we only have one component of $GP(n, 1) \setminus T$ with an odd number of edges when n is even.

Let n be odd.

Consider the subgraph obtained by removing all edges in the inner star-polygon except i_1i_2, and removing the edges o_1o_2, o_1o_n in the outer regular polygon.

To see that this subgraph is a spanning tree T, notice that the subgraph contains all vertices of $GP(n, 1)$ and is connected. The removal of all edges but i_1i_2, in the inner star-polygon results in possible cycles, o_1, i_1, i_n, o_n, o_1 and $o_1, o_2, \ldots, o_n, o_1$ for the subgraph. The subgraph then becomes a graph with no cycles with the removal of the edges o_1o_2 and o_1o_n.

For example, the corresponding spanning tree, T for the graph $GP(3, 1)$ and the components of the $GP(3, 1) \setminus T$ are shown in Figure 3.

Observe that $GP(n, 1) \setminus T$ has the following components:

The inner star-polygon with one edge removed, which has an even number of edges (n is odd implies that $n - 1$ is even)

The component containing the vertices o_1, o_2, o_n and the

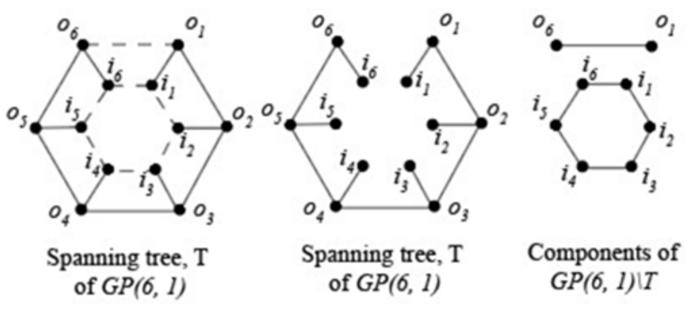


Figure 2: The process of deleting the edges, the corresponding spanning tree T of $GP(6, 1)$ and the components of $GP(6, 1) \setminus T$.

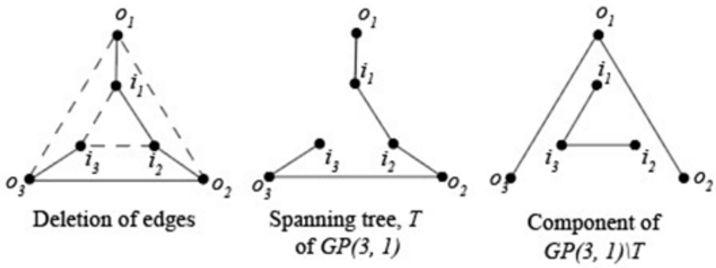


Figure 3: The process of deleting the edges, the corresponding spanning tree T of $GP(3, 1)$ and the components of $GP(3, 1) \setminus T$.

corresponding adjacent edges o_1o_2, o_1o_n which has an even number of edges.

Hence, for an odd n, we have no component of $GP(n, 1) \setminus T$ with an odd number of edges.

Therefore, for either case n is even or odd, $GP(n, 1)$ has a spanning tree T such that $GP(n, 1) \setminus T$ has at most one component with an odd number of edges. Hence, $GP(n, 1)$ is upper embeddable.

<u>Case II:</u> Let $k = 2$.

Let n be even.

For $n/2$ divisible by 2, consider the subgraph obtained by removing all edges in the inner star-polygon and the edge o_1o_2. This subgraph is a spanning tree T_1 for the graph as it contains all vertices and is connected with no cycles present. Removing all edges in the inner star polygon results in a possible cycle $o_1, o_2, \ldots, o_n, o_1$ and with the removal of o_1o_2, the subgraph becomes free of cycles.

For instance, consider the corresponding spanning tree, T_1 for the graph $GP(8, 2)$ and the components of the $GP(8, 2) \setminus T_1$ that are shown in Figure 4.

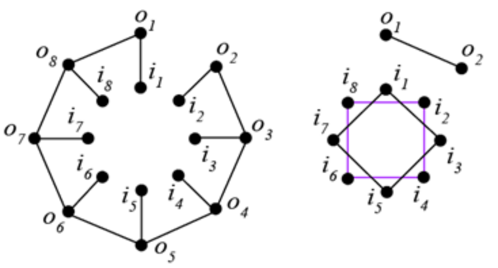


Figure 4: The corresponding spanning tree T_1 of $GP(8, 2)$ and the components of $GP(8, 2) \setminus T_1$.

Observe, $GP(n, 2) \setminus T$ has the following components:

Two cycle graphs of length $n/2$,

The component containing the vertices o_1, o_2 and the corresponding adjacent edge o_1o_2.

$GP(n, 2) \setminus T$ has exactly one component with an odd number of edges.

For $n/2$ not divisible by 2, consider the subgraph obtained by removing all edges but i_2i_4, i_1i_{n-1} in the inner star-polygon and removing the edges o_no_1, o_1o_2 and o_2o_3 in the outer regular polygon. It is easily seen that this connected subgraph is a spanning tree T_2.

The corresponding spanning tree, T_2 for the graph $GP(6, 2)$ and the components of the $GP(6, 2) \setminus T_2$ are shown in Figure 5 as an example.

