J.Natn.Sci.Foundation Sri Lanka 2023 52 (1): 17 - 34
DOI: http://dx.doi.org/10.4038/jnsfsr.v52i1.11227

RESEARCH ARTICLE

Statistical Quality Control

The modified control charts for monitoring the error detection of process control under different estimators

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Submitted: 10 June 2022; Revised: 16 September 2023; Accepted: 24 November 2023

Abstract: The reflected power function distribution (RPFD) has increasing importance in practical life due to its application in diversified fields of life. Organisations often face difficulty monitoring operations to identify and remove errors during production. That is why there is a need to introduce control charts that effectively monitor the processes, mainly when the number of errors follows RPFD and the manufacturing process is in control. The current study suggested memory-based control charts as a solution to the problem. The control charts are based on the estimation methods and play a remarkable role in enhancing the machine process reliability. The parameters of RPFD are evaluated through the percentile estimator (PE) and modified maximum likelihood estimator methods (MMLM). Further, we create memory-based control charts, *i.e.*, hybrid exponentially weighted moving average (HEWMA) and extended exponentially weighted moving average (EEWMA), using the PE and MMLM. The findings reflect that HEWMA control charts based on PE provide a better result in estimating the defects. The implications of the study will be helpful for practitioners and policy makers from reliability engineering, management sciences, and statisticians.

Keywords: Control charts, machine errors, manufacturing process, modified maximum likelihood estimator, percentile estimator, reflected power function distribution.

INTRODUCTION

The reflected power function was introduced by Zaka *et al*. (2020) as a modification to power function distribution. The reason for introducing the reflected power function distribution is to enhance the application of the power function distribution in reliability, medical, and engineering sciences. The probability density function (PDF) of the reflected power function distribution is given as

$$f(x) = \frac{\gamma(\theta - x)^{\gamma-1}}{\beta^{\gamma}}, \quad \theta - \beta < x < \theta, \text{ and } \beta, \theta, \gamma > 0,$$

where θ is the reflecting parameter that will reflect the distribution towards positive skewed to negative skewed or negative skewed to positive skewed. Also γ and β are the shape and scale parameters, respectively.

The RPFD is more flexible to fit the data sets obtained from the medical and engineering sciences,

The RPFD can be negative-skewed or positive-skewed, whereas the HRF can be J-shape, monotonically increasing and decreasing shapes, which shows its more flexible nature to fit any data set. The graphs for the probability density function and hazard rate function (HRF) are given in Figure 1 and Figure 2, respectively.

We see that the shape parameter of the RPFD plays a vital role in defining the shape of any process in reliability, medical, and engineering sciences. A little shift in the shape parameter affects the shape of the distribution of any process. It is essential to control these small shifts in the process. We use statistical process control (SPC)

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to control the small shifts in the process that follows reflected power function distribution.

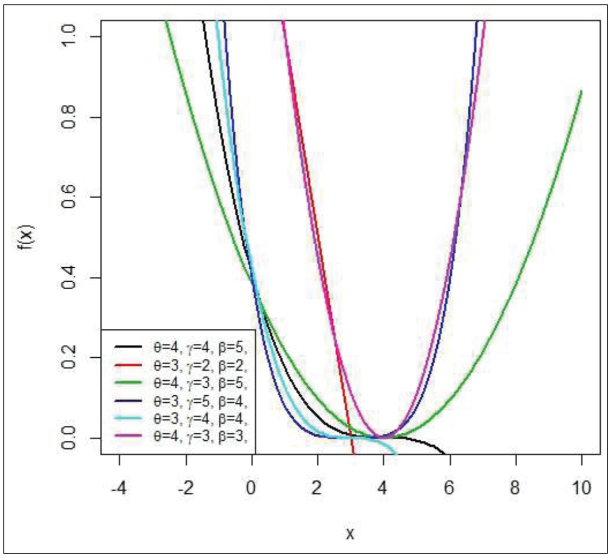

Figure 1: PDF plots for RPFD.

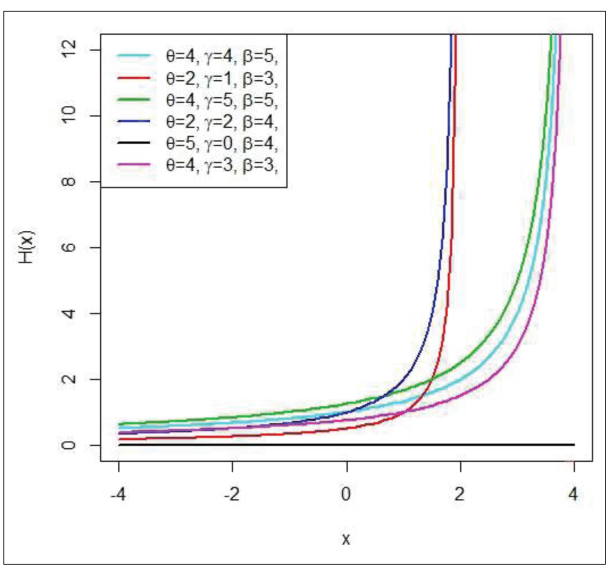

Figure 2: HRF plots for RPFD

Quality control issues have gained the attention of scholars and practitioners in the recent era. It is an important topic of discussion as the credibility of firms depends on it, and it is also considered a source of customer retention. But in real life, sometimes firms fail to accurately assess the error trends during production, affecting product quality and, ultimately, reputation. That is why the managers and reliability engineers are keen to determine the failure trends for the manufacturing units to confirm the validity of pre-production testing results in real-life. One of the examples is from the pharmaceutical industry, where life-saving drugs are manufactured. The monitoring of the production process is necessary at every stage. The companies adopt strict checking procedures to avoid any mishaps. So, the quality control check in these types of industries is essential not only for the firms' credibility but also for the well-being of humans.

Similarly, another example is from the software industry, which is expected to be error-free when delivered in the market. That is why it has become essential for the product to pass through the careful quality assessment processes during testing. Whenever defects in the machine appear, they will be handled skillfully so that the final product will be perfect. Therefore, the companies try to control the errors, but the underlying distribution of errors may not be normal and often follows a reflected power function distribution (RPFD).

The current study tries to introduce the control charts that can assist the firms in identifying and eliminating the variations in the production process through well-established monitoring of the product reliability when the number of errors follows RPFD.

Prior studies related to the control charts mostly followed the normality assumption, such as the hybrid exponential weighted moving average (HEWMA) control charts due to Shamma and Shamma (1992) and Haq (2013). When the distribution of the process is normal, the extended exponentially weighted moving averages (EEWMA) control chart was introduced by Naveed *et al* (2018). But what can organizations do if the normality assumption fails in practical life. There are very few studies available in the literature that highlighted this issue, including Qiu and Li (2011), Noorossana *et al*. (2016), Zhang *et al*. (2017), Lin *et al*. (2017), Erto *et al*. (2018), Li *et al*. (2018), and Liang *et al*. (2019). As many production processes do not necessarily follow the normality assumption, the study aims to introduce the memory-based control charts, *i.e.*, HEWMA and EEWMA control charts assume that the underlying process distribution follows an RPFD. Further, the proposed control charts are based on the different estimators of the shape parameter of RPFD. We use the shape parameter of the RPFD instead of the usual mean as the process parameter. The percentile estimator (PE) and modified maximum likelihood method (MMLM) estimator are discussed by Zaka *et al*. (2020) for the shape parameter of RPFD, which is used to construct the plotting statistics. Both are equally efficient, and the comparison is made to see which performs better in statistical process control. Monte Carlo simulation is used to estimate the shape parameter of the RPFD as well as the plotting statistics. Average run lengths are used to evaluate the performances of each of the control charts. It is observed that the proposed EEWMA control chart of the PE detects smaller shifts quicker than the HEWMA control charts.The simulation study and real-life application are discussed in the end.

MATERIALS AND METHODS

In this section, the percentile estimator (PE) and modified maximum likelihood method (MMLM) to estimate the shape parameter of RPFD are discussed using Zaka *et al*. (2020). Then, some traditional HEWMA and EEWMA are modified to monitor the shape parameter of RPFD using PE and MMLM.

Proposed estimators of process monitoring for an RPFD

The following section uses PE and MMLM to construct memory-based control charts to monitor the shape parameter of a process that follows an RPFD. From Zaka *et al*. (2020), it is assumed that the process random variables $x_1, x_2, x_3, ..., x_t$ are independently and identically distributed following RPFD, with the probability density function (PDF) and cumulative density function (CDF) given respectively by

$$f(x) = \frac{\gamma(\theta-x)^{\gamma-1}}{\beta^{\gamma}}, \theta-\beta < x < \theta, \text{and } \beta, \theta, \gamma > 0,$$

and

$$F(x) = 1 - \frac{(\theta-x)^{\gamma}}{\beta^{\gamma}},$$

where θ is the reflecting parameter that will reflect the distribution towards positive skewed to negative skewed or negative skewed to positive skewed. Also γ *and* β are the shape and scale parameters.

