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RESEARCH ARTICLE

The Geodesic-Transversal Problem on Complementary Prisms

M. G. Anand and T. Jeba Raj

Highlights

- For cycle graph C_n of order $n \geq 3$, we have $gt(C_n \overline{C}_n) = \begin{cases} n-1; & 3 \leq n \leq 6, \\ n; & n \geq 7. \end{cases}$
- For path graph P_n of order $n \geq 2$, we have $gt(P_n \overline{P}_n) = \begin{cases} n-1; & 2 \leq n \leq 9 \\ n; & n \geq 10 \end{cases}$
- For multipartite graph $K_{p1, p2,...pm}$ of order $n = \sum_{i=1}^{m} p_i$, we have $gt(K_{p_1 p_2 \cdots p_m} \overline{K}_{p_1 p_2 \cdots p_m}) = n - 1$
- For split graph S_{pq} of order $n = p + q$, $gt(S_{pq} \overline{S}_{pq}) = p + q$
- For butterfly network BF(n), we have $gt(BF(n)) \leq \left\lceil \frac{n+1}{3} \right\rceil$, $n \geq 1$

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Abstract

A maximal geodesic in a graph is a geodesic which is not a sub-path of a longer geodesic. A subset S of vertices of a graph G is called a geodesic transversal of G if every maximal geodesic of G intersects S. The geodesic transversal problem aims to find the minimum cardinality of such a set and its cardinality is denoted by gt(G). In this work, we determine gt(G) for complementary cycle graphs, complementary path graphs, complete multipartite graph, complete split graphs, wheel graphs, fan graphs and butterfly graphs, and their complementary prisms. We also find an upper bound of gt(G) of butterfly network.

Keywords: Butterfly network; Complementary prism; Cycles; Geodesic transversal; Paths

INTRODUCTION

The geodesic transversal problem is among the classic problems, such as the Vertex Cover Problem (Miller & Thatcher, 1972), Dominating Set Problem (Garey & Johnson, 1979), Metric Dimension Problem (Slater, 1975; Harary & Melter, 1976), Vertex Feedback Set Problem (Garey & Tarjan, 1978), and Hitting Set Problem (Brešar et al., 2019), which seek a minimum-cardinality vertex set satisfying specific properties. Each of these problems is NP-complete in general graphs (Karp, 1972), although polynomial-time algorithms can be applied to some restricted classes of graphs. The geodesic transversal problem was independently introduced by Manuel et al. (2022) and Peterin and Semanišin (2021). Subsequently, the geodesic transversal problem was solved for trees and cactus graphs by Manuel et al. (2022), for joins and lexicographic products by Peterin and Semanišin (2022), and for some networks by Manuel et al. (2023b). The concept of the geodesic packing number of a graph was introduced by Manuel et al. (2023a) and serves as a lower bound for gt(G). The geodesic packing number of trees and diagonal grids was also determined in that study.

Let G be a graph, and let $\overline{G}$ be its complement. Then complementary prism $G\overline{G}$ of G is the graph formed from the disjoint union of G and $\overline{G}$ by adding edges between corresponding vertices of G and $\overline{G}$ was discussed in Haynes et al. (2007). For example, $C_5\overline{C}_5$ is the Petersen graph. The topics studied on complementary prisms include the chromatic index (Zatesko et al., 2019), domination (Bertossi, 1984; Haynes et al., 2009; Rajeshkumar & Anto, 2022), cycle structure (Meierling et al., 2015), complexity properties (Duarte et al., 2017), spectral properties (Cardoso et al., 2018), convexity number (Neethu & Chandran, 2022), b-chromatic number (Bendali-Braham et al., 2019), monophonic convexity (Neethu et al., 2024), monophonic position number (Thomas et al., 2024), and general position number (Neethu et al., 2021). Other extensions in the literature on graph theory, especially in fuzzy graphs, can be observed in Anto et al. 2025.

The butterfly network is a fundamental interconnection topology that is widely examined for its recursive structure and efficient communication properties in parallel and distributed systems. Various studies have explored its computational and structural aspects, including its relation to cube-connected cycles (Preparata & Vuillemin, 1981), broadcast efficiency (Stöhr, 1992), and network resilience (Upfal, 1994). More recent graph-theoretic investigations have focused on algebraic parameters such as minimum rank and zero forcing number of butterfly networks (Grigorious et al., 2019), highlighting its continuing significance in both theoretical and applied research.

Initially, we focus on cycle graphs and path graphs which typically exhibits higher diameter, many of the following fall within the class of diameter two graphs. Moreover, since complementary prisms are diameter three graphs, we believe that our work will contribute for the unified understanding of graphs of diameter two and three. Given that the geodesic transversal problem is relatively recent

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and remains underexplored, this study aims to broaden its theoretical foundations. The geodesic transversal problem will have potential applications arise in diverse areas such as network monitoring and control (optimal placement of monitoring devices), transportation and traffic systems (deployment of surveillance and sensing units), biological and neural networks (identification of critical communication nodes), social network analysis (detection of influential intermediaries), and distributed sensor systems (efficient coverage of essential communication paths).

Section 2 presents the fundamental definitions, notation, and preliminary observations that serve as the foundation for the subsequent discussions. In Section 3, we establish a general result concerning the geodesic transversal number of the complements of cycles and paths, and further extend this result to their complementary prisms. Section 4 is devoted to determining the exact values of the geodesic transversal number for the complementary prisms of complete bipartite, complete multipartite, and complete split graphs. In Section 5, we address the problem for wheel graphs, fan graphs, and their complementary prisms. Finally, in section 6, we investigate the geodesic transversal number for the butterfly graph and its complementary prisms, and derive an upper bound for the geodesic transversal number of the butterfly network.

MATERIALS AND METHODS

The graphs in this paper are finite, undirected, and simple. Let G = (V(G), E(G)) be a graph, and n(G) denote the order of G. We will write u ~ v if the vertices u and v are adjacent. A geodesic on a graph G is the shortest path between two vertices in G. A geodesic is maximal if it is not a subpath of a longer geodesic. A set S of vertices of G is a geodesic transversal set of G if every maximal geodesic of G contains at least one vertex of S. In this case, we say that S hits (or covers) all maximal geodesics. The geodesic transversal number of G, denoted by gt(G), is the minimum cardinality of a geodesic transversal of G. Every geodesic transversal of G of cardinality gt(G) is called a gt(G)-set. In Manuel et al. (2022), it is shown that for a connected graph G, $1 \leq gt(G) \leq n(G) - 1$, and the bounds are sharp. To extend the concept to a disconnected graph, we consider isolated vertices as (trivial) maximal geodesics. If G is a disconnected graph, with components $G_1, G_2, \cdots, G_n$, then $gt(G) = \sum_{i=1}^{n} gt(G_i)$. A geodesic packing of a graph G is the set of vertex-disjoint maximal geodesics. The maximum cardinality of a geodesic packing is the geodesic packing number, denoted by gpack(G), and serves as a lower bound of gt(G).

G[X] denotes the induced subgraph of G with vertex set X ⊆ V(G). If the parent graph is clear from context, we simply write [X]. The distance $d_G(u, v)$ between vertices u and v is the length of a shortest path between u and v, called a u,v -path. An u, v-path of minimum length is also called a u, v−geodesic. The eccentricity of vertex u is

$ecc_G(u) = \max\{d_G(u, v) : v \in V(G)\}$.

Accordingly, the radius and diameter of G are $diam(G) = \min\{ecc_G(v): v \in V(G)\}$ and $diam(G) = \max\{ecc_G(v): v \in V(G)\}$ respectively.

The open neighborhood N[u] of u∈V(G) is {v ∈ V(G): u ~ v}, while the closed neighborhood N [u] is defined by N[u] = N(u) ∪{u}.

Let C_n denote an n-cycle with vertex set $\{v_1, v_2, ..., v_n\}$, and let I ⊆ [n] be a set of indices corresponding to vertices of $V(C_n)$. Denote by $C_n(I)$ the graph obtained from C_n by attaching a pendant vertex v'_i to each vertex $v_i \in V(C_n)$ for every i ∈ I. The geodesic transversal problem for $C_n(I)$ has been investigated in Manuel et al. 2022.

The complementary prism of a graph G, denoted by $G\overline{G}$, is constructed from the disjoint union of G and its complement $\overline{G}$ by adding the edges of a perfect matching between the corresponding vertices of G and $\overline{G}$ in Neethu et al. 2021. It is straightforward to verify that $G\overline{G} \cong \overline{G}G$, and that the vertex set of $G\overline{G}$ satisfies

$$V(G\overline{G}) = V(G) \uplus V(\overline{G}),$$

where ⊎ denotes the disjoint union.

For a vertex u ∈ V(G) ⊆ $V(G\overline{G})$ its unique adjacent vertex in u ∈ $V(\overline{G}) \subseteq V(G\overline{G})$ will be denoted by $\overline{u}$ and is referred to as the partner of uin $\overline{G}$. This notation naturally extends to subsets of vertices for any X ⊆ V(G), the set of partners of vertices in X is denoted by $\overline{X}$. We define, for u ∈ V(G), let $\overline{(\overline{u})} = u$ and for X ⊆ V(G), let $\overline{(\overline{X})} = X$.

Cycles and Paths

In this section, we discuss the geodesic transversal problem in complementary prisms of cycles and paths. In the process, we also discuss the problem in complements of cycles and paths. Let C_n, n ≥ 3 denote the cycle with n vertices with vertex set {0, 1, 2, ..., n - 1} with natural adjacency. It is easily observed that $gt(C_n) = 2$ and $gpack(C_n) = 1$.

A cycle graph is one of the simplest connected 2-regular graphs. This is a basic structure that determines many structural properties of a graph. They serve as a fundamental substructure in more complex graphs such as wheel graphs, prisms, and toroidal grids. Due to their balance of structural simplicity and combinatorial richness, cycle graphs provide a natural starting point for extending the study of the problem. It is essential to have clarity of the structure and number of maximal geodesics in cycles and in their complement to solve the problem for its complementary prism. For that, we now discuss the geodesic packing of $\overline{C}_n$.

Theorem 1: For n ≥ 3,

$$\text{gpack}(\bar{C}_n)=\begin{cases}3; & n=3,\\ 2; & n=4,\\ \left\lfloor\frac{n}{3}\right\rfloor; & n\geq 5.\end{cases}$$

Proof: We can observe that $\bar{C}_3 \cong \bar{K}_3$ and $\bar{C}_4 \cong K_2 \cup K_n$, so the result easily follows for n = 3 and n = 4. If n = 5, $\bar{C}_5 \cong \bar{C}_5$, then $1 = \text{gpack}(C_5) = \left\lfloor\frac{5}{3}\right\rfloor$.

Now we let n ≥ 6, let $\bar{i}$ and $\bar{j}$ be any two vertices of $\bar{C}_n$, if they are not adjacent in $\bar{C}_n$ they must be adjacent in C_n and as n ≥ 6 there exists a vertex k ∈ V (C_n) such that k is not adjacent to i and j in C_n. So $\bar{i}$ - $\bar{k}$ - $\bar{j}$ is a geodesic in $\bar{C}_n$. So diam($\bar{C}_n$) = 2 for n ≥ 6. In addition, every edge in $\bar{C}_n$ can be extended to a geodesic of length 2, so all the maximal geodesics of $\bar{C}_n$ are of length 2.

We can observe that a maximal geodesic in $\bar{C}_n$ will be of the form $\bar{i}$ - $\overline{(i+k)(\text{mod } n)}$ - $\overline{(i+1)(\text{mod } n)}$, wherei = 0, 1, ..., n - 1 and K = 3, 4, ..., n - 2. So, there are n - 4 distinct maximal geodesics starting at $\bar{i}$ in $\bar{C}_n$ and there are n(n - 4) distinct maximal geodesics in $\bar{C}_n$, n ≥ 6. We are interested here in finding the maximum number of maximal geodesics which are vertex-disjoint.

If n=6m, m≥1, then $\overline{(3k-3)(\text{mod } n)}$ - $\overline{3k(\text{mod } n)}$ - $\overline{(3k-2)(\text{mod } n)}$ and $\overline{(3k+1)(\text{mod } n)}$ - $\overline{(3k-1)(\text{mod } n)}$ - $\overline{(3k+2)(\text{mod } n)}$, where K= 1,2,..., $\left\lfloor\frac{n}{6}\right\rfloor$, are the vertex-disjoint maximal geodesics in $\bar{C}_n$. So, there are $2\left\lfloor\frac{n}{6}\right\rfloor = \left\lfloor\frac{n}{3}\right\rfloor$ vertex-disjoint maximal geodesics. Since there is no vertex is left to cover, $\left\lfloor\frac{n}{3}\right\rfloor$ is the maximum number of vertex-disjoint maximal geodesics of $\bar{C}_n$ in this case.

If n = 6m + 3, m ≥ 1, then

$\bar{0}$ - $\bar{3}$ - $\bar{1}$, $\bar{5}$ - $\bar{2}$ - $\bar{6}$, $\bar{7}$ - $\bar{4}$ - $\bar{8}$, $\overline{(6k-3)(\text{mod } n)}$ - $\overline{6k(\text{mod } n)}$ - $\overline{(6k-2)(\text{mod } n)}$, $\overline{(6k+1)(\text{mod } n)}$ - $\overline{(6k-1)(\text{mod } n)}$ - $\overline{(6k+2)(\text{mod } n)}$,

where K = 1,2,..., $\left\lfloor\frac{n}{6}\right\rfloor$, are the vertex-disjoint maximal geodesics and there are $2\left(\left\lfloor\frac{n}{6}\right\rfloor - 1\right)+3 = \left\lfloor\frac{n}{3}\right\rfloor$ of them. Since there is no vertex is left to cover, $\left\lfloor\frac{n}{3}\right\rfloor$ is the maximum number of vertex-disjoint maximal geodesics of $\bar{C}_n$ in this case.