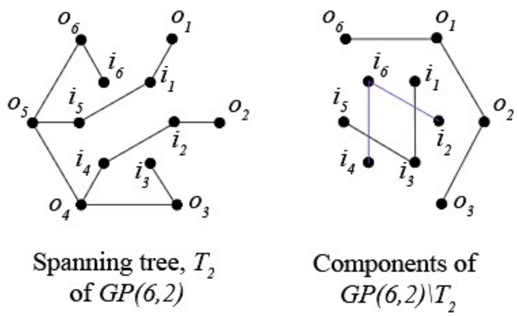


Figure 5: The corresponding spanning tree T_2 of $GP(6, 2)$ and the components of $GP(6, 2) \setminus T_2$.

$GP(n, 2) \setminus T_2$ has the following components:

Two graphs of length $n/2 - 1$ (which corresponds to the removal of edges in the inner star-polygon),

The component containing the vertices o_1, o_2 on o_3 and the corresponding adjacent edges o_no_1, o_1o_2 and o_2o_3.

Therefore, $GP(n, 2) \setminus T_2$ has exactly one component with an odd number of edges.

Thus, for an even n, in either case ($n/2$ is divisible by 2 and $n/2$ is not divisible by 2), $GP(n, 2)$ has a spanning tree T such that $GP(n, 2) \setminus T$ has exactly one component with an odd number of edges.

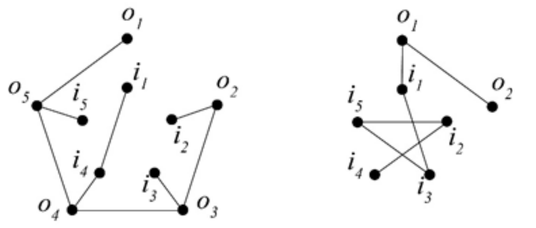
Let n be odd.

Consider the subgraph obtained by removing all edges but i_1i_{n-1} in the inner star-polygon and removing edges i_1o_1, o_1o_2 in the outer regular polygon. It is clear that this connected subgraph is a spanning tree T for the graph. $GP(n, 2) \setminus T$ has only one component containing all the previously removed edges and observe that this component has $n - 1 + 2 = n + 1$, an even number of edges.

For instance, consider the corresponding spanning tree, T for the graph $GP(5, 2)$ and the component of the $GP(5, 2) \setminus T$ shown in Figure 6 as an example.

Since, $GP(n, 2)$ has a spanning tree T such that $GP(n, 2) \setminus T$ has at most one component with an odd number of edges for any n, $GP(n, 2)$ is upper embeddable, by Theorem 2.

Thus, both $GP(n, 1) \setminus T$ and $GP(n, 2) \setminus T$ edge complements



Spanning tree, *T* of *GP(5,2)* Component of *GP(5,2)\T*

Figure 6: The corresponding spanning tree *T* of *GP*(5, 2) and the components of *GP*(5, 2) \ *T*.

have at most one component with an odd number of edges, ensuring the existence of splitting trees *T* for both *k* = 1, 2 of *GP*(*n*, *k*). Hence, *GP*(*n*, *k*) is upper embeddable for *k* = 1 and *k* = 2.

Using this result of the upper embeddability of *GP*(*n*, *k*) with the extended Euler's formula (Theorem 1), the following corollary is established.

Corollary 1: The maximum genus of *GP*(*n*, *k*) is given by,

$$\gamma_M\big(GP\ (n,k)\big) = \left\lfloor \frac{n+1}{2} \right\rfloor \quad (3)$$

where n ≥ 3 for *k* = 1 and n ≥ 5 for *k* = 2.■

Proof:

It is proved in theorem 3 that *GP*(*n*, *k*) is upper embeddable for the cases *k* = 1 (n ≥ 3) and *k* = 2 (n ≥ 5).

Observe that *GP*(*n*, *k*) has |*V*| = 2*n* vertices and |*E*| = 3*n* edges. Therefore, we have,

$$\gamma_M\big(GP\ (n,k)\big) = \left\lfloor \frac{\beta(G)}{2} \right\rfloor = \left\lfloor \frac{1 + |E| - |V|}{2} \right\rfloor = \left\lfloor \frac{1 + 3n - 2n}{2} \right\rfloor = \left\lfloor \frac{n+1}{2} \right\rfloor ■$$

CONCLUSION

The focus of our study was observing and obtaining results for the maximal embedding of generalized Petersen graphs, *GP*(*n*, *k*) for the cases *k* = 1 and *k* = 2. First, we proved the upper embeddability of generalized Petersen graphs, *GP*(*n*, 1) for n ≥ 3 and *GP*(*n*, 2) for n ≥ 5 via constructive proof. Utilizing these results, we have established the maximum genus of generalized Petersen graphs, *GP*(*n*, *k*) for the cases *k* = 1, 2 as $\gamma_M\big(GP\ (n,k)\big) = \left\lfloor \frac{n+1}{2} \right\rfloor$.

DECLARATION OF CONFLICT OF INTEREST

The authors declare no conflict of interest.

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