MMLM and PE estimators for the shape parameter of RPFD defined by Zaka *et al*. (2020) are given below.

$$\hat{\gamma}_{MMLM} = \left(\frac{n(1+ln\,(0.5\,))}{(n\,ln\,(\theta-\bar{x})-\sum_{i=1}^{n} ln\,(\theta-x_i))}\right). \quad ...(1)$$

$$\hat{\gamma}_{PE} = \frac{ln\left(\frac{1-H}{1-L}\right)}{ln\left(\frac{\theta-P_H}{\theta-P_L}\right)}. \quad ...(2)$$

Traditional hybrid exponentially weighted moving average (HEWMA) control chart

Let the distribution of the underlying process having the sequence $\{X_t\}$ be normal. Also, let $0 \le \lambda_i \le 1$ for i = 1, 2 be a known constant. Now, consider a new sequence HW_t as

$$\text{HW}_t = \lambda_1 \text{W}_t + (1-\lambda_1)\text{HW}_{t-1}, \quad ...(3)$$

where

$$\text{W}_t = \lambda_2 \hat{\mu}_t + (1-\lambda_2)W_{t-1}, \quad ...(4)$$

where

$\text{HW}_0 = W_0 = \mu$ (population mean) and HW_t is a plotting statistic. By placing (4) in (3), we get the following

$$HW_t = \lambda_1\lambda_2 \sum_{i=0}^{t-1} (1-\lambda_1)^i \sum_{j=0}^{t-i-1} (\frac{1-\lambda_1}{1-\lambda_2})^j\, Y_i +$$

$$\lambda_1 \sum_{i=0}^{t-1} (1-\lambda_1)^i (1-\lambda_2)^{t-i}\, \mu + (1-\lambda_1)^t \mu.$$

The mean and the variance of HW_t are given below as

$$E(HW_t) = \mu$$

where μ is the population mean and

$$V(HW_t) = \left(\frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_2)}\right)^2 \left[\sum_{i=1}^{2} (1-\lambda_1)^2(1-(1-\lambda_i)^{2t}/1-(1-\lambda_i)^2) - \frac{2(1-\lambda_1)(1-\lambda_2)\{1-(1-\lambda_1)^t(1-\lambda_2)^t\}}{1-(1-\lambda_1)(1-\lambda_2)}\right]\sigma^2,$$

Where σ^2 is the population variance.

The control limits for the HEWMA control chart are explained as

$$UCL_{\text{HW}_t} = \mu + L\sqrt{V(HW_t)}$$

and

$$UCL_{\text{HW}_t} = \mu + L\sqrt{V(HW_t)}.$$

Proposed HEWMA control chart based on PE of the shape parameter of RPFD

The HEWMA statistic using PE of the shape parameter of the RPFD is given by

$$HEW\mathrm{PE}_t = \lambda_1 EW\mathrm{PE}_t + (1-\lambda_1) HEW\mathrm{PE}_{t-1}, \quad ...(5)$$

where $EW\mathrm{PE}_t$ is a usual EWMA statistic based on PE, given as

$$EW\mathrm{PE}_t = \lambda_2 \hat{\gamma}_{\mathrm{PE(t)}} + (1-\lambda_2) EWPE_{t-1}, \quad ...(6)$$

and $HEW\mathrm{PE}_{t-1}$ is the statistic on previous time. Also λ_1 and λ_2 are smoothing constants here. The control limits are given for the HEWMA control chart using PE for the shape parameter of the RPFD; we get,

$$\mathrm{HEWPE}_{\boldsymbol{t}} = \lambda_1 \Big(\lambda_2 \hat{\gamma}_{\mathrm{PE}(t)} + (1-\lambda_2) \textstyle\sum_{j=0}^{1} \left(\frac{(1-\lambda_1)}{(1-\lambda_2)}\right)^j \hat{\gamma}_{PE(t-1)}$$

$$+ (1-\lambda_2)^2 \textstyle\sum_{j=0}^{2} \left(\frac{(1-\lambda_1)}{(1-\lambda_2)}\right)^j \hat{\gamma}_{PE(t-2)} +$$

$$(1-\lambda_2)^3 \textstyle\sum_{j=0}^{3} \left(\frac{(1-\lambda_1)}{(1-\lambda_2)}\right)^j \hat{\gamma}_{PE(t-3)} + \ldots +$$

$$(1-\lambda_2)^{t-1} \textstyle\sum_{j=0}^{t-1} \left(\frac{(1-\lambda_1)}{(1-\lambda_2)}\right)^j \hat{\gamma}_{PE(1)} \Big) +$$

$$\lambda_1 (1-\lambda_2)^t \textstyle\sum_{j=0}^{t} \left(\frac{(1-\lambda_1)}{(1-\lambda_2)}\right)^j \hat{\gamma}_{PE(0)} +$$

$$(1-\lambda_2)^t \mathrm{HEWPE}_{\boldsymbol{0}}.$$

$$\mathrm{LCL}_{\mathrm{HEWPE}_{\boldsymbol{t}}} = \gamma - \mathrm{L} * \frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_2)} \sqrt{\left(\sum_{i=1}^{2} \frac{(1-\lambda_i)^2(1-(1-\lambda_i)^{2t})}{1-(1-\lambda_i)^2} - \frac{2(1-\lambda_1)(1-\lambda_2)\{1-(1-\lambda_1)^t(1-\lambda_2)^t\}}{1-(1-\lambda_1)(1-\lambda_2)}\right) \mathrm{var}(\hat{\gamma}_{\mathrm{PE}})}$$

$$\mathrm{CL}_{\mathrm{HEWPE}_{\boldsymbol{t}}} = \gamma$$

$$\mathrm{UCL}_{\mathrm{HEWPE}_{\boldsymbol{t}}} = \gamma + \mathrm{L} * \frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_2)} \sqrt{\left(\sum_{i=1}^{2} \frac{(1-\lambda_i)^2(1-(1-\lambda_i)^{2t})}{1-(1-\lambda_i)^2} - \frac{2(1-\lambda_1)(1-\lambda_2)\{1-(1-\lambda_1)^t(1-\lambda_2)^t\}}{1-(1-\lambda_1)(1-\lambda_2)}\right) \mathrm{var}(\hat{\gamma}_{\mathrm{PE}})}.$$

Proposed HEWMA control chart of the shape parameter of RPFD using MMLM

Let $x_1, x_2, \ldots, x_n$ be a sequence of independent and identical random variables generated from a process which follows an RPFD with shape parameter "γ". It is important to note that we use the estimator of the shape parameter of the process instead of an average of observations or single observations by assuming that $E(\hat{\gamma}_{MMLM}) = \gamma$.

The HEWMA statistic by using MMLM of RPFD may be written as

$$HEW\mathrm{MMLM}_t = \lambda_1 EW_t + (1-\lambda_1) HEW\mathrm{MMLM}_{t-1} \quad ...(7)$$

where PEW_t is a usual EWMA statistic given as

$$EW\mathrm{MMLM}_t = \lambda \hat{\gamma}_{MMLM(t)} + (1-\lambda) EW\mathrm{MMLM}_{t-1},$$

where $\hat{\gamma}_{\mathrm{MMLM}}$ is the modified maximum likelihood estimator for RPFD and $EW\mathrm{MMLM}_{t-1}$ is the statistic on previous time. Also "λ" is a smoothing constant. We may get the control limits for HEWMA as

$$\mathrm{LCL}_{HEW\mathrm{MMLM}_t} = \gamma - \mathrm{L} \left(\frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_2)} \sqrt{\left(\textstyle\sum_{i=1}^{2} \frac{(1-\lambda_i)^2(1-(1-\lambda_i)^{2t})}{1-(1-\lambda_i)^2} - \frac{2(1-\lambda_1)(1-\lambda_2)\{1-(1-\lambda_1)^t(1-\lambda_2)^t\}}{1-(1-\lambda_1)(1-\lambda_2)}\right) \mathrm{Var}(\hat{\gamma}_{\mathrm{MMLM}})} \right), \quad ...(8)$$

$$\mathrm{CL}_{HEW\mathrm{MMLM}_t} = \gamma$$

$$\mathrm{UCL}_{HEW\mathrm{MMLM}_t} = \gamma + \mathrm{L}\left(\frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_2)}\sqrt{\left(\sum_{i=1}^{2}\frac{(1-\lambda_i)^2(1-(1-\lambda_i)^{2t})}{1-(1-\lambda_i)^2} - \frac{2(1-\lambda_1)(1-\lambda_2)\{1-(1-\lambda_1)^t(1-\lambda_2)^t\}}{1-(1-\lambda_1)(1-\lambda_2)}\right)\mathrm{Var}(\hat{\gamma}_{\mathrm{MMLM}})}\right). \quad ...(9)$$

Algorithm for HEWMA control charts under PE and MMLM

1. Generate random samples of size n = 150 on "X_t" from the RPFD where with parameters (β, γ, θ) = (1, 2, 1).
2. Compute $\hat{\gamma}_{*t}$ where * = PE, and MMLM.
3. Repeat steps 1 and 2 5000 times and compute $\hat{\gamma}_{*t}$, $E(\hat{\gamma}_{*t})$ and $V(\hat{\gamma}_{*t})$.
4. Repeat step 3 5000 times and compute $\hat{\gamma}_{*t}$.
5. Compute control limits to construct HEWMA control charts based on $\hat{\gamma}_{*t}$.
6. Compute the ARL value for each HEWMA control chart that is based on $\hat{\gamma}_{*t}$, given that process is an in-control state.
7. Now fix ARL_0 = 500 for the in-control state of the process, and search the suitable value of L so that ARL_0 for the in-control state of a process is achieved.
8. Now assume that the process parameter γ is shifted by its true value and compute ARL_S. This step is repeated for different shift values 0, 0.05, 0.10, 0.15, 0.20, 0.25, 0.25 0.30, and 0.35. The shifts are selected based on the change observed in the shape parameter during the simulation process. Also, compute ARL_1 in each case of shift values.
9. Plot ARL_s values against the values of shift that are used in step 7 & 8.
10. It is to be noted that the procedure of the HEWMA control chart is based on $\hat{\gamma}_{*t}$, observe whether the process following the RPFD is in-control or out of control. If the process is in-control, go to Step 1. Otherwise, record the Run Length, *i.e.*, the process remaining in control before being declared out-of-control.
11. Repeat this process 5000 times to obtain the ARLs, SDRLs, and fractiles.