If n ≡ 1(mod 3), then n = 6m+1 or n = 6m + 4, m ≥ 1. If n = 6m + 1, arrange the maximal geodesics as in the case of n = 6m. In this case, only one vertex will remain in V($\bar{C}_n$), which is not a part of any of the maximal geodesics. If n = 6m + 4, arrange the maximal geodesics as in the case of n = 6m + 3. As in the previous case, only one vertex will remain in V($\bar{C}_n$), which is not a part of any of the maximal geodesics. This remaining vertex alone cannot contribute a new vertex-disjoint maximal geodesic, so $\left\lfloor\frac{n}{3}\right\rfloor$ is the maximum number of vertex-disjoint maximal geodesics of $\bar{C}_n$ in this case.

Finally, if n - 2(mod3), then n = 6m + 2 or n = 6m + 5, m ≥ 1. If n = 6m + 2, arrange the maximal geodesics as in the case of n = 6m. In this case, only two vertices will remain in V($\bar{C}_n$), which are not a part of any of the maximal geodesics. If n = 6m + 5, arrange the maximal geodesics as in the case of n = 6m + 3. As in the previous case, only two vertices will remain in V($\bar{C}_n$), which are not a part of any of the maximal geodesics. This remaining two vertices alone cannot contribute a new vertex disjoint maximal geodesic, so $\left\lfloor\frac{n}{3}\right\rfloor$ is the maximum number of vertex-disjoint maximal geodesics of $\bar{C}_n$ in this case. So, in all cases, whenever n ≥ 6, gpack ($\bar{C}_n$) $\left\lfloor\frac{n}{3}\right\rfloor$.

We know gt(G) ≥ gpack(G), so we can conclude that gt ≥ ($\bar{C}_n$) $\left\lfloor\frac{n}{3}\right\rfloor$. Now we extend our study to the geodesic transversal number of complements of a cycle.

Corollary 1: For n ≥ 3,

$$\text{gt}(\bar{C}_n)=\begin{cases}3; & n=3,\\ 2; & n=5,\\ \left\lceil\frac{n}{2}\right\rceil; & n\geq 6.\end{cases}$$

Proof: As we mentioned in Theorem 1, $\bar{C}_3 \cong \bar{K}_3$, $\bar{C}_4 \cong K_2 \cup K_2$ and $\bar{C}_5 \cong C_5$. So, the result is clear for n = 3, 4, 5. Now let n ≥ 6 and S = $\left\{\bar{0}, \bar{2}, \bar{4}, \cdots, \overline{2\left\lfloor\frac{n-1}{2}\right\rfloor}\right\}$, clearly, |S| = $\left\lceil\frac{n}{2}\right\rceil$. For 0 ≤ i ≤ n - 2, one among two vertices $\bar{i}$ and $\overline{i+1}$ is in S, as we have discussed in detail about the maximal geodesics of $\bar{C}_n$ in Theorem 1, all maximal geodesics hits S. So, gt ($\bar{C}_n$) ≤ $\left\lceil\frac{n}{2}\right\rceil$.

Now, K ⊆ V ($\bar{C}_n$) be a geodesic transversal set of $\bar{C}_n$ such that |K| < $\left\lceil\frac{n}{2}\right\rceil$. Then there exists 0 ≤ i ≤ n - 1 such that $\bar{i}, \overline{i+1(\text{mod } n)} \notin K$. As we discussed in Theorem 1, there are n - 4 ≥ 2 distinct maximal geodesics between the consecutive vertices $\bar{i}$ and $\overline{i+1(\text{mod } n)}$, they are, $\bar{i}$ - $\overline{(i+k)(\text{mod } n)}$ - $\overline{(i+1)(\text{mod } n)}$, for K = 3, 4, ..., n - 2. If K hits these maximal geodesics, it must have n - 4 ≥ 2 vertices, namely, $\overline{(i+k)(\text{mod } n)}$, for

K = 3, 4, ..., n - 2 and they are consecutive as well. So, K lacks more than half of the vertices of $V(\overline{C}_n)$ and also needs at least 2 consecutive vertices of $V(\overline{C}_n)$. This leaves behind at least 4 consecutive vertices of $\overline{C}_n$ away from K, namely $j, (j+1)(\text{mod } n), (j+2)(\text{mod } n)$ and $\overline{(j+3)(\text{mod } n)}$, for some $0 \le j \le n-1$. This leaves a maximal geodesic $\bar{j} - \overline{(j+3)(\text{mod } n)} - \overline{(j+1)(\text{mod } n)}$, without vertices from K, which is a contradiction. Hence K cannot be a geodesic transversal set of $\overline{C}_n$ and $gt(\overline{C}_n) \left\lceil \frac{n}{2} \right\rceil$ for $n \ge 6$. Now we extend our study to complementary prisms of cycles. We can observe that for $n \ge 6$, all maximal geodesics of $\overline{C}_n$ and all paths of length 3 in C_n are maximal in $C_n \overline{C}_n$. Since $C_5 \overline{C}_5$ is congruent with Petersen's graph, $gt(C_5 \overline{C}_5) = 4$ and $gt(C_3 \overline{C}_3) = 2$ as it is isomorphic to $C_3(1, 2, 3)$ Now in case of $gt(C_4 \overline{C}4)$ all paths of length 2 in C_4 are maximal, so at least 2 vertices of C_4 are required in geodesic transversal set of $gt(C_4 \overline{C}_4)$. Also, one vertex from $V(\overline{C}_4)$ is required in a geodesic transversal set of $gt(C_4 C_4)$ so that it hits all other maximal geodesics, so $gt(C_4 \overline{C}_4) \ge 3$. We can see that the set $\{0, 2, \bar{1}\}$ hits all the maximal geodesics of $C_4 \overline{C}_4$, hence $gt(C_4 \overline{C}_4) = 3$. Finally, for $C_6 \overline{C}_6$, a set of vertices requires at least 3 vertices of $V(\overline{C}_6)$ and 2 vertices of $V(C_6)$ so that it hits maximal geodesics which lies entirely on $V(\overline{C}_6)$ or $V(C_6)$, so $gt(C_6 \overline{C}_6) \ge 5$. Moreover, the set $\{0, 2, \bar{3}, \bar{4}, \bar{5}\}$ hits all maximal geodesics, and hence $gt(C_6 \overline{C}_6) = 5$.

Theorem 2: For $n \ge 3$, $gt(C_n \overline{C}_n) = \begin{cases} n-1; & 3 \le n \le 6, \\ n; & n \ge 7. \end{cases}$

Proof: It is clear from our above discussion that the theorem follows for $3 \le n \le 6$. Now, let $G = C_n \overline{C}_n$ and $n \ge 7$. The possible types of geodesics in G are:

$M_1: i - \bar{i} - \overline{(i \pm 2)(\text{mod } n)}$, where $0 \le i \le n-1$,

$M_2: i - (i \pm 1)(\text{mod } n) - \overline{(i \pm 1)(\text{mod } n)}$, where $0 \le i \le n-1$,

$M_3: i - (i+1)(\text{mod } n) - (i+2)(\text{mod } n) - (i+3)(\text{mod } n)$, where $0 \le i \le n-1$,

$M_4: i - \bar{i} - \overline{(i+k)(\text{mod } n)} - (i+k)(\text{mod } n)$, where $0 \le i \le n-1$ and $3 \le k \le n-3$, and

$M_5: \bar{i} - \overline{(i+k)(\text{mod } n)} - \overline{(i+1)(\text{mod } n)}$, where $0 \le i \le n-1$ and $3 \le k \le n-2$.

Figure 1 illustrates the complementary prisms of cycle graphs, with each of the maximal geodesics discussed above distinguished by a distinct colour. If n is odd, let $X = \{\bar{0}, \bar{2}, ..., \overline{n-1}, 1, 3, ..., n-2\}$ and if n is even, let $X = \{\bar{0}, \bar{2}, ..., \overline{n-2}, 1, 3, ..., n-1\}$. In both cases $|X| = n$ and X hits all maximal geodesics in G. Hence $gt(G) \le n$.

Now, let $K \subseteq V(G)$ and $|K| < n$. We can assume that K is a geodesic transversal set of G. Since all maximal geodesics in $\overline{C}_n$ are maximal in G we need $\left\lceil \frac{n}{2} \right\rceil$ vertices from $V(\overline{C}_n)$ in geodesic transversal set of G. We have $|K \cap V(\overline{C}_n)| \ge \left\lceil \frac{n}{2} \right\rceil$, so $K \cap V(C_n) \le \left\lceil \frac{n-3}{2} \right\rceil$. If $|K \cap V(\overline{C}_n)| = n - 1$, then there exists $0 \le i \le n-1$, such that $\bar{i} \notin K \cap V(\overline{C}_n)$. Then K never hits maximal geodesic $\bar{i} - i - (i+1)(\text{mod } n)$. If $|K \cap V(\overline{C}_n)| < n - 1$, then there exist two vertices from $V(\overline{C}_n)$ which is not an element of vertex set K. If both the vertices are consecutive, namely, $\bar{i}$ and $\overline{(i+1)(\text{mod } n)}$ for some $0 \le i \le n-1$. Then, K needs at least n - 3 vertices from $V(\overline{C}_n)$ so that it hits all maximal geodesic of type $\bar{i} - \overline{(i+k)(\text{mod } n)} - \overline{(i+1)(\text{mod } n)}$, where $3 \le k \le n-2$. Also, K requires at least two more vertices from $V(C_n)$, namely i and $(i+1)(\text{mod } n)$, so that it hits maximal geodesics $\bar{i} - i - (i-1)(\text{mod } n)$, $\bar{i} - i - (i+1)(\text{mod } n)$, $\overline{(i+1)(\text{mod } n)} - (i+1)(\text{mod } n) - i$ and $\overline{(i+1)(\text{mod } n)} - (i+1)(\text{mod } n) - (i+2)(\text{mod } n)$, this completes the quota of n - 1 vertices in K. But this leaves 4 consecutive vertices $(i+2)(\text{mod } n), (i+3)(\text{mod } n), (i+4)(\text{mod } n)$ and $(i+5)(\text{mod } n)$ away from K. Hence a maximal geodesic $(i+2)(\text{mod } n) - (i+3)(\text{mod } n) - (i+4)(\text{mod } n) - (i+5)(\text{mod } n)$ never hits K, which is not possible.

So, now we know if two vertices of $V(\overline{C}_n)$ are not in K, then it cannot be consecutive. So, for all $\bar{j} \in V(\overline{C}_n)$, such that $\bar{j} \notin K$, K must contain the partner vertex, j of $\bar{j}$, so that it hits maximal geodesics $\bar{j} - j - (j \pm 1)(\text{mod } n)$. For that, set K requires at least n vertices which is a contradiction. Hence, K cannot be a geodesic transversal set of G, so $gt(G) \ge n$, which confirms the result.

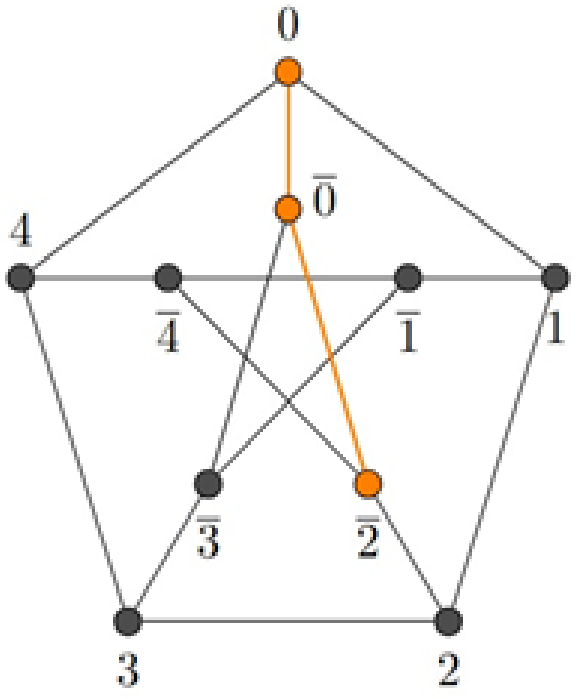


Figure 1 (A): $C_5 \overline{C}_5$

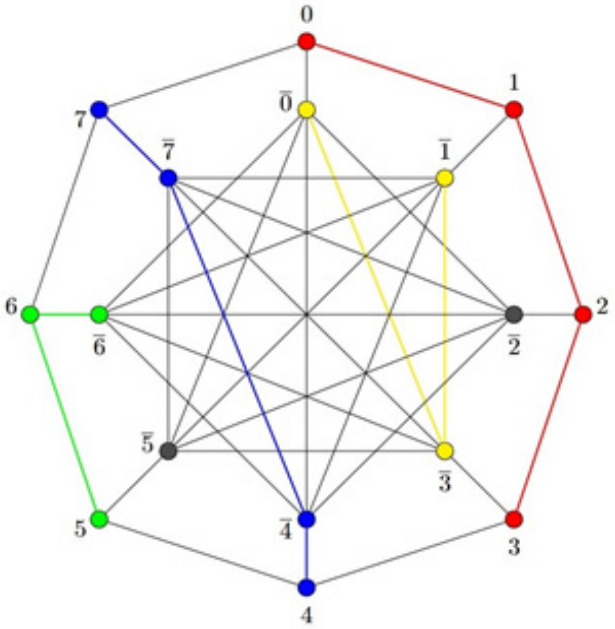


Figure 1 (B): $C_n\bar{C}_n$

Figure 1: The complementary prism of cycle graphs illustrating the distinct possible maximal geodesics types in $C_n\bar{C}_n$, each represented in a distinct colour

Next, we continue our study to complementary prisms of paths. For $n \geq 2$, let P_n denote the path with n vertices and $V(P_n) = \{0, 1, 2, ..., n - 1\}$ with natural adjacency. A path graph P_n can be obtained by deleting an edge from the cycle graph C_n, and may therefore be viewed as an acyclic reduction of a cycle. Moreover, while a path admits a unique geodesic between every pair of vertices, a cycle may admit two distinct geodesics of equal length. Consequently, the extension of our study to path graphs is naturally and closely connected to our earlier analysis of cycle graphs. It is easily observed that $gt(P_n) = 1$. Initially, we find the geodesic transversal number of complement of a path graph.