The traditional extended exponentially weighted moving averages (EEWMA) control chart

When the distribution of process is normal, the EEWMA control chart was introduced by Naveed *et al*. (2018). The EEWMA control chart by Naveed *et al*. (2018) is given as

$$\mathrm{Z_t} = \lambda_1\mathrm{T_t} - \lambda_2\mathrm{T_{t-1}} + (1-\lambda_1+\lambda_2)\mathrm{Z_{t-1}}$$

where $0 \le \lambda_1 \le 1$ and $0 \le \lambda_2 \le \lambda_1$. T_{t-1}. represents the previous value of the variable and Z_{t-1} denotes the previous value of statistic.

The mean and the variance are given as

$$E(Z_t) = \mu$$

and

$$var(Z_t) = \sigma^2\left[\left(\lambda_1{}^2+\lambda_2{}^2\right)\left\{\frac{1-(1-\lambda_1+\lambda_2)^{2t}}{2(\lambda_1-\lambda_2)-(\lambda_1-\lambda_2)^2}\right\} - 2a\lambda_1\lambda_2\left\{\frac{1-(1-\lambda_1+\lambda_2)^{2t-2}}{2(\lambda_1-\lambda_2)-(\lambda_1-\lambda_2)^2}\right\}\right],$$

respectively.

Proposed EEWMA control chart based on PE of the shape parameter of RPFD

The EEWMA statistic using PE of the shape parameter of RPFD using Zaka *et al*. (2020) and Naveed *et al*. (2018) is given by

$$EEW\mathrm{PE}_t = \lambda_1\hat{\gamma}_{\mathrm{PE(t)}} - \lambda_2\hat{\gamma}_{\mathrm{PE(t-1)}} + (1-\lambda_1+\lambda_2)EEW\mathrm{PE}_{t-1}.$$

For t = 1

$$EEW\mathrm{PE}_1 = \lambda_1\hat{\gamma}_{\mathrm{PE(1)}} - \lambda_2\hat{\gamma}_{\mathrm{PE(0)}}(1-\lambda_1+\lambda_2)EEW\mathrm{PE}_0.$$

For t = 2

$$EEW\mathrm{PE}_2 = \lambda_1\hat{\gamma}_{\mathrm{PE(2)}} - \lambda_2\hat{\gamma}_{\mathrm{PE(1)}}(1-\lambda_1+\lambda_2)EEW\mathrm{PE}_1.$$

$$EEW\mathrm{PE}_2 = \lambda_1\hat{\gamma}_{\mathrm{PE(2)}} - \lambda_2\hat{\gamma}_{\mathrm{PE(1)}} + (1-\lambda_1+\lambda_2)\big(\lambda_1\hat{\gamma}_{\mathrm{PE(1)}} - \lambda_2\hat{\gamma}_{\mathrm{PE(0)}} + (1-\lambda_1+\lambda_2)EEW\mathrm{PE}_0\big).$$

Let $a = (1-\lambda_1+\lambda_2)$

$$EEW\mathrm{PE}_2 = \lambda_1\hat{\gamma}_{\mathrm{PE(2)}} + b\hat{\gamma}_{\mathrm{PE(1)}} - a\lambda_2\hat{\gamma}_{\mathrm{PE(0)}} + a^2EEW\mathrm{PE}_0.$$

$$EEW\mathrm{PE}_2 = \lambda_1\hat{\gamma}_{\mathrm{PE(2)}} + (a\lambda_1-\lambda_2)\hat{\gamma}_{\mathrm{PE(1)}} - a\lambda_2\hat{\gamma}_{\mathrm{PE(0)}} + a^2EEW\mathrm{PE}_0.$$

Let $b = (a\lambda_1 - \lambda_2)$.

$$EEW\mathrm{PE}_2 = \lambda_1\hat{\gamma}_{\mathrm{PE(2)}} + b\hat{\gamma}_{\mathrm{PE(1)}} - a\lambda_2\hat{\gamma}_{\mathrm{PE(0)}} + a^2EEW\mathrm{PE}_0.$$

$$EEWPE_3 = \lambda_1 \hat{\gamma}_{PE(3)} + b\hat{\gamma}_{PE(2)} + ab\hat{\gamma}_{PE(1)} - a^2\lambda_2\hat{\gamma}_{PE(0)} + a^3 EEWPE_0.$$

$$EEWPE_t = \lambda_1 \hat{\gamma}_{PE(t)} + b\hat{\gamma}_{PE(t-1)} + ab\hat{\gamma}_{PE(t-2)} + a^2 b\lambda_2\hat{\gamma}_{PE(t-3)} + a^3\lambda_2\hat{\gamma}_{PE(0)} + \dots + a^{t-2} b\hat{\gamma}_{PE(1)} - a^{t-1}\lambda_2\hat{\gamma}_{PE(0)} + a^t EEWPE_0.$$

By taking expectation, we get

$$E(EEWPE_t) = \lambda_1\gamma + b\gamma + ab\gamma + a^2 b\lambda_2\gamma + a^3\lambda_2\gamma + \dots + a^{t-2} b\gamma - a^{t-1}\lambda_2\gamma + a^t\gamma.$$

Replacing $b = (a\lambda_1 - \lambda_2)$,

$$E(EEWPE_t) = \gamma\big(\lambda_1 + (a\lambda_1 - \lambda_2) + a(a\lambda_1 - \lambda_2) + a^2(a\lambda_1 - \lambda_2) + \dots + a^{i-2}a^2(a\lambda_1 - \lambda_2) - a^{t-1}\lambda_2 + a^t\big).$$

$$E(EEWPE_t) = \gamma\{\lambda_1(1 + a + a^2 + a^3 + \dots + a^{t-1}) - \lambda_2(1 + a + a^2 + a^3 + \dots + a^{t-1}) + a^t\}.$$

$$E(EEWPE_t) = \gamma\{(\lambda_1 - \lambda_2)(1 + a + a^2 + a^3 + \dots + a^{t-1}) + a^t\}.$$

By using geometric series, we get

$$E(EEWPE_t) = \gamma\left\{(\lambda_1 - \lambda_2)\left(\frac{1-a^t}{1-a}\right) + a^t\right\}.$$

So, $E(EEWPE_t) = \gamma\{1 - a^t + a^t\}$

$$E(EEWPE_t) = \gamma \qquad \dots(10)$$

Since we know $Var(\hat{\gamma}_{PE(t)}) = V_{PE} = E(\hat{\gamma}_{PE} - \gamma)^2$, we get

$$Var(EEWPE_t) = \lambda_1^2 V_{PE} + b^2 V_{PE} + a^2 b^2 V_{PE} + a^4 b^2 \lambda_2^2 V_{PE} + \dots + a^{2(t-2)} b^2 V_{PE} - a^{2(t-1)}\lambda_2^2 V_{PE}.$$

Let $b = (a\lambda_1 - \lambda_2)$. Then,

$$Var(EEWPE_t) = V_{PE}\{\lambda_1^2 + (a\lambda_1 - \lambda_2)^2 + a^2(a\lambda_1 - \lambda_2)^2 + a^4(a\lambda_1 - \lambda_2)^2 + \dots + a^{2(t-2)}(a\lambda_1 - \lambda_2)^2 - a^{2(t-1)}\lambda_2^2\}.$$

$$Var(EEWPE_t) = V_{PE}\{\lambda_1^2 + a^2\lambda_1^2 - 2a\lambda_1 + \lambda_2^2 + a^4\lambda_1^2 - 2a^3\lambda_1 + a^2\lambda_2^2 + \left(a^6\lambda_1^2 - 2a^5\lambda_1^2 + a^4\lambda_2^2\right) + \dots + a^{2(t-2)}\lambda_1^2 - 2a^{2t-3}\lambda_1 + a^{2(t-2)}\lambda_2^2 + a^{2(t-1)}\lambda_2^2\}.$$

$$Var(EEWPE_t) = V_{PE}\{\lambda_1^2 + a^2\lambda_1^2 - 2a\lambda_1 + \lambda_2^2 + a^4\lambda_1^2 - 2a^3\lambda_1 + a^2\lambda_2^2 + \left(a^6\lambda_1^4 - 2a^5\lambda_1^3 + a^4\lambda_2^4\right) + \dots + a^{2(t-2)}\lambda_1^2 - 2a^{2t-3}\lambda_1 + a^{2(t-2)}\lambda_2^2 + a^{2(t-1)}\lambda_2^2\}.$$

$$Var(EEWPE_t) = V_{PE}\left\{(\lambda_1^2 + \lambda_2^2)\left(\frac{1-a^{2t}}{1-a^2}\right) - 2a\lambda_1\lambda_2\left(\frac{1-a^{2t-2}}{1-a^2}\right)\right\}.$$

$$Var(EEWPE_t) = V_{PE}\left\{(\lambda_1^2 + \lambda_2^2)\left(\frac{1-(1-\lambda_1+\lambda_2)^{2t}}{1-(1-\lambda_1+\lambda_2)^2}\right) - 2a\lambda_1\lambda_2\left(\frac{1-(1-\lambda_1+\lambda_2)^{2t-2}}{1-(1-\lambda_1+\lambda_2)^2}\right)\right\} \qquad \dots(11)$$

The control limits are given as

$$UCL_{EEWPE_t} = \gamma + L\sqrt{V_{PE}\left\{(\lambda_1^2 + \lambda_2^2)\left(\frac{1-(1-\lambda_1+\lambda_2)^{2t}}{1-(1-\lambda_1+\lambda_2)^2}\right) - 2a\lambda_1\lambda_2\left(\frac{1-(1-\lambda_1+\lambda_2)^{2t-2}}{1-(1-\lambda_1+\lambda_2)^2}\right)\right\}}$$

$$CL_{EEWPE_t} = \gamma$$

$$LCL_{EEWPE_t} = \gamma - L\sqrt{V_{PE}\left\{(\lambda_1^2 + \lambda_2^2)\left(\frac{1-(1-\lambda_1+\lambda_2)^{2t}}{1-(1-\lambda_1+\lambda_2)^2}\right) - 2a\lambda_1\lambda_2\left(\frac{1-(1-\lambda_1+\lambda_2)^{2t-2}}{1-(1-\lambda_1+\lambda_2)^2}\right)\right\}}$$

Proposed EEWMA control chart for shape parameter of RPFD using MMLM

Let $x_1, x_2, \dots, x_n$ be a sequence of independent and identical random variables generated from a process which follows a RPFD with shape parameter γ. It is important to note that we use MMLM of the shape parameter of the process instead of the average of observations assuming that $E(\hat{\gamma}_{MMLE}) = \gamma$. .

The EEWMA statistic using MMLM of the shape parameter of RPFD using Zaka *et al*. (2020) and Naveed *et al*. (2018) is given by

By using the mean and variance for the EEWMA statistic given in (10) and (11), the control limits are given as

$$UCL_{\mathrm{EEWMMLM}_t} = \gamma + L\sqrt{V_{\mathrm{MMLM}}\left\{\left({\lambda_1}^2 + {\lambda_2}^2\right)\left(\frac{1-(1-\lambda_1+\lambda_2)^{2t}}{1-(1-\lambda_1+\lambda_2)^2}\right) - 2a\lambda_1\lambda_2\left(\frac{1-(1-\lambda_1+\lambda_2)^{2t-2}}{1-(1-\lambda_1+\lambda_2)^2}\right)\right\}}$$

$$CL_{\mathrm{EEWMMLM}_t} = \gamma$$

$$UCL_{\mathrm{EEWMMLM}_t} = \gamma + L\sqrt{V_{\mathrm{MMLM}}\left\{\left({\lambda_1}^2 + {\lambda_2}^2\right)\left(\frac{1-(1-\lambda_1+\lambda_2)^{2t}}{1-(1-\lambda_1+\lambda_2)^2}\right) - 2a\lambda_1\lambda_2\left(\frac{1-(1-\lambda_1+\lambda_2)^{2t-2}}{1-(1-\lambda_1+\lambda_2)^2}\right)\right\}}$$

$$CL_{\mathrm{EEWMMLM}_t} = \gamma$$

Algorithm for EEWMA control charts under PE and MMLM

1. Generate a random sample of size n = 150 on X_t from the RPFD, *i.e.*, $x = \theta - \beta(1- R)^{\frac{1}{\gamma}}$ with parameters (β,γ,θ) = (1, 2, 1).
2. Compute $\hat{\gamma}_*$ where * = PE and MMLM.
3. Repeat steps 1 and 2 5000 times and compute $E(\hat{\gamma}_*)$ and $V(\hat{\gamma}_*)$.
4. Repeat step 3 5000 times and compute the average of $E(\hat{\gamma}_*)$ and $V(\hat{\gamma}_*)$.
5. Compute control limits for EEWMA control chart based on $\hat{\gamma}_*$.
6. Compute ARL value for each EEWMA control chart that based on $\hat{\gamma}_*$ given that process is in the in-control state.
7. Now fix ARL_0 = 500 for in-control state of the process and search the suitable value of L so that ARL_0 for in-control state of process is achieved.
8. Now assume that the process parameter γ is shifted from its true value and then compute ARL_1. This step is repeated for different shift values 0, 0.05, 0.10, 0.15, 0.20, 0.25, 0.30, and 0.35. The shifts are selected based on the change observed in shape parameter during the simulation process. Also compute ARL_1 for each of the shift values.
9. Plot ARL_s values against the values of shift which are used in step 7 and 8.
10. It is to be noted that the procedure of EEWMA control chart is based on $\hat{\gamma}_*$ observe whether the process following the RPFD is in-control or out of control. If the process is in-control, go to Step 1. Otherwise, record the run length (RL), *i.e.*, the process remained in-control before it is declared to be out-of-control.
11. Repeat this process 5000 times to obtain the ARLs, SDRLs, and fractiles.

RESULTS AND DISCUSSION

Interpretation

From Figure 3 to Figure 8, we observe that EEWMA control charts based on PE and MMLM for the shape parameter of RPFD perform more efficiently as compared to the PE and MMLM based HEWMA control charts. In order to compare the performance of EEWMA and HEWMA control charts based on PE and MMLM, different choices of λ_1 (0.3, 0.5 and 0.9) and λ_2 (0.2, 0.6, and 0.75) are used. The ARL values for the EEWMA control chart based on MMLM and PE are given in Table 3 and Table 4. From Table 4, it is observed that for larger of values of λ_1, ARL values are getting larger. For instance, taking, λ_1 = 0.30, λ_2 = 0.20 and shift = 0.05, ARL for EEWMA based on PE is 5.291, and taking λ_1 = 0.90, λ_2 = 0.20, and shift = 0.05, ARL is 6.312. Now from this picture it is clearly noted that the ARL for the EEWMA control chart increases if the value of λ_1 is increased, and this is similar to the kind of performance of the MMLM based EEWMA control chart for larger values of λ_1. However this conclusion remains the same for the PE based HEWMA control chart if, in Table 2, we compare the ARLs taking λ_1 = 0.30, λ_2 = 0.20, and shift = 0.05 with the ARLs taking λ_1 = 0.50, λ_2 = 0.20, and shift = 0.4. One point that is interesting to note is that the PE based EEWMA control chart is consistently getting smaller ARLs than those of the MMLM based EEWMA and HEWMA control charts for each value of λ_1 as well as for each shift value. It is clear from Figure 4 to Figure 6 that the line of ARLs for the PE based EEWMA control chart remains below the line of ARLs for the MMLM based EEWMA control chart. So we can say that the PE based EEWMA control chart performs more efficiently than the MMLM based EEWMA control charts, to detect an out of control state of a process having RPFD.

The ARLs of EEWMA based on MMLM and PE presented respectively in Tables 3 and 4 are compared with the ARLs of HEWMA based on MMLM and PE. It is clearly observed that the ARLs of PE based EEWMA are smaller than the ARLs of PE based HEWMA, taking shift to be 0.05, $\lambda_1 = 0.3$, and $\lambda_2 = 0.20$. For instance, see that the ARL for PE based EEWMA is 5.291 at shift = 0.4, $\lambda_1 = 0.3$, $\lambda_2 = 0.2$, while the ARL for PE based HEWMA is 18.191 at shift = 0.05 and $\lambda = 0.20$. So it is clear from the ARLs presented in Tables 1 to 4 that ARLs for the PE based EEWMA control charts remains smaller than the PE based HEWMA control chart as well as the MMLM based EEWMA and HEWMA control charts.

Furthermore, it is also observed that use of PE makes each type of control chart more efficient as compared to the control charts that are based on MMLM for monitoring a process which is following RPFD.

Table 1: ARL_s using MMLM estimators for the parameters of RPFD using HEWMA control chart.