Theorem 3: For $n \geq 2$, $gt(\bar{P}_n)=\begin{cases} 2; n = 2 \text{ or } n = 3, \\ 1; n = 4, \\ \left\lfloor \frac{n}{2} \right\rfloor; n \geq 5. \end{cases}$

Proof: We can also observe that $\bar{P}_2 \cong \bar{K}_2$, $\bar{P}_3 \cong K_1 \cup K_2$ and $\bar{P}_4 \cong P_4$. So, we can easily conclude that $gt(\bar{P}_2) = gt(\bar{P}_3) = 2$ and $gt(\bar{P}_4) = 1$. Now, for $n \geq 5$, the maximal geodesics in $\bar{P}_n$ are:

$M_1 : \bar{0} - \overline{(n - 1)} - \bar{1}$,

$M_2 : \overline{(n - 1)} - \bar{0} - \overline{(n - 2)}$, and

$M_3 : \bar{i} - \overline{(i + k)(\text{mod } n)} - \overline{(i + 1)(\text{mod } n)}$, where $3 \leq k \leq n - 2$ and $0 \leq i \leq n - 2$.

Consider the set $S = \left\{\bar{1}, \bar{3}, \cdots, \overline{2\left\lfloor \frac{n}{2} \right\rfloor - 1}\right\}$. Clearly $|S| = \left\lfloor \frac{n}{2} \right\rfloor$ and hits all the maximal geodesics of $\bar{P}_n$. So, $gt(\bar{P}_n) \leq \left\lfloor \frac{n}{2} \right\rfloor$.

Now, let $K \subseteq V\ \bar{P}_n()$ be a geodesic transversal set of $\bar{P}_n$ such that $|K| < \left\lfloor \frac{n}{2} \right\rfloor$. Then there exists $0 \leq i \leq n - 1$ such that $\overline{(i - 1)(\text{mod } n)}, \bar{i}, \overline{(i + 1)(\text{mod } n)} \notin K$. If $i \neq 0$ and $i \neq n - 1$ then there exist $n - 3 \geq \left\lfloor \frac{n}{2} \right\rfloor$ distinct maximal geodesics between the vertices $\overline{(i - 1)(\text{mod } n)}$, and $\overline{(i + 1)(\text{mod } n)}$. If $i = 0$ or $i = n - 1$, then there exist maximal geodesics of type M_1 or M_2 which never hits K. Both cases contradict the geodesic transversal property of K. So, K is not a geodesic transversal set of $\bar{P}_n$, therefore $gt(\bar{P}_n) \geq \left\lfloor \frac{n}{2} \right\rfloor$ and hence the result is valid.

Next, we can solve the geodesic transversal problem for complementary prisms of path graphs. Similar to the case of cycle graphs, the complementary prisms of path graphs also exhibit analogous types of maximal geodesics. However, in the case of path graphs, the number of maximal geodesics corresponding to each type is increased.

Theorem 4: Let $n \geq 2$, then

$$gt(P_n\bar{P}_n)=\begin{cases} n - 1; 2 \leq n \leq 9, \\ n; n \geq 10. \end{cases}$$

Proof: We can observe that $P_2\bar{P}_2 \cong P_4$ and $P_3\bar{P}_3 \cong C_4(1)$. Now sets $\{\bar{0}, 1, 3\}, \{\bar{0},1,3,\bar{4}\}, \{\bar{0}, \bar{1}, 2, 4, \bar{5}\}, \{\bar{0}, \bar{1}, 2, 4, \bar{5}, \bar{6}\}, \{\bar{0}, \bar{1}, \bar{2}, 3, 5, \bar{6}, \bar{7}\}$ and $\{\bar{0}, \bar{1}, \bar{2}, 3, 5, \bar{6}, \bar{7}, \bar{8}\}$ are geodesic transversal sets of $P_4\bar{P}_4, P_5\bar{P}_5, P_6\bar{P}_6, P_7\bar{P}_7, P_8\bar{P}_8$ and $P_9\bar{P}_9$, respectively. Hence the result is true for $2 \leq n \leq 9$. We can observe that for $n \geq 10$, all maximal geodesics in $\bar{P}_n$ and all paths of length 3 in P_n are maximal in $P_n\bar{P}_n$. In addition to them, other maximal geodesics of $P_n\bar{P}_n$ are:

$M_1 : \bar{i} - i - (i + 1)$, where $0 \leq i \leq n - 2$,

$M_2 : \bar{i} - i - (i - 1)$, where $1 \leq i \leq n - 1$,

$M_3 : i - \bar{i} - \overline{(i + 2)}$, where $0 \leq i \leq n - 3$.

$M_4 : i - \bar{i} - \overline{(i - 2)}$, where $2 \leq i \leq n - 1$, and

$M_5 : i - \bar{i} - \overline{(i + k)} - (i + k)$, where $0 \leq i \leq n - 4$ and $3 \leq k \leq n - i - 1$.

Figure 2 depicts the complementary prisms of path graphs, highlighting all its maximal geodesics, each represented by a distinct colour. If n is odd, let $X = \{0, 1, 2, 3, \cdots n - 1\}$ and if n is even, let $X = \{0, \bar{1}, 2, \bar{3}, \cdots \overline{n - 1}\}$. In both cases $|X| = n$ and X hits all maximal geodesics of $P_n\bar{P}_n$. Hence $gt(P_n\bar{P}_n) \leq n$.

Now, let $K \subseteq V(P_n \bar{P}_n)$ be a geodesic transversal set of $P_n \bar{P}_n$ such that $|K| < n$. Then there exist $0 \le i \le n-1$ such that, vertices $i, \bar{i} \notin K$. We can observe that the edge $i\bar{i}$ can be extended to all other vertices of $V(P_n)$ or their partners to obtain a maximal geodesic. Since $|K|<n$, it follows that, for each $0 \le j \le n-1$ and $i \ne j$, exactly one of j or $\bar{j}$ belongs to the set K. But the set K specifically needs vertex (i + j) and vertex (i - j) so that it hits maximal geodesic M_1 and M_2. So, vertex $\overline{(i+1)}$ or vertex $\overline{(i-1)}$ will not be an element of set K. So, now we are in the situation where we need at least n - 3 vertices of $V(\bar{P}_n)$ so that the set K hits all maximal geodesics of types $\bar{0} - \overline{(n-1)} - \bar{1}$, $\overline{(n-1)} - \bar{0} - \overline{(n-2)}$, and $\bar{j} - \overline{(j+k)(\mathrm{mod}\, n)} - \overline{(j+1)(\mathrm{mod}\, n)}$, where $3 \le k \le n-2$ and $0 \le j \le n-2$. But, this leaves a path of length 3 in P_n, that is a maximal geodesic in $P_n \bar{P}_n$ which never hits K. This is in contradiction to the geodesic transversal property of K. So, $gt(P_n \bar{P}_n) \ge n$, and therefore the result is valid.

Bipartite, multipartite and split graphs

In this section, we discuss the problem for complementary prisms of various partite graphs. First, we solve the problem for complementary prisms of a complete bipartite graph, then for a complete multipartite graph, and conclude with complete split graphs. In pursuit, we will also discuss the problem for a complete multipartite graph as well.

The extension of our earlier investigations on cycle graphs C_n and path graphs P_n naturally leads to the broader and structurally richer class of bipartite graphs. A fundamental characterization asserts that a graph is bipartite if and only if it contains no odd cycle. On the other hand, every path graph P_n, being acyclic, is inherently bipartite. Motivated by these observations, our present study extends these insights from cycles and paths to complete bipartite graphs, whose highly regular structure provides a natural and significant setting for further theoretical development.

Let $K_{n,m}$ be a complete bipartite graph with partition P and Q with cardinality n and m, respectively. It is already known that $gt(K_{n,m}) = \min\{n, m\}$ and $\bar{K}_{n,m} \cong \bar{K}_n \cup \bar{K}_m$. A complete bipartite graph $K_{n,m}$, has diameter 2 and all its geodesics of length 2 are maximal in its complementary prisms. Whereas $K_{n,m}$ has only geodesics of length one and is not maximal in its complementary prism. So, possible maximal geodesics of $\bar{K}_{n.m} K_{n.m}$ are: $M_1 : u - v - w$, $M_2 : u - \bar{u} - \bar{v}$, and $M_3 : \bar{u} - u - v - \bar{v}$ for $u, v, w \in V(K_{n,m})$.

Theorem 5: Let $K_{n,m}$ be a complete bipartite graph, then $gt(K_{n,m} \bar{K}_{n,m}) = n+m-1$.

Proof: If $n = m = 1$ then $K_{n,m} \bar{K}_{n,m} \cong P_4$, then the result is true. Now, let n and m not both be equal to one and P, Q be partition of the bipartite graph $K_{n,m}$. Without loss of generality, we let $n \le m$ so, $m \ge 2$. Let $S = P \cup \bar{Q} - \{\bar{u}\}$, for some $u \in Q$. Then $|S| = n + m - 1$ and S hits all maximal geodesics of $K_{n,m} \bar{K}_{n,m}$, so $gt(K_{n,m} \bar{K}_{n,m}) \le n+m-1$.

Now, let $K \subseteq V(K_{n,m} \bar{K}_{n,m})$, $|K| < n + m - 1$ and K be a geodesic transversal set of $K_{n,m} \bar{K}_{n,m}$. Figure 3 provides a visual representation of the complementary prism of a complete bipartite graph, with the maximal geodesics distinguished by distinct colours. Since all maximal geodesics of $K_{n,m}$ are maximal in $K_{n,m} \bar{K}_{n,m}$, we have $|K \cap V(K_{n,m})| \ge n$, and hence $|K \cap V(\bar{K}_{n,m})| \le m - 2$.

We claim that $P \subseteq K$ or $Q \subseteq K$. If not, that is $P \not\subseteq K$ and $Q \not\subseteq K$, then there exist $u \in P$ and $v \in Q$ such that $u, v \notin K$. So, K must contain $\bar{u}$ or $\bar{v}$, so that it hits the geodesic $\bar{u} - u - v - \bar{v}$. But as $|K| \le n + m - 2$ there exists another vertex $w \in V(K_{n,m})$ different from u and v such that $w \notin K$ Without loss of generality, let $w \in P$, then geodesic u - v - w never hits K, which is not possible, which proves the claim.

If $P \subseteq K$, then there exists u, $v \in Q$ such that $\bar{u}, \bar{v} \notin K$. So, it is essential that $u, v \in K$, so that K hits geodesics of types $u - \bar{u} - \bar{v}$ and $v - \bar{v} - \bar{u}$. So, for any pair of vertices in $\bar{Q} \setminus K$, its partners eventually need to be in K to maintain the geodesic transversal property of K. For that K needs at least m - 1 more vertices, which is not possible. The same conclusion occurs if $Q \subseteq K$, and so K is not a geodesic transversal set of $K_{n,m} \bar{K}_{n,m}$, which results in a contradiction. So, $gt(K_{n,m} \bar{K}_{n,m}) \ge n+m-1$ and hence the result.

A natural progression from bipartite graphs leads to a more general class of multipartite graphs, in which the vertex set is partitioned into 2 or more mutually disjoint independent sets, and edges occur only between vertices belonging to different partitions. In this sense, multipartite graphs may be viewed as a direct generalization that preserves the fundamental idea of vertex partition.

Consider the graph $K_{p_1,p_2,\ldots,p_m}$, where $p_i \ge 1$ for all $i \in [m]$. This graph is a complete multipartite graph whose vertex set is partitioned into $m \ge 2$ independent sets denoted by $P_i = \{u_{i1}, u_{i2}, \ldots, u_{ip_i}\}$, where $1 \le i \le m$. Here, $|P_i| = p_i$ for each $i \in [m]$, and the total number of vertices is $n = \sum_{i=1}^{m} p_i$. It can be observed that a complete multipartite graph has diameter two, and its complement is isomorphic to $K_{p_1} \cup K_{p_2} \cup \ldots \cup K_{p_m}$. Furthermore, it follows that $gt(\bar{K}_{p_1,p_2,\ldots,p_m}) = n - m$. Let $A = \{P_i | p_i = 1\}$ and $B = \{P_i | p_i \ge 1\}$, and denote $\alpha = |A|$ and $\beta = |B|$. We can observe that $\alpha + \beta = m$.

Theorem 6: Let $K_{p1, p2,\ldots,pm}$, where $p_i \ge 1$ for all $i \in [m]$ and $m \ge 2$, be a complete multipartite graph. Then

$$gt\left(K_{p_1,p_2,\ldots,p_m}\right) = \begin{cases} n - \max\limits_{P_i \in B}\{\beta, |P_i|\}; \alpha = 0, \\ n - \max\limits_{P_i \in B}\{\beta+1, |P_i|\}; \alpha \ne 0. \end{cases}$$

Proof: In a complete multipartite graph $K_{p1, p2,\ldots,pm}$, an edge between two vertices of $\bigcup_{P_i \in A} P_i$ is a maximal geodesic, so we cannot avoid two vertices of $\bigcup_{P_i \in A} P_i$ in the geodesic transversal set of $K_{p1, p2,\ldots,pm}$. If $p_i = 1$ for all $i \in [m]$, then

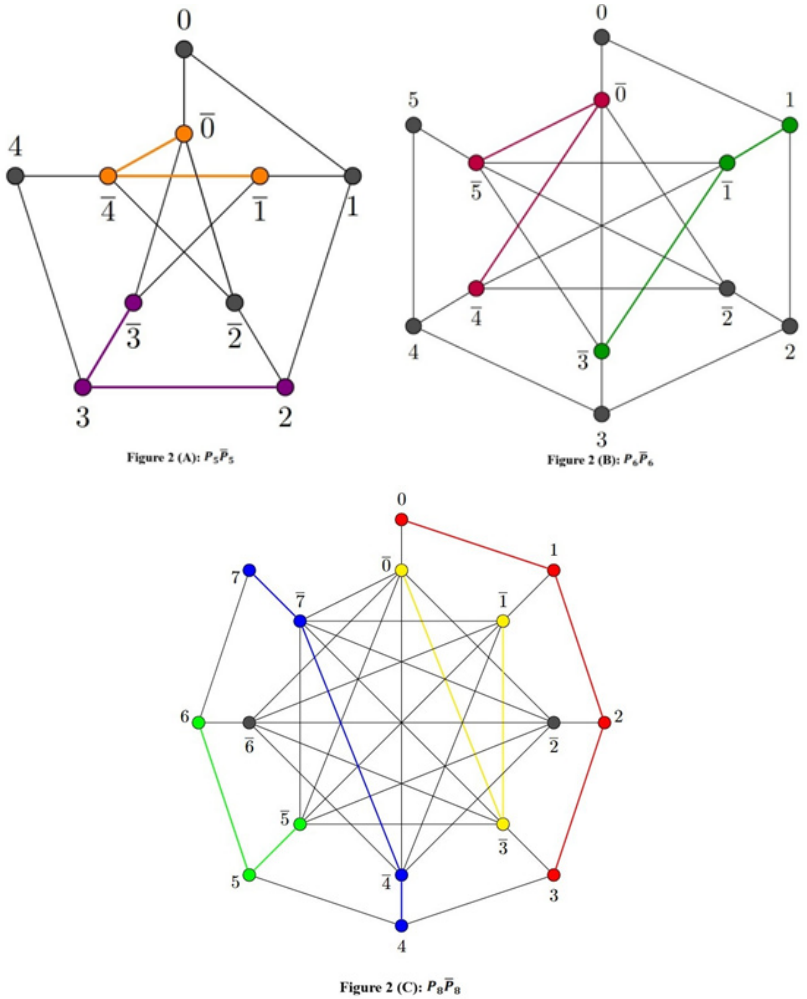


Figure 2: The complementary prism of path graphs illustrating the distinct possible maximal geodesics types in $P_n\bar{P}_n$, each represented in a distinct colour

$K_{p_1,p_2,\ldots,p_m} \cong K_n$ and $gt(K_m) = m - 1$. In this case, $\alpha = m = n$ and $\beta = 0$, which upholds the established result. Now we are only interested in the case in which $p_i \geq 2$for some $i \in [m]$.

Let $X = V(K_{p1,\ p2,\ldots,pm})\backslash\{u_{il} \mid P_i \in B\}$ and $Y = V(K_{p1,\ p2,\ldots,pm})\backslash P_j$, for some $P_j \in B$. Clearly, X/P_k, for some $P_k \in \mathbf{A}$ and Y are geodesic transversal sets in $K_{p1,\ p2,\ldots,pm}$. Hence, $gt(K_{p1,\ p2,\ldots,pm}) \leq n - \max_{P_i \in \mathbf{B}}\{\beta+1, |P_i|\}$. If $\alpha = 0$, then $\mathbf{A} = \phi$, then X and Y are geodesic transversal sets in $K_{p1,p2,\ldots,pm}$. So, $gt(K_{p_1,p_2,\ldots,p_m}) \leq n - \max_{P_i \in \mathbf{B}}\{\beta, |P_i|\}$.

Now, for the converse part, let $K \subseteq V(K_{p_1,p_2,\ldots,p_m})$ be such that $V(K_{p1,\ p2,\ldots,pm})\backslash K$ is a geodesic transversal set of $K_{p1,\ p2,\ldots,pm}$. We now divide the proof into two cases,

Case 1: Let $\alpha \neq 0$ and $|K| > \max_{P_i \in \mathbf{B}}\{\beta+1, |P_i|\}$.

Sub case 1: If $\beta+1 = \max_{P_i \in \mathbf{B}}\{\beta+1, |P_i|\}$, then $|K| > \beta+1$. Since $V(K_{p1,\ p2,\ldots,pm})\backslash K$ is a geodesic transversal set in $K_{p1,\ p2,\ldots,pm}$, set K cannot have more than one vertex from $\cup_{P_i \in \mathbf{A}} P_i$. If $u \in K \cap \cup_{P_i \in \mathbf{A}} P_i$, then as $|K| > \beta+1$ there exist two vertices $x, y \in P_j \cap K$ for some $P_j \in \mathbf{B}$. This would result in a maximal geodesic $x - u - y$ that is entirely in K, which would contradict the geodesic transversal property of $V(K_{p1,\ p2,\ldots,pm})\backslash K$. Now, if $K \cap \cup_{P_i \in \mathbf{A}} P_i = \phi$, then as $\beta + 1 = \max_{P_i \in \mathbf{B}}\{\beta+1, |P_i|\}$, all vertices of K cannot be entirely from some $P_i \in \mathbf{B}$. In this scenario, we must have two distinct $P_i, P_j \in \mathbf{B}$ such that $x, y \in P_i \cap K$ and $z \in P_i \cap K$. This would generate maximal geodesic $x - z - y$ that never hits $V(K_{p1,\ p2,\ldots,pm})\backslash K$ again, resulting in a contradiction.

Sub case 2: If there exists some $P_j \in B$ such that $|P_j| = \max_{P_i \in \mathbf{B}}\{\beta+1, |P_i|\}$, then $|K| > |P_j|$. Clearly, all elements of K cannot be from some $P_k \in B$ alone, and all elements of K cannot be from distinct partitions of $V(K_{p1,\ p2,\ldots,pm})$. So, there exists two distinct partitions P_i and P_K of $V(K_{p1,\ p2,\ldots,pm})$ such that vertices $x,y \in P_i \cap K$ and $z \in P_k \cap K$, generate a maximal geodesic x-z-y that lies entirely in K. This results in the contradiction of geodesic transversal property of $V(K_{p1,\ p2,\ldots,pm})\backslash K$.

Thus, $gt(K_{p_1,p_2,\ldots,p_m}) \geq n - \max_{P_i \in \mathbf{B}}\{\beta+1, |P_i|\}$, and hence the result holds.

Case 2: Let $\alpha = 0$ and $|K| > \max_{P_i \in B}\{\beta, |P_i|\}$.

If $\beta = \max_{P_i \in B}\{\beta, |P_i|\}$, then |K| > β. Then all vertices of K cannot be entirely from some $P_j \in B$. Same happens if there exists some $P_k \in B$ such that $|P_j| = \max_{P_i \in B}\{\beta, |P_i|\}$. In both cases, we will obtain a maximal geodesic that lies entirely in K, which would contradict the geodesic transversal property of $V(K_{p1, p2,...,pm})\setminus K$. Thus, $gt(K_{p_1,p_2,...,p_m}) \geq n - \max_{P_i \in B}\{\beta, |P_i|\}$, and the result is valid for both cases.

As in all previous cases, we now extend our study to complementary prisms of multipartite graphs. The complementary prisms of a multipartite graph exhibit similar kinds of maximal geodesics as in bipartite graphs. However, the connectivity of each vertex in a partition increased significantly in multipartite and its complementary prisms. This leads to a corresponding rise in number of maximal geodesics compared to the bipartite case.

Theorem 7: Let $K_{p1, p2,...,pm}$ be a complete multipartite graph of order $n = \sum_{i=1}^{m} p_i$, where $p_i \geq 1$ for all $i \in [m]$ and $m \geq 2$, then $gt(K_{p_1,p_2,...,p_m} \overline{K}_{p_1,p_2,...,p_m}) = n - 1$.

Proof: We can observe that maximal geodesics of $K_{p1, p2,...,pm} \overline{K}_{p1, p2,...,pm}$ are of the following forms:

$M_1 : u_{ij} - u_{rq} - u_{is}$, where $i, r \in [m], j, s \in [p_i]$ and $q \in [p_r]$,

$M_2 : \bar{u}_{ij} - u_{ij} - u_{rs} - \bar{u}_{rs}$ where $i, r \in [m], j \in [p_i]$ and $s \in [p_r]$, and

$M_3 : u_{ij} - \bar{u}_{ij} - \bar{u}_{is}$, where $i \in [m]$ and $j, s \in [p_i]$.

Figure 4 depicts the complementary prism of a complete multipartite graph, in which maximal geodesics described above are represented by distinct colours. Consider the set $\{u_{11}\} \cup V(K_{p1,p2,...,pm})\setminus\{u_{11}, u_{21}\}$ and it hits all maximal geodesics of $K_{p1,p2,...,pm} \overline{K}_{p1,p2,...,pm}$, so $gt(K_{p1,p2,...,pm} \overline{K}_{p1,p2,...,pm}) \leq n-1$. Now, let $K \subseteq V(K_{p1, p2,...,pm} \overline{K}_{p1, p2,...,pm})$ be such that $|K| < n - 1$. Then there exist $i, j \in [m]$ and $r, s \in [p_i]$ or $[p_j]$ such that $\{u_{ir}, \bar{u}_{ir}, u_{js}, \bar{u}_{js}\} \cap K = \phi$. If i=j, then vertices u_{ir} and u_{js} are not adjacent but their partners $\bar{u}_{ir}$ and $\bar{u}_{js}$ are adjacent. This would result in a maximal geodesic of type $M_3 : u_{ir} - \bar{u}_{ir} - \bar{u}_{js}$ which never hits K. Similarly, if $i \neq j$, then vertices u_{ir} and u_{js} are adjacent but their partners $\bar{u}_{ir}$ and $\bar{u}_{js}$ are not adjacent. This would result in a maximal geodesic $M_2 : \bar{u}_{ir} - u_{ir} - u_{js} - \bar{u}_{js}$ which never hits K. Hence K can never be a geodesic transversal set of $G\overline{G}$, so $gt(K_{p1, p2,...,pm} \overline{K}_{p1, p2,...,pm}) \geq n - 1$ So, the result is valid.

We can conclude the section after discussing the problem for complementary prisms of complete split graphs. A split graph is a graph whose vertex set can be partitioned into a clique and an independent set, thereby combining two extremal structures within a single framework. In particular, split graphs are closely related to bipartite graphs: while bipartite graph prohibits all edges within each partite set, split graphs relax this condition by allowing one part to induce a complete subgraph, thereby extending the bipartite paradigm.

Let $S_{p, q}$ be a complete split graph with a vertex set partition $P = \{u_1, ..., u_p\}$ and $Q = \{v_1,...,v_q\}$ with cardinality P and q, respectively. Here, $[P] \cong K_p$ and $[Q] \cong K_q$ and all the vertices in P are adjacent to all the vertices in Q. It is clear that $gt(S_{p,q}) = p$ and $\bar{S}_{p,q} \cong \bar{K}_p \cup K_q$. The complementary prism of split graphs exhibits some comparable types of maximal geodesics as in case of bipartite graphs. Along with them there are some additional variations as well. Due to the significant increase in connectivity of the vertices, the number of maximal geodesics in complementary prism of a split graph grows substantially.

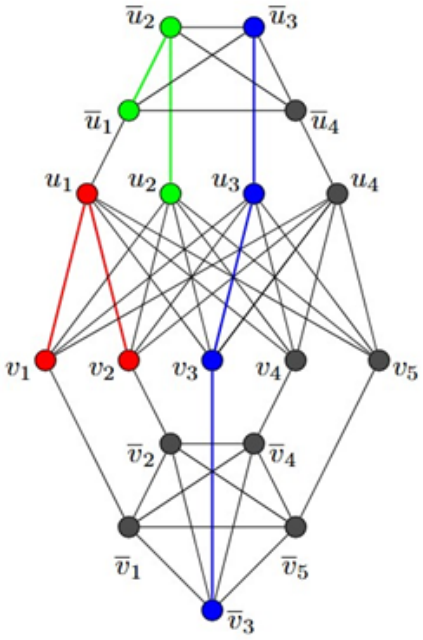


Figure 3: The complementary prism of a complete bipartite graph $K_{4,5}\overline{K}_{4,5}$, illustrating the distinct possible maximal geodesic types in the graph, each represented in a distinct colour

Theorem 8: Let $S_{p,q}$ be a complete split graph with order n = p + q. Then

$gt(S_{p,q}\bar{S}_{p,q}) = n - 1$.

Proof: We can observe that maximal geodesics of the complementary prism of a split graph are:

$M_1 : v_i - u_k - v_j$,

$M_2 : v_i - \bar{v}_i - \bar{v}_j$,

$M_3 : \bar{u}_k - u_k - u_l - \bar{u}_l$, and

$M_4 : \bar{u}_k - u_k - v_i - \bar{v}_i$, where i, j ∈ [q] and K, 1 ∈[p].