Estimation methods		Shift 0	0.05	0.10	0.15	0.20	0.25	0.30	0.35
HEWMA	ARL	500.848	23.08	13.36	12.57	1.10	1.005	1.001	1
λ_1= 0.30,	SDRL	483.706	7.5235	1.408	0.5807	0.3130	0.0772	0.0316	0
λ_2= 0.20	P10	53.90	6	2	1	1	1	1	1
L= 3.10	P25	157.50	8	2	1	1	1	1	1
	P50	358.50	13	3	2	1	1	1	1
	P75	712.25	18	4	2	1	1	1	1
	P90	1117.50	24	5	2	2	1	1	1
HEWMA	ARL	500.865	30.533	13.19	11.38	1.045	1.002	1	1
λ_1= 0.30,	SDRL	511.57	20.6742	1.605	0.530	0.2074	0.0446	0	0
λ_2= 0.60	P10	46.00	5	1	1	1	1	1	1
L= 3.08	P25	149.75	9	2	1	1	1	1	1
	P50	334.50	17	3	1	1	1	1	1
	P75	712.25	33	4	2	1	1	1	1
	P90	1159.00	52	5	2	1	1	1	1
HEWMA	ARL	500.618	34.77	13.38	11.41	1.051	1.002	1.001	1
λ_1= 0.30,	SDRL	512.501	20.548	1.768	0.551	0.2201	0.0446	0.031	0
λ_2= 0.75	P10	46.90	4	1	1	1	1	1	1
L= 3.08	P25	144.25	11	2	1	1	1	1	1
	P50	329.5	18	3	1	1	1	1	1
	P75	713.25	38	4	1	1	1	1	1
	P90	1164.6	59	6	1	1	1	1	1
HEWMA	ARL	500.836	24.50	13.21	11.46	1.067	1.003	1.001	1
λ_1= 0.5,	SDRL	492.8587	9.544	1.4527	0.5558	0.2501	0.0547	0.0316	0
λ_2= 0.20	P10	56.80	4	1	1	1	1	1	1
L= 8.70	P25	153.75	7	2	1	1	1	1	1
	P50	358.50	11	3	1	1	1	1	1
	P75	707.00	17	4	2	1	1	1	1
	P90	1124.10	28	5	2	1	1	1	1
HEWMA	ARL	500.748	44.614	13.661	11.453	1.059	1.002	1.001	1
λ_1= 0.5,	SDRL	496.0317	30.778	2.1005	0.5693	0.2357	0.04469	0.0316	0
λ_2= 0.60	P10	53.90	6.0	1	1	1	1	1	1
	P25	146.75	11.0	2	1	1	1	1	1
	P50	337.50	25.0	3	1	1	1	1	1
	P75	708.75	52.0	5	2	1	1	1	1
	P90	1156.20	82.1	6	2	1	1	1	1

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Estimation methods		Shift 0	0.05	0.10	0.15	0.20	0.25	0.30	0.35
HEWMA	ARL	500.222	57.909	14.239	11.526	1.077	1.005	1.001	1
$\lambda_1 = 0.5$,	SDRL	495.0974	45.73822	2.846	0.6047	0.26672	0.07056	1.001	0
$\lambda_2 = 0.75$	P10	60.0	6	1	1	1	1	1	1
L = 11.65	P25	147.0	14	2	1	1	1	1	1
	P50	337.5	33	4	1	1	1	1	1
	P75	711.0	64	5	2	1	1	1	1
	P90	1217.7	107	8	2	1	1	1	1
HEWMA	ARL	500.382	31.324	13.361	11.424	1.053	1.002	1.001	1
$\lambda_1 = 0.90$,	SDRL	515.024	16.76437	1.6695	0.5572	0.2241	0.0446	0.0316	0
$\lambda_2 = 0.20$	P10	46.00	5	1	1	1	1	1	1
L = 9.725	P25	142.00	9	2	1	1	0	1	1
	P50	332.00	16	3	1	1	1	1	1
	P75	712.25	29	4	2	1	1	1	1
	P90	1160.10	46	6	2	1	1	1	1
HEWMA	ARL	500.865	82.193	15.892	11.666	1.12	1.006	1.001	1
$\lambda_1 = 0.90$,	SDRL	486.214	66.673	4.607	0.7368	0.3281	0.0772	0.03162	0
$\lambda_2 = 0.60$	P10	64.9	9.0	2	1	1	1	1	1
L = 4.14	P25	154.0	22.0	3	1	1	1	1	1
	P50	343.5	53.0	5	2	1	1	1	1
	P75	688.0	101.0	8	2	1	1	1	1
	P90	1174.4	160.1	12	3	2	1	1	1
HEWMA	ARL	500.2	98.447	18.41	11.81	1.146	1.007	1.001	1
$\lambda_1 = 0.90$,	SDRL	490.6586	86.438	7.653	0.9214	0.3561	0.08341	0.0316	0
$\lambda_2 = 0.75$	P10	63.9	11.00	2.0	1	1	1	1	1
L = 12.00	P25	155.0	29.00	3.0	1	1	1	1	1
	P50	346.0	66.00	6.0	2	1	1	1	1
	P75	693.0	132.25	11.0	2	1	1	1	1
	P90	1178.7	206.00	18.10	3	2	1	1	1

Table 2: ARL_S using PE estimators for the parameters of RPFD using HEWMA control chart.

Estimation method		Shift 0	0.05	0.10	0.15	0.20	0.25	0.30	0.35
HEWMA	ARL	500.291	18.191	15.41	1	1	1	1	1
$\lambda_1 = 0.30$,	SDRL	493.5543	2.246	0.5120	0	0	0	0	0
$\lambda_2 = 0.20$	P10	61.00	2	1	1	1	1	1	1
L = 5.80	P25	139.75	3	1	1	1	1	1	1
	P50	341.00	5	1	1	1	1	1	1
	P75	666.50	6	2	1	1	1	1	1
	P90	1190.50	8	2	1	1	1	1	1
HEWMA	ARL	500.291	27.575	15.161	1	1	1	1	1
$\lambda_1 = 0.30$,	SDRL	515.416	2.604	0.3757	0	0	0	0	0
$\lambda_2 = 0.60$	P10	51.00	2	1	1	1	1	1	1
L = 7.144	P25	141.75	3	1	1	1	1	1	1
	P50	349.50	4	1	1	1	1	1	1
	P75	686.75	6	1	1	1	1	1	1
	P90	1158.00	8	2	1	1	1	1	1

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Estimation method		Shift 0	0.05	0.10	0.15	0.20	0.25	0.30	0.35
HEWMA	ARL	500.948	21. 83	15.164	1	1	1	1	1
λ_1= 0.30,	SDRL	505.03	2.938	0.375	0	0	0	0	0
λ_2= 0.75	P10	51.90	2	1	1	1	1	1	1
L = 7.27	P25	141.75	3	1	0	1	1	1	1
	P50	349.0	4	1	1	1	1	1	1
	P75	681.0	6	1	1	1	1	1	1
	P90	1170.4	9	2	1	1	1	1	1
HEWMA	ARL	500.48	26.814	14.28	1	1	1	1	1
λ_1= 0.5,	SDRL	493.1145	2.4391	0.4602	0	0	0	0	0
λ_2= 0.20	P10	65.00	2	1	1	1	1	1	1
L = 7.00	P25	145.75	3	1	1	1	1	1	1
	P50	363.50	5	1	1	1	1	1	1
	P75	673.50	6	2	1	1	1	1	1
	P90	1232.20	8	2	1	1	1	1	1
HEWMA	ARL	500.326	26.104	14.294	1	1	1	1	1
λ_1= 0.5,	SDRL	496.0043	3.5320	0.375	0	0	0	0	0
λ_2= 0.60	P10	60	2	1	1	1	1	1	1
L = 7.35	P25	161	3	1	1	1	1	1	1
	P50	349	4	1	1	1	1	1	1
	P75	648	7	1	1	1	1	1	1
	P90	1195	10	2	1	1	1	1	1
HEWMA	ARL	500.405	25.456	11.166	1	1	1	1	1
λ_1= 0.5,	SDRL	480.0373	4.0849	0.37760	0	0	0	0	0
λ_2= 0.75	P10	52.00	2	1	1	1	1	1	1
L = 7.44	P25	150	3	1	1	1	1	1	1
	P50	361.00	4	1	1	1	1	1	1
	P75	703.25	7	1	1	1	1	1	1
	P90	1195.20	10	2	1	1	1	1	1
HEWMA	ARL	500.174	24.34	11.228	1	1	1	1	1
λ_1= 0.90,	SDRL	490.6912	2.519	0.429	0	0	0	0	0
λ_2= 0.20	P10	56.00	2	1	1	1	1	1	1
L = 7.25	P25	142.75	3	1	1	1	1	1	1
	P50	363.00	4	1	1	1	1	1	1
	P75	708.25	6	1	1	1	1	1	1
	P90	1185.40	8	2	1	1	1	1	1
HEWMA	ARL	500.452	26.589	11.18	1	1	1	1	1
λ_1= 0.90,	SDRL	471.5114	5.013	0.3996	0	0	0	0	0
λ_2= 0.60	P10	50.90	2	1	1	1	1	1	1
L = 7.59	P25	150.75	3	1	1	1	1	1	1
	P50	366.50	5	1	1	1	1	1	1
	P75	708.50	9	1	1	1	1	1	1
	P90	1172.60	13	2	1	1	1	1	1
HEWMA	ARL	500.005	28.112	11.191	1	1	1	1	1
λ_1= 0.90,	SDRL	478.2393	6.746	0.4274	0	0	0	0	0
λ_2= 0.75	P10	52.0	2	1	1	1	1	1	1
L = 7.66	P25	155.0	3	1	1	1	1	1	1
	P50	353.0	6	1	1	1	1	1	1
	P75	712.50	11	1	1	1	1	1	1
	P90	1192.0	17	2	1	1	1	1	1

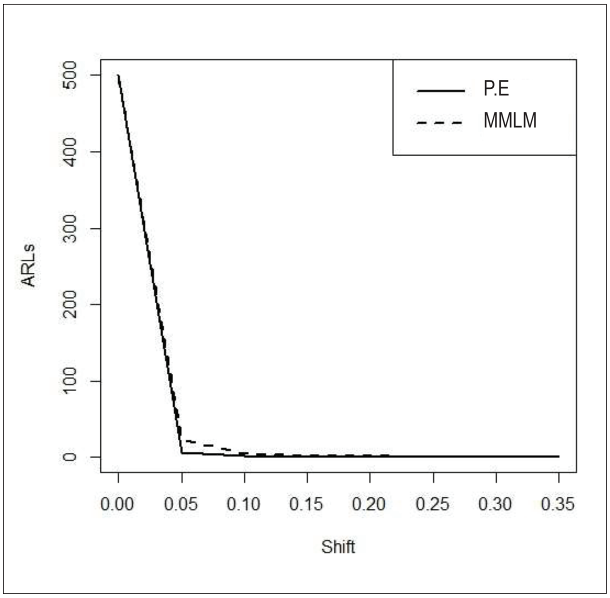

Figure 3: ARLs for the shape parameter of RPFD under HEWMA control chart and λ_1 = 0.90, λ_2=0.20

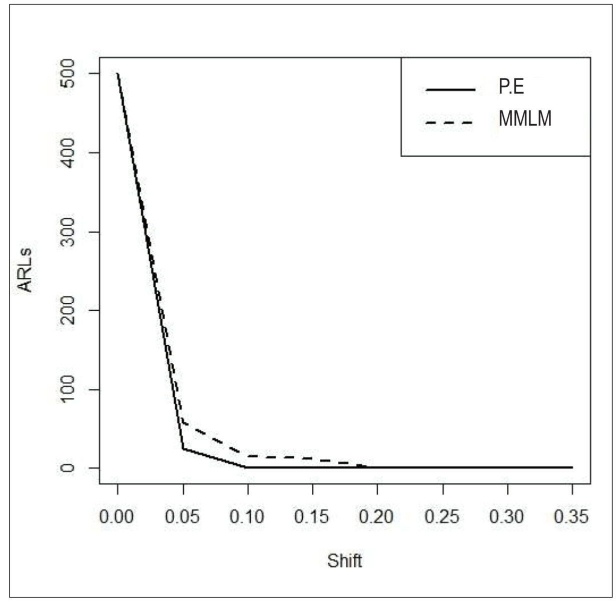

Figure 4: ARLs for the shape parameter of RPFD under HEWMA control chart and λ_1 = 0.50, λ_2=0.75.

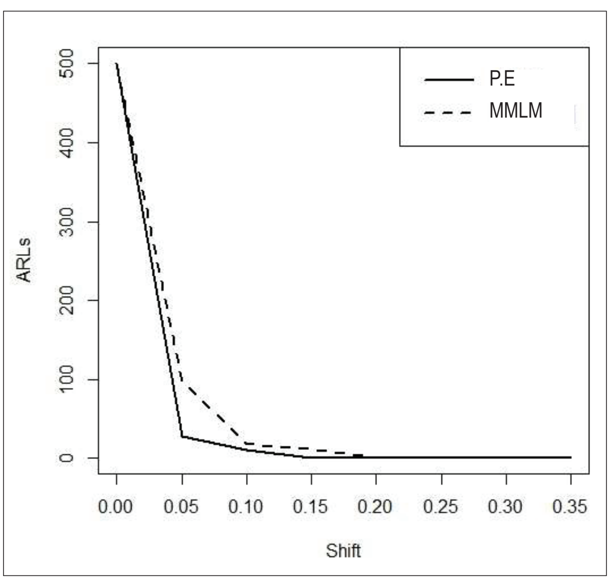

Figure 5: ARLs for the shape parameter of RPFD under HEWMA control chart and λ_1 = 0.90, λ_2=0.75 .

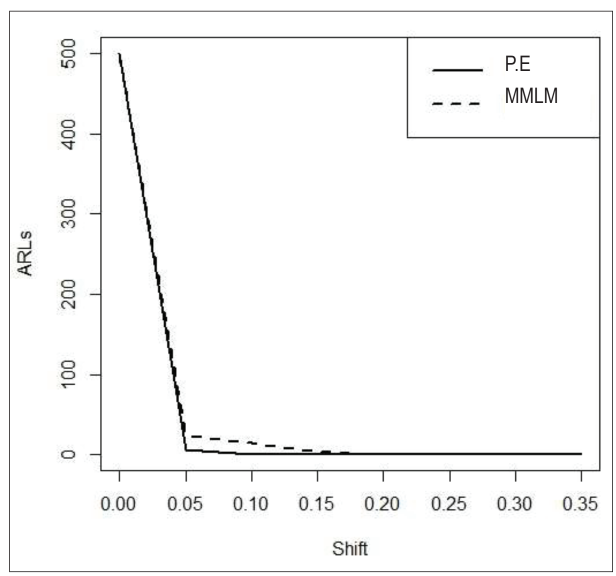

Figure 6: ARLS for MMLM and PE Based EEWMA control charts taking λ_1= 0.90 ,λ_2= 0.20

Table 3: ARLS for MMLM Based EEWMA control charts

Estimation methods		Shift 0	0.05	0.10	0.15	0.20	0.25	0.30	0.35
MMLM	ARL	500.4	5.291	2.41	1.571	1.11	1.006	1.001	1
λ_1= 0.30,	SDRL	483.7063	2.246	0.5120	0.5807	0.3130	0.0772	0.03162	0
λ_2= 0.20	P10	53.90	2	1	1	1	1	1	1
L = 4.30	P25	157.50	3	1	1	1	1	1	1
	P50	358.50	5	1	2	1	1	1	1
	P75	712.25	6	2	2	1	1	1	1
	P90	1117.50	8	2	2	2	1	1	1
MMLM	ARL	500.6	5.575	2.161	1.388	1.045	1.002	1	1
λ_1= 0.30,	SDRL	511.57	2.604	0.3757	0.5307	0.2074	0.0446	0	0
λ_2= 0.60	P10	46.00	2	1	1	1	1	1	1
L = 3.98	P25	149.75	3	1	1	1	1	1	1
	P50	334.50	4	1	1	1	1	1	1
	P75	712.25	6	1	2	1	1	1	1
	P90	1159.00	8	2	2	1	1	1	1
MMLM	ARL	500.1	5.783	2.164	1.419	1.051	1.002	1.001	1
λ_1= 0.30,	SDRL	512.5012	2.938	0.375	0.55112	0.2201	0.0446	0.031	0
λ_2= 0.75	P10	46.90	2	1	1	1	1	1	1
L = 3.98	P25	144.25	3	1	1	1	1	1	1
	P50	329.5	4	1	1	1	1	1	1
	P75	713.25	6	1	1	1	1	1	1
	P90	1164.6	9	2	1	1	1	1	1
MMLM	ARL	500.3	5.814	2.28	1.464	1.067	1.003	1.001	1
λ_1= 0.5,	SDRL	492.8587	2.4391	0.4602	0.5558	0.2501	0.0547	0.0316	0
λ_2= 0.20	P10	56.80	2	1	1	1	1	1	1
L = 8.80	P25	153.75	3	1	1	1	1	1	1
	P50	358.50	5	1	1	1	1	1	1
	P75	707.00	6	2	2	1	1	1	1
	P90	1124.10	8	2	2	1	1	1	1
MMLM	ARL	500.8	6.104	2.294	1.453	1.059	1.002	1.001	1
λ_1= 0.5,	SDRL	496.0317	3.5320	0.375	0.5693	0.2357	0.04469	0.0316	0
λ_2= 0.60									
	P10	53.90	2	1	1	1	1	1	1
	P25	146.75	3	1	1	1	1	1	1
	P50	337.50	4	1	1	1	1	1	1
	P75	708.75	7	1	2	1	1	1	1
	P90	1156.20	10	2	2	1	1	1	1
MMLM	ARL	500	7.456	2.166	1.526	1.077	1.005	1.001	1
λ_1= 0.5,	SDRL	495.0974	4.0849	0.3776	0.6047	0.26672	0.07056	1.001	0
λ_2= 0.75,	P10	60.0	2	1	1	1	1	1	1
L = 10.65	P25	147.0	3	1	1	1	1	1	1
	P50	337.5	4	1	1	1	1	1	1
	P75	711.0	7	1	2	1	1	1	1
	P90	1217.7	10	2	2	1	1	1	1