Figure 5 presents the complementary prism of a complete split graph, with all the maximal geodesics identified above indicated by distinct colours. Consider the set of vertices $S = V(S_{p,q}) \setminus \{u_i\}$ for some $i \in [p]$. Clearly, S hits all the maximal geodesics of $S_{p,q}\bar{S}_{p,q}$ and $|S| = n - 1$.

Hence, $gt(S_{p,q}\bar{S}_{p,q}) \leq n - 1$. Now we let $K \subseteq V(S_{p,q}\bar{S}_{p,q})$ be such that $|K| < n - 1$, then there exist two vertices x and y in $V(S_{p,q})$ such that $\{x,y,\bar{x},\bar{y}\} \cap K = \emptyset$. If $\{x,y\} \subseteq P$, then there exist a maximal geodesic of type $M_3 : \bar{u}_i - u_i - u_j - \bar{u}_j$, for some $i, j \in [p]$ in $S_{p,q}\bar{S}_{p,q}$ that never hits Y. If $\{x, y\} \subseteq Q$, then there exists a maximal geodesic of type $M_2 : v_i - \bar{v}_i - \bar{v}_j$, for some $i, j \in [q]$ in $S_{p,q}\bar{S}_{p,q}$ that never hits Y. Finally, if $x \in P$ and $y \in Q$, then there is a maximal geodesic of type $M_4 : \bar{u}_i - u_i - v_j - \bar{v}_j$, for some $i \in [p], j \in [q]$ in $S_{p,q}\bar{S}_{p,q}$ that never hits K. So, K cannot be a geodesic transversal set of $S_{p,q}\bar{S}_{p,q}$. Hence, $gt(S_{p,q}\bar{S}_{p,q}) \geq n - 1$, which concludes the theorem.

Wheel and Fan graphs

We now look for the geodesic transversal number of complementary prisms of wheel graphs and fan graphs. In addition to that, we will solve the problem for wheel graphs and fan graphs as well. A wheel graph denoted by W_n, $n \geq 4$ is obtained by joining all vertices of a cycle C_{n-1} to a new vertex, say a, where $V(C_{n-1}) = \{0, 1, \ldots, n - 2\}$ with natural adjacency. This construction makes wheel graphs an extension of cycle graphs, preserving the cyclic structure while introducing a vertex that significantly enriches the graph's connectivity. Motivated by the techniques developed in the study of cycle graphs, we naturally led to investigating the problem in wheel graphs.

Theorem 9: For $n \geq 6$,

$$gt(W_n) = \begin{cases} \frac{n+4}{3}; & n = 3m + 2, m \geq 2, \\ \left\lceil \frac{n+1}{3} \right\rceil; & \text{otherwise.} \end{cases}$$

Proof: We can observe that $W_4 \cong K_4$ and hence $gt(W_4) = 3$. In case of W_5, {0,1} is a geodesic transversal with the least possible cardinality, so $gt(W_5) = 2$. Now we focus on $n \geq 6$, the possible maximal geodesics of W_n are of the form:

$M_1 : i - (i + 1)(\mathrm{mod}\, n - 1) - (i + 2)(\mathrm{mod}\, n - 1)$, for $0 \leq i \leq n - 2$, and

$M_2 : i - a - j$ for $0 \leq i, j \leq n - 2$ and $|i - j| \geq 2$.

For $m \geq 2$, if $n = 3m$, let $S = \{2, 5, \ldots, n - 4\} \cup \{0, a\}$, if $n = 3m + 1$, then let $S = \{2, 5, \ldots, n - 2\} \cup \{a\}$ and if $n = 3m + 2$, then let $S = \{2, 5, \ldots, n - 2\} \cup \{0, a\}$. For $n = 3m + 2$, $|S| = \frac{n+4}{3}$ and is a geodesic transverse of W_n so $gt(W_n) \leq \frac{n+4}{3}$. For the other two cases, $|S| = \left\lceil \frac{n+1}{3} \right\rceil$ and is again a geodesic transverse of W_n so $gt(W_n) \leq \left\lceil \frac{n+1}{3} \right\rceil$.

For the converse, let $K \subseteq V(W_n)$ be a geodesic transversal of W_n. Suppose that $|K| < \frac{n+4}{3}$ when $n = 3m + 2$, and $|K| < \left\lceil \frac{n+1}{3} \right\rceil$ otherwise. If $a \notin K$, then K needs $n - 3$ vertices from $V(W_n)$, so that it hits all the maximal geodesics in W_n. But $n - 3 \geq \frac{n+4}{3}$ and $n - 3 \geq \left\lceil \frac{n+1}{3} \right\rceil$ in the respective cases and hence $a \in K$. Now, there will be a vertex $i (\neq a) \in V(W_n)$ such that $i, (i + 1)(\mathrm{mod}\, n - 1), (i + 2)(\mathrm{mod}\, n - 1) \notin K$ forms a maximal geodesic of the form M_1. Thus, we have a contradiction and K cannot be a geodesic transversal of W_n in both cases. So, $gt(W_n) \geq \frac{n+4}{3}$ if $n = 3m + 2$ and $gt(Wn) \geq \left\lceil \frac{n+1}{3} \right\rceil$ if otherwise. Hence, the result is valid.

Next, we present the solution for the problem in the case of fan graphs. A fan graph denoted by F_n, $n \geq 3$ is obtained by joining all vertices of a path graph P_{n-1} to a new vertex, say a, where $V(P_{n-1}) = \{0, 1, \ldots, n - 2\}$. A fan graph has a construction similar to that of a wheel graph. As a wheel graph is formed by joining every vertex of a cycle to a single universal vertex, similarly a fan graph is obtained by replacing the cycle with a path graph while retaining the universal vertex. A fan graph may also be derived from a wheel graph by the deletion of single edge from the cycle, thereby transforming the cyclic rim into a path. This structural relationship between the two classes, motivates to extend the results of wheel graphs to fan graphs under appropriate modifications.

Theorem 10: For $n \geq 4$, $gt(F_n) = \left\lceil \frac{n}{3} \right\rceil$.

Proof: We can observe that $F_3 \cong K_3$ and the set {a, 1} is a geodesic transversal set of F_4, from which we can clearly see that $gt(F_3) = gt(F_4) = 2$. Now we focus only on the case $n \geq 5$. The possible maximal geodesics of F_n will be of the forms:

$M_1 : i - (i + 1) - (i + 2)$, and $M_2 : i - a - (i + k)$, where $0 \leq i \leq n - 4$ and $2 \leq k \leq n - 2 - i$.

Let $S = \left\{a, 2, 5, 8, \ldots, 3\left\lceil \frac{n-1}{3} \right\rceil - 1\right\}$. Clearly $|S| = \left\lceil \frac{n}{3} \right\rceil$ and S hits all maximal geodesics of F_n. So, $gt(F_n) \leq \left\lceil \frac{n}{3} \right\rceil$.

Now, we let $K \subseteq V(F_n)$ be a geodesic transversal of F_n and $|K| < \left\lceil \frac{n}{3} \right\rceil$. If $a \notin K$, then K requires at least $n - 3 \geq \left\lceil \frac{n}{3} \right\rceil$ vertices to preserve its geodesic transversal property, which is not possible. So, $a \in K$, and therefore there are at most $\left\lceil \frac{n}{3} \right\rceil - 2$ vertices in $K \cap V(P_{n-1})$. Then there exists $0 \leq i \leq n - 4$ such that $i, i + 1, i + 2 \notin K$ produces a maximal geodesic of type M_1 that does not hit the set K. Hence, we

have a contradiction and $gt(F_n) \geq \left\lceil \frac{n}{3} \right\rceil$, which concludes the proof.

Now, we are in a position to address the problem concerning complementary prisms of wheel graphs. The presence of central vertex in wheel graphs enhances the connectivity and thus gives a different result on its geodesic transversal problem, compared to cycle graphs. In case of complementary prisms of wheel graphs, although the maximal geodesics arising from the complement of the rim are preserved, the remaining maximal geodesics are altered. Furthermore, the central vertex contributes to a significant number of maximal geodesics.

Theorem 11: For $n \geq 4$,

$$gt(W_n \bar{W}_n) = \begin{cases} n-1; 4 \leq n \leq 6, \\ n; n \geq 7. \end{cases}$$

Proof: We can observe that sets $\{\bar{0}, \bar{1}, \bar{2}\}$, $\{a, 0, \bar{1}, 2\}$, and $\{0, 1, \bar{2}, \bar{3}, 4\}$ are geodesic transversal sets of $W_4\bar{W}_4$, $W_5\bar{W}_5$, and $W_6\bar{W}_6$, respectively. So, the result is clear if $4 \leq n \leq 6$. Now we focus only on the case where $n \geq 7$. Note that $\bar{W}_n \cong \bar{C}_{n-1} \cup \{\bar{a}\}$ and that all maximal geodesics of W_n and $\bar{C}_{n-1}$ are maximal in their complementary prisms. So, by Corollary 1, geodesic transversal of $W_n\bar{W}_n$ requires $\left\lceil \frac{n}{2} \right\rceil$ vertices from $V(\bar{W}_n)$. In addition, for $0 \leq i \leq n-2$, $W_n\bar{W}_n$ contains maximal geodesics of types: $M_1: i - \bar{i} - \overline{(i \pm 1) (\mathrm{mod}\, n-1)}$, $M_2: i - \bar{i} - \overline{i \pm k (\mathrm{mod}\, n-1)}$, for $2 \leq k \leq n-3$, $M_3: \bar{i} - \overline{(i+k) (\mathrm{mod}\, n-1)} - \overline{(i+1) \mathrm{mod}(n-1)}$, $3 \leq k \leq n-3$ and $M_4: \bar{a} - a - i - \bar{i}$. Figure 6 depicts all the complementary prisms of wheel graphs and highlights its every maximal geodesic using distinct colours.

Consider the set,

$S = \left\{0, 2, 4, \ldots, \overline{2\left\lfloor \frac{n-2}{2} \right\rfloor}\right\} \cup \left\{a, 1, 3, \ldots, 2\left\lfloor \frac{n-1}{2} \right\rfloor - 1\right\}$, we can observe that it hits all maximal geodesics of $W_n\bar{W}_n$. Clearly $|S| = n$ and therefore $gt(W_n\bar{W}_n) \leq n$. In contrast, let $K \subseteq V(W_nW_n)$ be a geodesic transversal of $W_n\bar{W}_n$ and $|K| < n$. We need $\left\lceil \frac{n}{2} \right\rceil$ from $V(\bar{W}_n)$ to cover the maximal geodesics of $\bar{W}_n$ leaving the $\left\lceil \frac{n-2}{2} \right\rceil$ vertices in K to work with. If $a \notin K$, then K requires $n - 2 > \left\lceil \frac{n-2}{2} \right\rceil$ vertices from $V(W_n)$, so that it hits all maximal geodesics of W_n, which is impossible, so $a \in K$.

So now we have $|K \cap V(\bar{W}_n)| \geq \left\lceil \frac{n}{2} \right\rceil$ and $1 \leq |K \cap V(W_n)| \leq \left\lceil \frac{n-2}{2} \right\rceil$. If $|K \cap V(\bar{W}_n)| = n-2$, then there exists $0 \leq i \leq n-2$, such that $\bar{i} \notin K \cap V(\bar{W}_n)$. Then K never hits maximal geodesic of type $M_1: \bar{i} - i - \overline{(i \pm 1) (\mathrm{mod}\, n-1)}$. If $|K \cap V(\bar{W}_n)| = n-2$, then there exist two vertices from $V(\bar{W}_n)$ which are not an element of vertex set K. If both the vertices are consecutive, namely $\bar{i}$ and $\overline{(i+1) (\mathrm{mod}\, n-1)}$ for some $0 \leq i \leq n-2$. Then K needs at least n - 4 vertices from $V(\bar{W}_n)$, so that it hits all maximal geodesic of type $M_3: \bar{i} - \overline{(i+k) (\mathrm{mod}\, n-1)} - \overline{(i+1) (\mathrm{mod}\, n-1)}$, where $3 \leq k \leq n-3$. Also, for K to cover maximal geodesics $\bar{i} - i - \overline{(i-1) (\mathrm{mod}\, n-1)}$, $\bar{i} - i - \overline{(i+1) (\mathrm{mod}\, n-1)}$, $\overline{(i+1) (\mathrm{mod}\, n-1)} - \overline{(i+1) (\mathrm{mod}\, n-1)} - i$, $\overline{(i+1) (\mathrm{mod}\, n-1)} - \overline{(i+1) (\mathrm{mod}\, n-1)} - \overline{(i+2) (\mathrm{mod}}$ and $i - \overline{(i+1) (\mathrm{mod}\, n-1)} - \overline{(i+2) (\mathrm{mod}\, n-1)}$, it requires at least two more vertices from $V(W_n)$, namely i and $(i+1) (\mathrm{mod}\, n)$, this along with vertex a, completes the quota of n - 1 vertices in K. But this leaves 3 consecutive vertices $(i+2) (\mathrm{mod}\, n-1)$, $(i+3) (\mathrm{mod}\, n-1)$ and $(i+4) (\mathrm{mod}\, n-1)$ away from K. Hence a maximal geodesic $(i+2) (\mathrm{mod}\, n-1) - (i+3) (\mathrm{mod}\, n-1) - (i+4) (\mathrm{mod}\, n-1)$ never hits K.

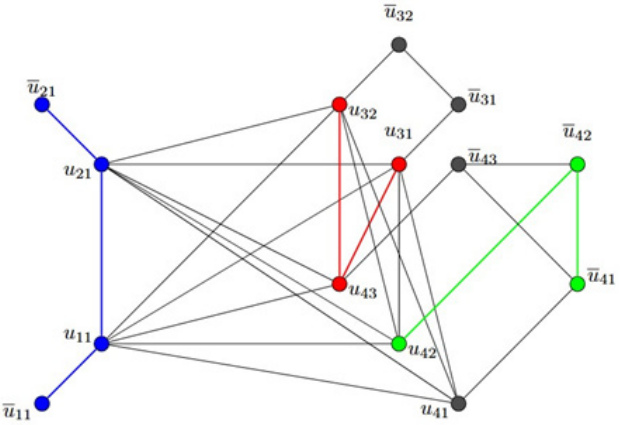


Figure 4: The complementary prism of a complete multipartite graph $K_{1,1,2,3}\bar{K}_{1,1,2,3}$, illustrating the distinct possible maximal geodesic types in the graph, each represented in a distinct colour

So, now we know if two vertices of $V(\bar{W}_n)$ is not in K, then it cannot be consecutive. So, for all $\bar{j} \in V(\bar{W}_n)$, such that $\bar{j} \notin K$, K must contain the partner vertex, j of $\bar{j}$, so that it hits all maximal geodesics $j - \bar{j} - \overline{(j \pm 1) (\mathrm{mod}\, n-1)}$. For that, set K, along with vertex a requires at least n vertices, which is a contradiction. Hence, K cannot be a geodesic transversal set of $W_n\bar{W}_n$, so $gt(W_n\bar{W}_n) \geq n$, which confirms the result.

At this stage, we are prepared to investigate the geodesic transversal problem of complementary prisms of fan graphs. Maximal geodesics in complementary prisms of fan graphs closely resemble those in complementary prisms of wheel graphs and fan graphs. They preserve almost all maximal geodesics of the underlying fan graphs, as well as those arising from the complement of the rim. Consequently, a result analogous to that obtained for wheel graphs is anticipated.

Theorem 12: For $n \geq 3$,

$$gt(F_n\bar{F}_n)=\begin{cases} n-1; & 3 \leq n \leq 6 \\ n; & n \geq 7. \end{cases}$$

Proof: We can observe that sets $\{\bar{0}, \bar{1}\}$, $\{a, \bar{0}, 1, 2\}$, $\{0, \bar{1}, \bar{2}, 3\}$ and $\{\bar{0}, \bar{1}, 2, 3, 4\}$ are geodesic transversal sets of $F_3\bar{F}_3$, $F_4\bar{F}_4$, $F_5\bar{F}_5$ and $F_6\bar{F}_6$, respectively. Thus, we can find $gt(F_3\bar{F}_3)=2$ $gt(F_4\bar{F}_4)=3$, $gt(F_5\bar{F}_5)=4$ and $gt(F_6\bar{F}_6)= 5$. Now we only consider the case where $n \geq 7$. It is clear that all maximal geodesics of F_n and all maximal geodesic of length 3 of $\bar{P}_{n-1}$ are maximal in $F_n\bar{F}_n$. So, the remaining possible maximal geodesics in $F_n\bar{F}_n$ are of the form: $M_1: i - i - (i+k)$, where $0 \leq i \leq n-4$ and $2 \leq k \leq n-i-2$, $M_2: \bar{i} - i - (i \pm 1)$, where $0 \leq i \leq n-3$, and $M_3: \bar{a} - a - i - \bar{i}$, where $0 \leq i \leq n-2$. Figure 7 illustrates the complementary prisms of fan graphs, highlighting each maximal geodesic in a distinct colour.

Let,

$$S = \left\{\bar{1}, \bar{3}, \ldots, \overline{2\left\lfloor\frac{n-1}{2}\right\rfloor - 1}\right\} \cup \left\{a, 0, 2, \ldots, 2\left\lfloor\frac{n-2}{2}\right\rfloor\right\},$$

clearly $|S| = n$ and S hits all maximal geodesics in $F_n\bar{F}_n$. So, $gt(F_nF_n) \leq n$.

Now, we assume that $K \subseteq V(F_n\bar{F}_n)$ is a geodesic transversal set of $F_n\bar{F}_n$ such that $|K| < n$. Since all maximal geodesics of $\bar{P}_{n-1}$ are maximal in the $F_n\bar{F}_n$, $\left\lfloor\frac{n-1}{2}\right\rfloor$ vertices of $V(\bar{F}_n)$ must be in K. If $a \notin K$, then we require additional $n-3$ vertices of $V(F_n)$ to cover maximal geodesics of type $i - a - i + k$, where $0 \leq i \leq n-4$ and $2 \leq k \leq n-2-i$, which does not fit the predefined cardinality of K so, $a \in K$. Since, $|K| < n$ there exist $0 \leq i \leq n-2$ such that, vertices $i, \bar{i} \notin K$. We can observe that the edge $i\bar{i}$ can be extended to all other vertices of $V(F_n)$ or their partners to obtain a maximal geodesic. It follows that, for each $0 \leq j \leq n-2$ and $i \neq j$, exactly one of j or $\bar{j}$ belongs to the set K. But the set K specifically needs vertex $(i+1)$ and vertex $(i-1)$so that it hits maximal geodesic M_2. So, vertex $\overline{(i+1)}$ or vertex $\overline{(i-1)}$ will not be an element of set K. So, now we are in the situation where we need at least $n-4$ vertices of $V(\bar{F}_n)$ so that the set K hits all maximal geodesics of types $\bar{0} - \overline{(n-1)} - \bar{1}$, $\overline{(n-1)} - \bar{0} - \overline{(n-2)}$, and $\bar{j} - \overline{(j+k)(\mathrm{mod}\, n)} - \overline{(j+1)(\mathrm{mod}\, n)}$, where $3 \leq k \leq n-3$ and $0 \leq j \leq n-2$. But, this leaves a path of length 2 in F_n, that is a maximal geodesic in $F_n\bar{F}_n$ which never hits K. This is in contradiction to the geodesic transversal property of K. So, $gt(F_n\bar{F}_n) \geq n$, and therefore the result is valid.

Butterfly graph and Butterfly network

In this concluding section, we examine the geodesic transversal problem in the context of the butterfly graph, its complementary prism, and the butterfly network. By adopting the methodologies utilized in the previous sections, we find the geodesic transversal number for both the butterfly graph and its complementary prisms. Finally,

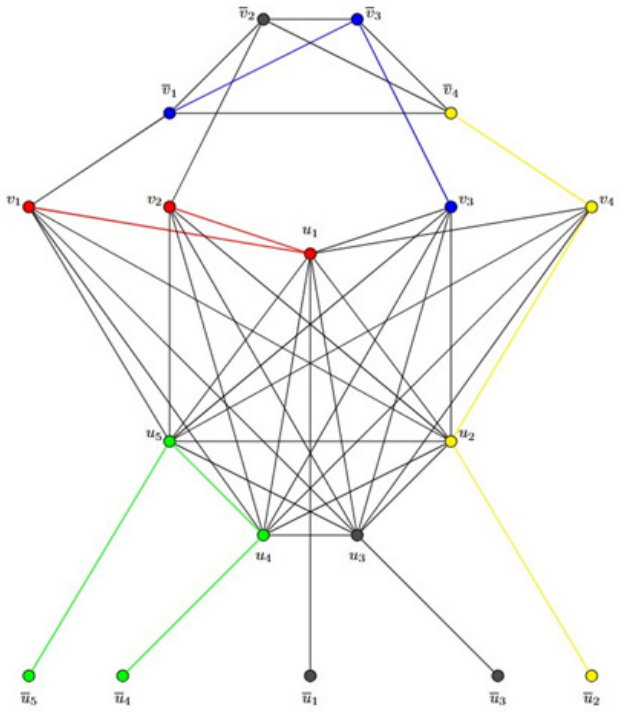


Figure 5: The complementary prism of a complete split graph $S_{5,4}\bar{S}_{5,4}$, illustrating the distinct possible maximal geodesics types in the graph, each represented in a distinct colour

we establish an upper bound for the geodesic transversal number of the butterfly network, thus concluding the discussion.

The graph obtained by joining all the vertices of paths P_n, $n \geq 2$ and P_m, $m \geq 2$ to a vertex, say a, where $V(P_n)$ and $V(P_m)$ are disjoint, is called a butterfly graph and is denoted by B(n, m). Let $V(P_n) = \{u_0, u_1, \ldots, u_{n-1}\}$ and $V(P_m) = \{v_0, v_1, \ldots, v_{m-1}\}$. A butterfly graph is a natural extension of a fan graph. As a fan graph connects all vertices of a path graph to a single vertex, the butterfly graph joins all vertices of two disjoint path graphs to a single vertex. Motivated by the insights and techniques developed in the study of path graphs and fan graphs, we can explore the corresponding properties of butterfly graphs, which would unify the earlier results within a boarder framework.

Theorem 13: For $n, m \geq 2$,

$$gt(B(n,m))=\begin{cases} 3; & n=m=2, \\ \left\lfloor\frac{n}{3}\right\rfloor + 2; & n \neq m = 2, \\ \left\lfloor\frac{n}{3}\right\rfloor + \left\lfloor\frac{m}{3}\right\rfloor + 1; & \text{otherwise.} \end{cases}$$

Proof: If $n=m=2$, then $u_0 - u_1$, $v_0 - v_1$ and $v_i - a - u_j$, where $i, j \in \{0, 1\}$ are the possible maximal geodesics in B(n, m), to cover them all we require at least 3 vertices and so $gt(B(2, 2)) = 3$. Thus, the result is valid for the case $n=m=2$.

We can observe that for B(n, m) the possible maximal geodesics are of the following forms: $M_1: u_i - u_{i+1} - u_{i+2}$, where $0 \le i \le n-3$, $M_2: v_j - v_{j+1} - v_{j+2}$, where $0 \le j \le m-3$, $M_3: u_i - a - u_{i+k}$, where $0 \le i \le n-3$ and $2 \le k \le n-1-i$, $M_4: v_j - a - v_{j+k}$, where $0 \le j \le m-3$ and $2 \le k \le m-1-j$, and $M_5: u_i - a - v_j$, where $0 \le i \le n-2$ and $0 \le j \le m-2$.

If m = 2, then $v_0 - v_1$ will be an additional maximal geodesic in B(n, m). Now, if $n \ne m = 2$, then $S = \{v_0, a, u_2, u_5, \ldots, u_{3\lfloor \frac{n}{3} \rfloor - 1}\}$ hits all maximal geodesics in B(n, 2) and $|S| = \lfloor \frac{n}{3} \rfloor + 2$, so $gt(B(n, 2)) \le \lfloor \frac{n}{3} \rfloor + 2$. Now, we let $K \subseteq V(B(n, 2))$ be a geodesic transversal set of B(n, 2) and $|K| < \lfloor \frac{n}{3} \rfloor + 2$. Since $v_0 - v_1$ is a maximal geodesic in $B(n, 2)$ one among v_0, v_1 must be in K, say $v_0 \in K$. If $a \notin K$, then K requires at least $n + 1 > \lfloor \frac{n}{3} \rfloor + 1$ vertices to cover all maximal geodesics of type $v_1 - a - u_i$, where $0 \le i \le n-1$, $u_i - u_{i+1} - u_{i+2}$, where $0 \le i \le n-3$, and $u_i - a - u_{i+k}$, where $0 \le i \le n-3$ and $2 \le k \le n-1-i$, which is not possible and hence $a \in K$. So, we can have at most $\lfloor \frac{n}{3} \rfloor - 1$ vertices in $\{u_0, u_1, \ldots, u_{n-1}\} \cap K$. But this is less than one-third of the total vertices in $\{u_0, u_1, \ldots, u_{n-1}\} \cap K$. Then there exists $0 \le i \le n-3$ such that $u_i - u_{i+1} - u_{i+2}$ does not hits K, which will contradict the geodesic transversal property of K. So, $gt(B(n, 2)) \ge \lfloor \frac{n}{3} \rfloor + 2$ and hence the result.

Now, if $n, m > 2$. Let,

$$S = \{a, u_2, u_5, \ldots, u_{3\lfloor \frac{n}{3} \rfloor - 1}\} \cup \{v_2, v_5, \ldots, v_{3\lfloor \frac{m}{3} \rfloor - 1}\}$$

we can observe that $|S| = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 1$, clearly S hits all the maximal geodesics of B(n, m) and hence $gt(B(n, m)) \le \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 1$. Now for the converse, let $K \subseteq V(B(n, m))$ be a geodesic transversal set of B(n, m) and $|K| < \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 1$. If $a \notin K$, then K requires all vertices in the set V(B(n, m)) \ {a} except one to cover all maximal geodesics of type $u_i - a - v_j$, where $0 \le i \le n-2$ and $0 \le j \le m-2$. Thus, set K needs $n + m - 2 > \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 1$ vertices to do so, which is not possible and hence $a \in K$. So, we are left with at most $\lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor - 1$ vertices from K to cover all maximal geodesics of types M_1 and M_2. But this is less than one-third of the total vertices in V(B(n, m)).

Hence, it results in the existence of $0 \le i \le n-3$ or $0 \le j \le m-3$ such that $u_i - u_{i+1} - u_{i+2}$ or $v_j - v_{j+1} - v_{j+2}$ does not hits K, resulting in another contradiction. So, $gt(B(n, m)) \ge \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 1$ and hence the result.