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Estimation Methods		Shift 0	0.05	0.10	0.15	0.20	0.25	0.30	0.35
MMLM	ARL	500.2	6	3.228	1.2	1.053	1.002	1.001	1
λ_1= 0.90,	SDRL	515.02	2.519	0.429	2.519	0.2241	0.0446	0.0316	0
λ_2= 0.20,	P10	46.00	2	1	2	1	1	1	1
L = 9.725	P25	142.00	3	1	3	1	0	1	1
	P50	332.00	4	1	4	1	1	1	1
	P75	712.25	6	1	6	1	1	1	1
	P90	1160.10	8	2	8	1	1	1	1
MMLM	ARL	500.6	8.589	4.18	1.666	1.12	1.006	1.001	1
λ_1= 0.90,	SDRL	486.214	5.013	0.3996	0.7368	0.3281	0.0772	0.03162	0
λ_2= 0.60	P10	64.9	2	1	1	1	1	1	1
L= 4.14	P25	154.0	3	1	1	1	1	1	1
	P50	343.5	5	1	2	1	1	1	1
	P75	688.0	9	1	2	1	1	1	1
	P90	1174.4	13	2	3	2	1	1	1
MMLM	ARL	500	12.112	4.191	1.816	1.146	1.007	1.001	1
λ_1= 0.90,	SDRL	490.6586	8.746	0.4274	0.9214	0.3561	0.08341	0.0316	0
λ_2= 0.75	P10	63.9	2	1	1	1	1	1	1
L =12.00	P25	155.0	3	1	1	1	1	1	1
	P50	346.0	8	1	2	1	1	1	1
	P75	693.0	11	1	2	1	1	1	1
	P90	1178.7	17	2	3	2	1	1	1

Table 4: ARLS for PE Based EEWMA control charts.

Estimation method		Shift 0	0.05	0.10	0.15	0.20	0.25	0.30	0.35
PE	ARL	500.01	5.01	1.05	1	1	1	1	1
λ_1= 0.35,	SDRL	490.525	3.050	0.896	0	0	0	0	0
λ_2= 0.25	P10	61.00	3	1	1	1	1	1	1
L= 4.2	P25	138.5	4	1	1	1	1	1	1
	P50	352.00	5	1	1	1	1	1	1
	P75	600.50	6	1	1	1	1	1	1
	P90	1050.50	8	1	1	1	1	1	1
PE	ARL	500.01	3.575	2.161	1	1	1	1	1
λ_1= 0.35,	SDRL	505.416	1.604	0.567	0	0	0	0	0
λ_2= 0.60	P10	52.20	2	1	1	1	1	1	1
L= 4.3	P25	140.5	3	1	1	1	1	1	1
	P50	350.50	4	1	1	1	1	1	1
	P75	686.75	6	1	1	1	1	1	1
	P90	1158.00	8	2	1	1	1	1	1

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Estimation method		0	0.05	0.10	0.15	Shift 0.20	0.25	0.30	0.35
PE	ARL	500.01	4.783	1.164	1	1	1	1	1
λ_1= 0.35,	SDRL	504.03	2.938	0.375	0	0	0	0	0
λ_2= 0.75	P10	53.90	2	1	1	1	1	1	1
L = 4.22	P25	138.75	3	1	1	1	1	1	1
	P50	355.0	4	1	1	1	1	1	1
	P75	675.0	6	1	1	1	1	1	1
	P90	1150.4	9	2	1	1	1	1	1
PE	ARL	500.01	5.814	1.12	1	1	1	1	1
λ_1= 0.5,	SDRL	490.05	2.4391	0.4602	0	0	0	0	0
λ_2= 0.25	P10	55.00	2	1	1	1	1	1	1
L = 5.00	P25	185.75	3	1	1	1	1	1	1
	P50	353.50	5	1	1	1	1	1	1
	P75	670.50	6	2	1	1	1	1	1
	P90	1232.20	8	2	1	1	1	1	1
PE	ARL	500.01	4.105	1.34	1	1	1	1	1
λ_1= 0.5,	SDRL	496.0043	2.520	0.375	0	0	0	0	0
λ_2= 0.60	P10	60	2	1	1	1	1	1	1
L = 5.05	P25	161	3	1	1	1	1	1	1
	P50	349	4	1	1	1	1	1	1
	P75	648	7	1	1	1	1	1	1
	P90	1195	10	2	1	1	1	1	1
PE	ARL	500.01	5.456	1.166	1	1	1	1	1
λ_1= 0.5,	SDRL	485.0373	4.0849	0.37760	0	0	0	0	0
λ_2= 0.75	P10	51.50	2	1	1	1	1	1	1
L = 5.2	P25	150	3	1	1	1	1	1	1
	P50	360.00	4	1	1	1	1	1	1
	P75	702.25	7	1	1	1	1	1	1
	P90	1185.20	10	2	1	1	1	1	1
PE	ARL	500.01	6.312	1.208	1	1	1	1	1
λ_1= 0.90,	SDRL	493.912	3.419	0.429	0	0	0	0	0
λ_2= 0.20	P10	52.00	3	1	1	1	1	1	1
L = 5.25	P25	132.75	4	1	1	1	1	1	1
	P50	361.50	5	1	1	1	1	1	1
	P75	705.15	6	1	1	1	1	1	1
	P90	1175.40	8	2	1	1	1	1	1
PE	ARL	500.01	6.589	1.15	1	1	1	1	1
λ_1= 0.90,	SDRL	471.5114	5.013	0.4996	0	0	0	0	0
λ_2= 0.60	P10	50.90	2	1	1	1	1	1	1
L= 4.59	P25	150.75	3	1	1	1	1	1	1
	P50	366.50	5	1	1	1	1	1	1
	P75	708.50	9	1	1	1	1	1	1
	P90	1172.60	13	2	1	1	1	1	1
PE	ARL	500	8.112	1.191	1	1	1	1	1
λ_1= 0.90,	SDRL	478.2393	6.746	0.4274	0	0	0	0	0
λ_2= 0.75,	P10	52.0	2	1	1	1	1	1	1
L= 4.62	P25	155.0	3	1	1	1	1	1	1
	P50	353.0	6	1	1	1	1	1	1
	P75	712.50	11	1	1	1	1	1	1
	P90	1192.0	17	2	1	1	1	1	1

Table 5: Simulated Data

EEWMA for MMLM λ_1=0.90	L=12	λ_2=0.75	EEWMA for PE λ_1=0.90	L=7.66	λ_2=0.75
$EEWMMLM_t$	LCL	UCL	$EEWPE_t$	LCL	UCL
2.792763	0.2380000	3.802000	2.174939	1.389540	2.630460
2.757910	0.1320048	3.907995	2.055011	1.352634	2.667366
2.890396	0.1240272	3.915973	1.884097	1.349857	2.670143
2.031693	0.1234880	3.916512	1.851996	1.349669	2.670331
1.607408	0.1234533	3.916547	1.845604	1.349657	2.670343
2.457326	0.1234511	3.916549	2.209770	1.349656	2.670344
2.706825	0.1234509	3.916549	2.468172	1.349656	2.670344
2.578589	0.1234509	3.916549	2.109744	1.349656	2.670344
2.277098	0.1234509	3.916549	2.124111	1.349656	2.670344
2.033351	0.1234509	3.916549	2.082389	1.349656	2.670344
2.089141	0.1234509	3.916549	2.135626	1.349656	2.670344
2.007338	0.1234509	3.916549	1.958650	1.349656	2.670344
2.080522	0.1234509	3.916549	1.994993	1.349656	2.670344
1.995012	0.1234509	3.916549	2.039405	1.349656	2.670344
1.985600	0.1234509	3.916549	1.999159	1.349656	2.670344
2.404916	0.1234509	3.916549	1.954418	1.349656	2.670344
1.881724	0.1234509	3.916549	2.070869	1.349656	2.670344
2.660258	0.1234509	3.916549	2.120229	1.349656	2.670344
2.194834	0.1234509	3.916549	2.278065	1.349656	2.670344
2.066108	0.1234509	3.916549	2.277322	1.349656	2.670344
2.170824	0.1234509	3.916549	2.277124	1.349656	2.670344
1.925193	0.1234509	3.916549	2.230325	1.349656	2.670344
1.711615	0.1234509	3.916549	2.109841	1.349656	2.670344
2.086559	0.1234509	3.916549	2.112682	1.349656	2.670344
2.466933	0.1234509	3.916549	2.196142	1.349656	2.670344
3.217173	0.1234509	3.916549	2.478410	1.349656	2.670344
3.268923	0.1234509	3.916549	2.428640	1.349656	2.670344
3.453694	0.1234509	3.916549	2.249904	1.349656	2.670344
2.431717	0.1234509	3.916549	2.218243	1.349656	2.670344
1.925132	0.1234509	3.916549	2.212262	1.349656	2.670344
2.943783	0.1234509	3.916549	2.649364	1.349656	2.670344
3.242825	0.1234509	3.916549	2.959301	1.349656	2.670344
3.089214	0.1234509	3.916549	2.529532	1.349656	2.670344
2.728023	0.1234509	3.916549	2.546798	1.349656	2.670344
2.436000	0.1234509	3.916549	2.496781	1.349656	2.670344
2.502835	0.1234509	3.916549	2.560622	1.349656	2.670344
2.404841	0.1234509	3.916549	2.348429	1.349656	2.670344
2.492511	0.1234509	3.916549	2.391968	1.349656	2.670344
2.390080	0.1234509	3.916549	2.445245	1.349656	2.670344
2.378799	0.1234509	3.916549	2.396851	1.349656	2.670344
2.881183	0.1234509	3.916549	2.343292	1.349656	2.670344
2.254360	0.1234509	3.916549	2.482961	1.349656	2.670344
3.187066	0.1234509	3.916549	2.542158	1.349656	2.670344
2.629467	0.1234509	3.916549	2.731320	1.349656	2.670344
2.475245	0.1234509	3.916549	2.730486	1.349656	2.670344
2.600721	0.1234509	3.916549	2.730273	1.349656	2.670344
2.306430	0.1234509	3.916549	2.674169	1.349656	2.670344
2.050553	0.1234509	3.916549	2.529708	1.349656	2.670344
2.499746	0.1234509	3.916549	2.533056	1.349656	2.670344
2.955474	0.1234509	3.916549	2.633164	1.349656	2.670344