Now we investigate the problem for the complementary prism of butterfly graphs. Since a butterfly graph is a generalization of fan graphs, most maximal geodesic types of complementary prisms of fan graphs are preserved in complementary prisms of butterfly graph. However, the enhanced connectivity of butterfly graphs gives rise to many additional types of maximal geodesics. We can observe that no maximal geodesic in $\bar{B}(2, 2)$ is maximal in its complementary prism.

Theorem 14: For n, m ≥ 2,

$$gt(B(n, m)\bar{B}(n, m)) = \begin{cases} n+2; & n \le 7, m = 2, \\ n+m+1; & \text{otherwise.} \end{cases}$$

Proof: For notational convenience, we take $V(B(n, m)) = \{a\} \cup V(P_n) \cup \overline{V(P_m)}$. If n≤7, m=2, then let $S = \{a, v_0, v_1\} \cup \{u_{r-1}, u_{r+1}\} \cup \overline{V(P_n) \setminus \{u_{r-1}, u_r, u_{r+1}\}}$ where $r = \lfloor \frac{n}{2} \rfloor$. Clearly, |S| = n + 2 and hits all maximal geodesics of B(n, m)$\bar{B}$(n, m). A subset of

VB(n, m)$\bar{B}$(n, m) with cardinality less than n + 2 will leave behind two vertices of VB(n, m) along with their partners, say $\{x, y, \bar{x}, \bar{y}\}$. For given vertices x and y, irrespective of their adjacency, we can construct maximal geodesics of types $x - y - \bar{y}$, $\bar{x} - x - y - \bar{y}$ or $\bar{x} - \bar{y} - y$ each of which never hits a vertex set with cardinality less than n + 2. It follows that the result holds true in this case.

If $n, m \ne 2$, then all maximal geodesics of B(n, m) and $\bar{B}$(n, m) are maximal in B(n, m)$\bar{B}$(n, m). In addition, there are maximal geodesics of types $M_1: \bar{a} - a - u_i - \bar{u}_i$ for $0 \le i \le n-1$, $M_2: \bar{a} - a - v_j - \bar{v}_j$ for $0 \le j \le m-1$, $M_3: u_i - u_{i+1} - \bar{u}_{i+1}$ for $0 \le i \le n-2$, $M_4: v_i - v_{i+1} - \bar{v}_{j+1}$ for $0 \le j \le m-2$, $M_5: u_i - \bar{u}_i - \bar{v}_j$ for $0 \le i \le n-1$ and $0 \le j \le m-1$, $M_6: v_j - \bar{v}_j - \bar{u}_i$ for $0 \le i \le n-1$ and $0 \le i \le m-1$, $M_7: u_i - \bar{u}_i - \bar{u}_{i+k(\text{mod } n)}$ for $0 \le i \le n-1$ and K≥2, and $M_8: v_j - \bar{v}_j - \bar{v}_{j+k(\text{mod } m)}$ for $0 \le j \le m-1$ and k ≥ 2.

If m = 2, then B(n, m)$\bar{B}$(n, m) will be lacking a maximal geodesic involving only vertices from $\overline{V(P_m)}$, all other maximal geodesics will be the same as in the previous case.

let

$S_1: \{\bar{u}_1, \bar{u}_3, \ldots, \bar{u}_{2\lfloor \frac{n}{2} \rfloor - 1}\}$, $S_2: \{\bar{v}_1, \bar{v}_3, \ldots, \bar{v}_{2\lfloor \frac{m}{2} \rfloor - 1}\}$ and $S = S_1 \cup S_2 \cup [V(B(n, m)) \setminus \overline{S_1 \cup S_2}]$. Clearly, S is a geodesic transversal set of B(n, m)$\bar{B}$(n, m), |S| = n + m + 1 and hence $gt(B(n, m)\bar{B}(n, m)) \leq n + m + 1$. Now, for the converse, let $K \subseteq V(B(n, m)\bar{B}(n, m))$ be a geodesic transversal set of B(n, m)$\bar{B}$(n, m) such that $|K| < n + m + 1$, that is, K can have at most n + m vertices. Then there exists a vertex $x \in V(B(n, m)\bar{B}(n, m))$ such that $x, \bar{x} \notin K$.

Case 1: If x = a, then to cover maximal geodesics of type $u_i - a - v_j$ set K requires all vertices of $V(B(n, m)) \setminus \{a\}$, except two. Without loss of generality, let u_l and u_{l+1} for some $0 \leq l \leq n - 1$, be two vertices of $V(B(n, m)) \setminus \{a\}$ that need not be necessarily in set K. Now, K needs 2 more vertices to cover the maximal geodesics $u_l - u_{l+1} - u_{l+1}$, $u_{l+1} - u_l - \bar{u}_l$, $\bar{a} - a - u_l - \bar{u}_l$ and $\bar{a} - a - u_{l+1} - \bar{u}_{l+1}$. Finally, K needs at least 1 more vertices to cover the geodesics of types $\bar{u}_i - \bar{u}_{i+k} - \bar{u}_{i+1}$, $\bar{v}_j - \bar{v}_{j+k} - \bar{v}_{j+1}$, $\bar{u}_i - \bar{u}_{i+k} - \bar{v}_j$ and $\bar{v}_j - \bar{v}_{j+k} - \bar{u}_i$ which exceeds the prescribed upper bound on |K|, thereby yielding a contradiction.

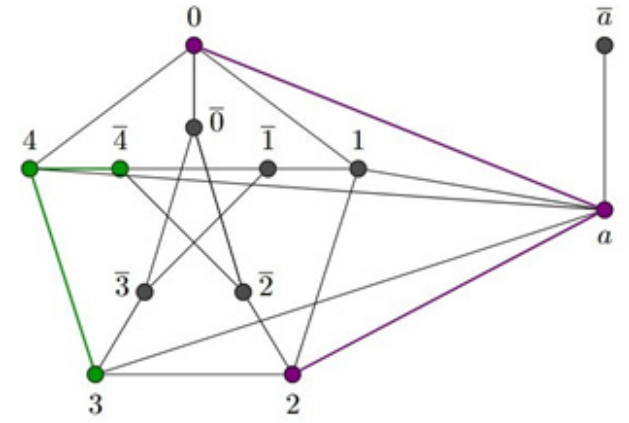


Figure 6 (A): $W_6\bar{W}_6$

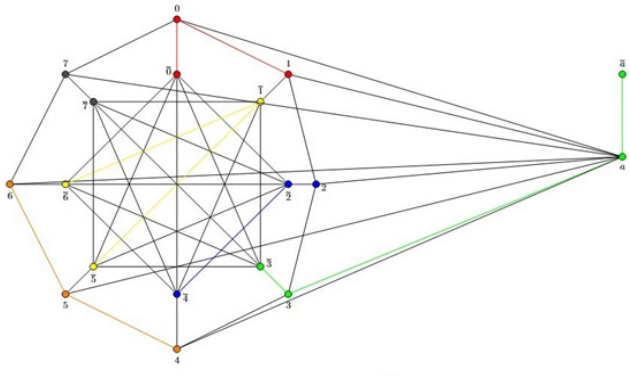


Figure 6 (B): $W_9\bar{W}_9$

Figure 6: The complementary prism of wheel graphs illustrating the distinct possible maximal geodesics types in the graph $W_n\bar{W}_n$, each represented in a distinct colour

Case 2: If $x \neq a$, without loss of generality, let $x = u_i$ for some $0 \leq i \leq n - 1$. Then, to cover all maximal geodesics of type $u_i - \bar{u}_i - \bar{v}_j$, k requires all m vertices of set $\{\bar{v}_0, \bar{v}_1, \ldots, \bar{v}_{m-1}\}$. Now, to cover all maximal geodesics of type $u_{i\pm1} - u_i - u_i$, K requires vertices $\{u_{i-1}, u_{i+1}\}$ and to cover maximal geodesics of type $u_i - \bar{u}_i - \bar{u}_{i+k}$, K requires vertex set $\overline{V(P_n) \setminus \{u_{i-1}, u_i, u_{i+1}\}}$, which makes another n - 1 vertices. Finally, to cover maximal geodesic $\bar{a} - a - u_i - \bar{u}_i$, K require one more vertex which completes its quota of n + m vertices. Now it is impossible for K to cover maximal geodesics of type $v_j - v_{j+1} - v_{j+2}$, hence we have a contradiction. Since a contradiction occurs in both cases, the result follows as asserted.

Now we discuss the problem for the butterfly network. The butterfly network is a fundamental class of interconnection networks that has been extensively studied in graph theory. The structural features render the butterfly network a canonical model in study of routing, sorting, and network design problem. Butterfly networks provide rich examples of path structures, embedding, and various covering properties. Hence, the butterfly network serves as an important framework within which diverse combinatorial phenomena can be explored.

The butterfly network $BF(n), n \geq 1$, is a graph network that contains n + 1 rows of vertices; each row contains 2^n vertices. (i, j) represents j^{th} vertex in i^{th} row, where $1 \leq i \leq n + 1$ and $1 \leq j \leq 2^n$. The vertex (i, j_1) and $(i + 1, j_2)$ in BF(n) for $1 \leq i \leq n$ is adjacent if and only if

a) $j_1 = j_2$, or

b) $|j_1 - j_2| = 2^{(i-1)}$, for $k2^i < j_1, j_2 \leq (k + 1)2^i$ and $0 \leq k \leq 2^{n-1} - 1$.

Some examples of butterfly networks are illustrated in Figure 8. We can observe that $|V(BF(n))| = (n + 1)2^n$ and that every vertex (i, j) is adjacent to the 4 vertices, $(i + 1, j), (i - 1, j), (i + 1, j \pm 2^{(i-1)})$ and $(i - 1, j \pm 2^{(i-2)})$ where $2 \leq i \leq n$ and $1 \leq j \leq 2^n$ but the vertices of the first and last rows are adjacent to only two vertices. The vertex (1, j) is adjacent to (2, j) and $(2, j \pm 1)$ and the vertex (n + 1, j) is adjacent to (n, j) and $(n, j \pm 2^{(n-1)})$. In Figure 8(A), we have BF(1) which is isomorphic to C_4. In Figure 8(B) and Figure 8(C) we can observe the complexity of BF(2) and BF(3), respectively.

There are even-length cycles consisting of a pair of vertices from the $(n + 1)^{th}$ row or a pair from the 1^{st} row. The cycles are as follows:

If $1 \leq i, j \leq 2^{(n-1)}$ or $2^{(n-1)} < i, j \leq 2^n$,

$(n+1, i) - (n, i) - \cdots - (n, j) - (n+1, j) - (n, j \pm 2^{(n-1)}) - \cdots - (n, i \pm 2^{(n-1)}) - (n+1, i)$

If $1 \leq i \leq 2^{(n-1)}$ and $j > 2^{(n-1)}$,

$(n+1, i) - (n, i) - \cdots - (n, j-2^{(n-1)}) - (n+1, j) - (n, j) - \cdots - (n, i+2^{(n-1)}) - (n+1, i)$

If i and j are odd,

$(1, i) - (2, i) - \cdots - (2, j) - (1, j) - (2, j+1) - \cdots - (2, i+1) - (1, i)$

If i and j are even,

$(1, i) - (2, i-1) - \cdots - (2, j-1) - (1, j) - (2, j) - \cdots - (2, i) - (1, i)$

If i is odd and j is even,

$(1, i) - (2, i) - \cdots - (2, j-1) - (1, j) - (2, j) - \cdots - (2, i-1) - (1, i)$

Now, we are clear about some cycles in the butterfly network; using this, we can conclude about some of its maximal geodesics in the next theorem.

Theorem 15: Every geodesic between two vertices of the $(n + 1)^{th}$ row or between any two vertices of the first row is a maximal geodesic.

Proof: Let P be a geodesic between two vertices $(n+1, i)$ and $(n + 1, j)$ in BF(n). The adjacent vertices of the vertex $(n +1, i)$, namely (n, i) and $(n, i + 2^{(n-1)} (\mathrm{mod}\ 2^n))$. One of these vertices will be followed by $(n + 1, i)$ in P.

Let the length of P be K. Then by previous observation P is in a cycle C of length 2k and C contains both adjacent vertices of $(n + 1, i)$. If $(n + 1, j)$ is followed by (n, i) in P, then P can only be extended to $(n, i + 2^{(n-1)} (\mathrm{mod}\ 2^n))$, but from C we can find a shorter $(n, i + 2^{(n-1)} (\mathrm{mod}\ 2^n)) - (n + 1, j)$ path than P. Similarly, if $(n +1, i)$ is followed by $(n, i + 2^{(n-1)} (\mathrm{mod}\ 2^n))$ in P, then P can only be extended to (n, i), but from C we can find a shorter $(n, i) - (n +1, j)$ path than P. Hence P cannot be extended to a longer geodesic on the left. Similarly, P cannot be extended to a longer geodesic on the right. Hence P is a maximal geodesic.

Now, let P be geodesic $(1, i) - (1, j)$ in BF(n) and let the length of P be K. The vertex $(1, i)$ is adjacent to only two vertices and they are $(2, i)$ and $(2, i \pm 1)$. We know that P is part of a cycle C of length 2k and contains all adjacent vertices of $(1, i)$. If $(1, i)$ is followed by $(2, i)$ in P, we can find a geodesic $(2, i \pm 1) - (1, j)$ shorter than P in C. If $(1, i)$ is followed by $(2, i \pm 1)$ in P, we can find a geodesic $(2, i) - (1, j)$ shorter than P in C. Hence P cannot be extended to a longer geodesic on the left. Similarly, P cannot be extended to a longer geodesic on the right. Hence P is a maximal geodesic.

We can also observe that every vertex of the butterfly network, say (r,s) is a part of the maximal geodesic of length 2n. In general, $(n + 1, s) - (n, s) - \cdots - (r, s) - \cdots - (1, s) - (2, s \pm 1) - (3, s \pm 1) - \cdots - (n+1, s \pm 1) - (n + 1, s)$, is the maximal geodesic of BF(n) of length 2n containing vertex (r, s). Now we prelude to find an upper bound for the geodesic transversal number of a butterfly network.

Theorem 16: For $n \geq 1$, $\mathrm{gt}(\mathrm{BF}(n)) \leq \left\lceil \frac{n+1}{3} \right\rceil 2^n$

Proof: If $n = 1$, then $\mathrm{BF}(n) \cong C_4$, and hence $\mathrm{gt}(\mathrm{BF}(n)) = 2$ and the result is true. If $n = 2$, then BF(n)will have 3 rows and each row have 4 vertices. Let S be the set of all vertices in the 2^{nd} row. Then all the edges in BF(n) hits , and so will all the maximal geodesics. Hence $\mathrm{gt}(\mathrm{BF}(n)) \leq |S| = 4$ and the result holds.

Now, we are concerned only about $n \geq 3$, then let S be the collection of vertices of the 2^{nd} and n^{th} rows, together with the vertices of the fewest additional rows chosen from rows 4 to n - 2 such that any one of the three adjacent rows remains in S entirely. Then $|S| = \left\lceil \frac{n+1}{3} \right\rceil 2^n$.

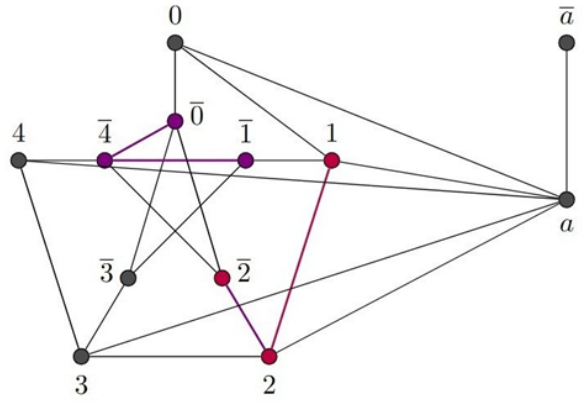


Figure 7 (A): $F_6\bar{F}_6$

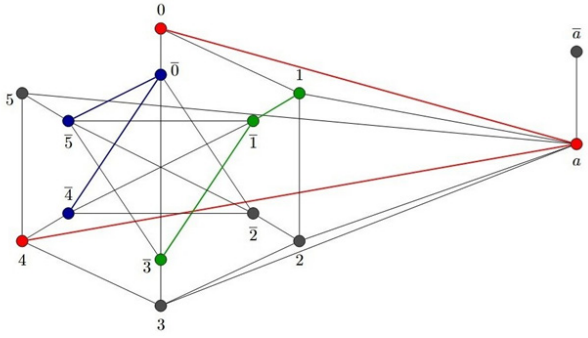


Figure 7 (B): $F_7\bar{F}_7$

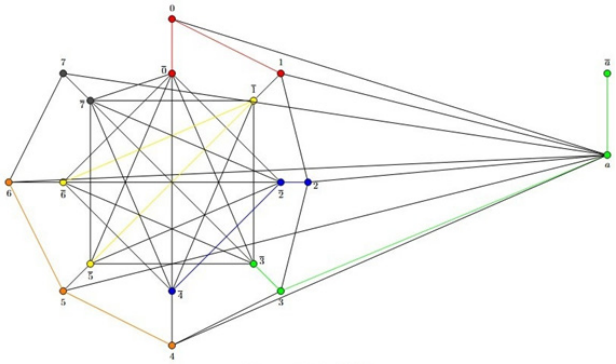


Figure 7 (C): $F_9\bar{F}_9$

Figure 7: The complementary prism of fan graphs illustrating the distinct possible maximal geodesic types in the graph $F_n\bar{F}_n$, each represented in a distinct colour

Let M be a geodesic that does not hits S. Then M will be between two adjacent rows in BF(n). A geodesic between two adjacent rows will have a maximum length of 2. If the rows are i and i - 1, then M will have a general form;

$M: (i-1, j) - (i, j) - (i-1, j \pm 2^{(i-2)})$ or

$M: (i+1, j) - (i, j) - (i+1, j \pm 2^{(i-1)})$.

But both geodesics can be extended to a longer geodesic, respectively

$(1, j) - \cdots - (i-1, j) - (i, j) - (i-1, j \pm 2^{(i-2)}) - \cdots - (1, j \pm 2^{(i-2)})$

or

$(n+1, j) - \cdots - (i+1, j) - (i, j) - (i+1, j \pm 2^{(i-1)}) - \cdots - (n+1, j \pm 2^{(i-1)})$

Therefore, M is not maximal. So, a geodesic that does not hit S is not maximal. So, S is a geodesic transversal set in BF(n). Hence $gt(BF(n)) \leq |S| = \left\lceil \frac{n+1}{3} \right\rceil 2^n$.

The set of vertices S mentioned in Theorem 15 is a minimal geodesic transversal set of BF(n). We can observe that in case of $1 \leq n \leq 2$, S is the minimum cardinality geodesic transversal set of BF(n). Now we focus only in the case of $n \geq 3$. For that remove a vertex from the 2nd row of S, say (2, i). Then $(i, j) - (2, j) - (1, j \pm 1)$ is a maximal geodesic that does not hits S. Now, if we remove a vertex in the n^{th} row of S, say (n, j), then $(n+1, j) - (n, j) - (n+1, j \pm 2^{(n-1)})$ is a maximal geodesic that does not hit S.

Let $4 \leq i \leq n-2$ and i^{th} row entirely be in S. Now we remove a vertex, say (i, j) in the i^{th} row from S. Then there will be two rows near the i^{th} row that is not part of S. Without loss of generality, let them be row i - 1 and row i - 2. Hence, the maximal geodesic

$P: (i-1, j \pm 2^{(i-2)}) - (i, j) - (i-1, j) - (i-2, j) - (i-1, j \pm 2^{(i-3)})$

never hits S.

We know that $(i-1, j \pm 2^{(i-2)})$ will have 3 adjacent vertices other than (i, j), and are $(i, j \pm 2^{(i-2)})$, $(i-2, j \pm 2^{(i-2)})$ and $(i-2, j \pm 2^{(i-2)} \pm 2^{(i-3)})$. In addition, $(i-1, j \pm 2^{(i-3)})$ will have 3 adjacent vertices other than $(i-2, j)$ and they are $(i, j \pm 2^{(i-3)})$, $(i-2, j \pm 2^{(i-3)})$ and $(i, j \pm 2^{(i-3)} \pm 2^{(i-2)})$. Then the possible geodesics that connect adjacent vertices of $(i-1, j \pm 2^{(i-2)})$ and vertex $(i-1, j \pm 2^{(i-3)})$ are; $(i, j \pm 2^{(i-2)}) - (i-1, j) - (i-2, j \pm 2^{(i-3)}) - (i-1, j \pm 2^{(i-3)})$,

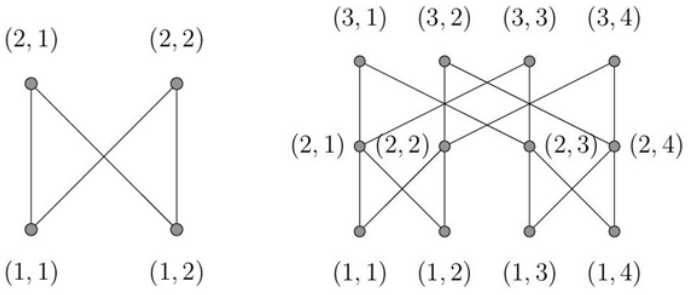


Figure 8 (A): BF(1)

Figure 8 (B): BF(2)

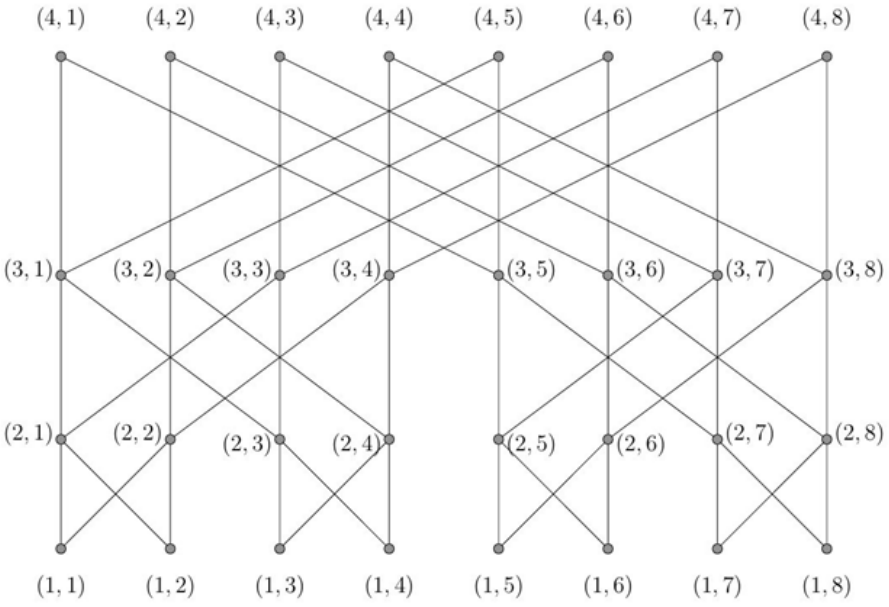


Figure 8 (C): BF(3)

Figure 8: Example configurations of the butterfly network BF(n)

$(i-2, j \pm 2^{(i-2)}) - (i-1, j \pm 2^{(i-2)} \pm 2^{(i-3)}) - (i, j \pm 2^{(i-3)}) - (i-1, j \pm 2^{(i-3)})$, and $(i-2, j \pm 2^{(i-2)} \pm 2^{(i-3)}) - (i-1, j \pm 2^{(i-2)} \pm 2^{(i-3)}) - (i, j \pm 2^{(i-3)}) - (i-1, j \pm 2^{(i-3)})$

The possible geodesics connecting adjacent vertices of $(i-1, j \pm 2^{(i-3)})$ and vertex $(i-1, j \pm 2^{(i-2)})$ are; $(i, j \pm 2^{(i-3)}) - (i-1, j \pm 2^{(i-3)} \pm 2^{(i-2)}) - (i-2, j \pm 2^{(i-2)}) - (i-1, j \pm 2^{(i-2)})$, $(i-2, j \pm 2^{(i-3)}) - (i-1, j) - (i, j \pm 2^{(i-2)}) - (i-1, j \pm 2^{(i-2)})$,

and

$(i, j \pm 2^{(i-3)} \pm 2^{(i-2)}) - (i-1, j \pm 2^{(i-3)} \pm 2^{(i-2)}) - (i-2, j \pm 2^{(i-3)} \pm 2^{(i-2)}) - (i-1, j \pm 2^{(i-2)})$

All of these geodesics have a length less than that of P. So, P cannot be extended to a longer geodesic, and so P is maximal. So, if we avoid a vertex (i, j) from S, it loses its geodesic transversal property.

RESULTS AND DISCUSSION

The available studies on the geodesic transversal problem investigated the problem for various graph classes (Peterin & Semanišin, 2021 & Manuel et al., 2022), graph networks (Manuel et al., 2023b), and graph operations (Peterin & Semanišin, 2022). Our study, initially motivated by cycle graphs C_n and path graphs P_n, of diameter $\left\lfloor \frac{n}{2} \right\rfloor$ and n - 1, respectively. However, the study gradually shifted its focus to bipartite, multipartite, split, wheel, fan, and butterfly graphs, viewed as generalizations of cycles and paths, to examine how the results vary with increasing connectivity among vertices. Interestingly, most of these graphs have diameter two, which arises as a coincidence rather than a deliberate restriction. Notably, complementary prisms themselves form graphs of diameter three. This work also obtains an upper bound for the butterfly network, which will complement the existing results on network graphs. These results indicate that the diameter of a graph plays an important role in determining its geodesic transversal number, which will be a key to upcoming studies.

CONCLUSION

In this work, we have investigated the geodesic transversal problem in complementary prisms of graphs of order n, and observed that the geodesic transversal number in all examined cases attains either n - 1 or n. This consistent behaviour encourages one to explore whether or not this characterization extends to all graphs. Through this analysis, we have gained a clearer structural understanding of maximal geodesics in graphs of diameter two and three. These insights indicate promising directions for further research, and we propose, as a continuation of this work, a systematic study of the geodesic transversal number for graphs with diameter two and three, to establish more general results in this setting. A key result is an upper bound for the geodesic transversal number of the butterfly network, shown to be approximately one-third of its order. Future work will focus on sharpening these bounds and determining exact values for wider classes of graphs, with the aim of a more unified understanding of the geodesic transversal problem.

DECLARATION OF CONFLICTS OF INTERESTS

The authors confirm that they have no conflicts of interest to declare.

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AUTHOR CONTRIBUTION

Designing and planning of the research: **A.M.G. & T.J.R.;** Concept, literature search, data analysis, article preparation, and correspondence: **A.M.G.;** Reviewed all results and approved the final version: **T.J.R.** All authors have accepted responsibility for the content of the manuscript.

DATA AVAILABILITY STATEMENT

No datasets were generated or analysed during the current study. The results presented in this paper are purely theoretical, and all relevant information is contained within the manuscript.

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