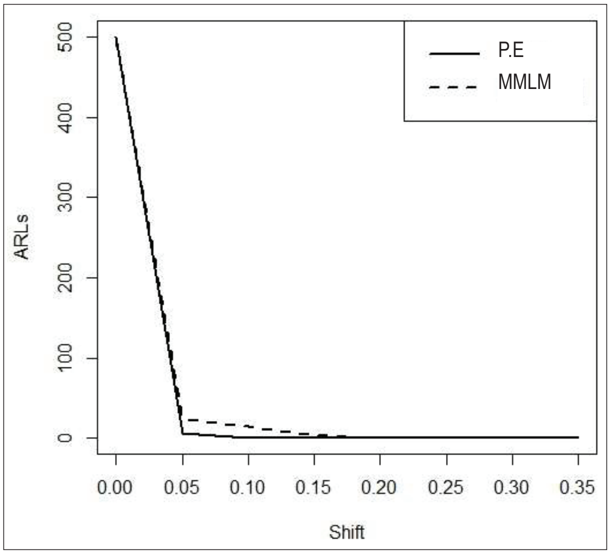

Figure 7: ARLS for MMLM and PE Based EEWMA control charts taking λ_1= 0.50 ,λ_2= 0.75.

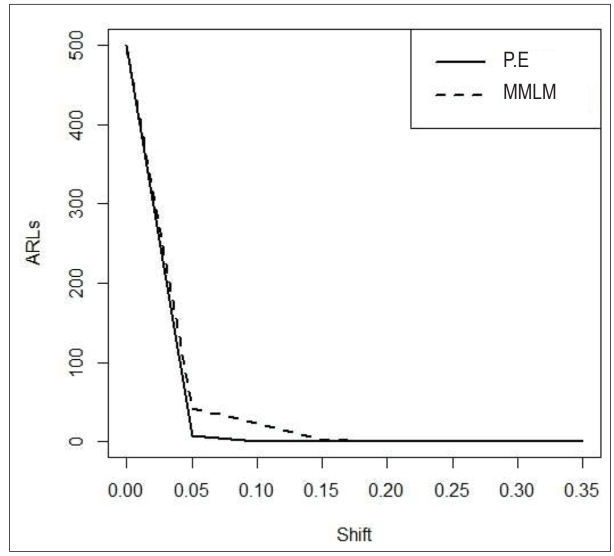

Figure 8: ARLS for MMLM and PE Based EEWMA control charts taking λ_1= 0.90 ,λ_2= 0.75.

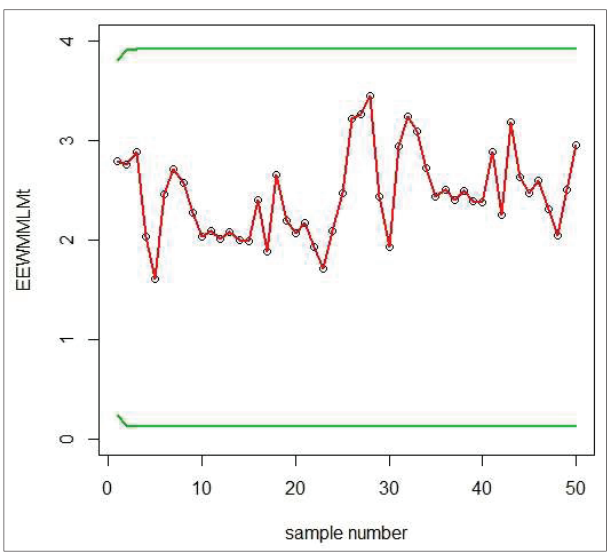

Figure 9: Graph of simulated data of the proposed EEWMA control chart under MMLM when λ_1= 0.90 and λ_2= 0.75

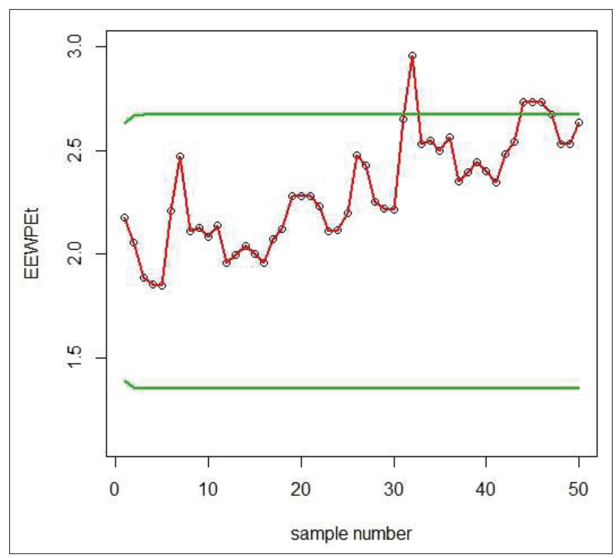

Figure 10: Graph of simulated data of the proposed EEWMA control chart under PE when λ_1= 0.90 and λ_2= 0.75

Simulation

In order to see the working procedure of the proposed control charts, a simulation study was carried out. For this purpose, we generated 25 observations from an RPFD for the in-control process, and the next 25 observations were generated from the shifted process. The estimated values of the proposed EEWMA statistic (MMLM and PE) were computed for the selected levels of the proposed control charts parameters with λ_1 = 0.90 and λ_2 = 0.75. The data and values of the proposed and existing statistics are listed in Table 5, and plotted values of these statistics are shown in Figures 9 and 10. In Figure 10, we noted that the proposed EEWMA control chart under PE detected a shift at the 32nd sample, while in Figure 9, the EEWMA control chart under MMLM could not detect the shift.

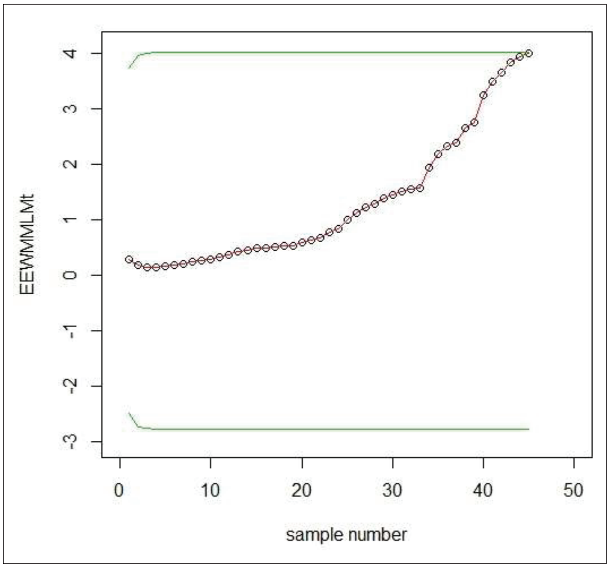

Figure 11: Graph of real data of the EEWMA control chart under MMLM when L=12, λ_1= 0.90 and λ_2= 0.75

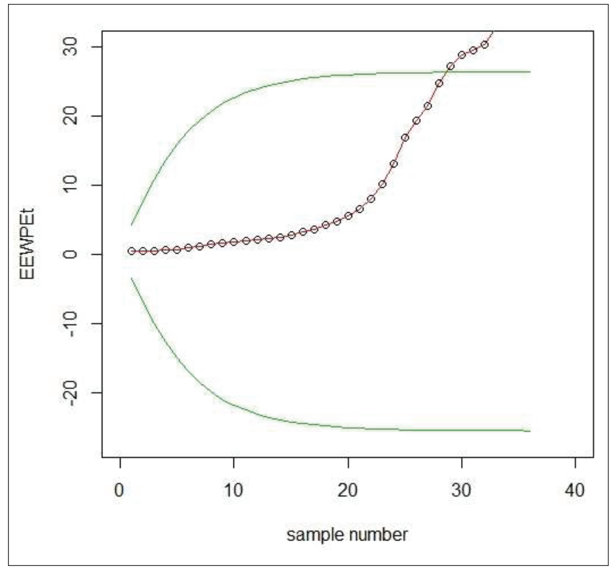

Figure 12: Graph of real data of the EEWMA control chart under PE when L=7.66, λ_1= 0.90 and λ_2= 0.75

Table 6: Earnings per share (EPS) of the National Refinery Ltd.

Year	1984	1985	1986	1987	1988	1989	1990	1991	1992	1993	1994	1995	1996
EPS	0.8	1.8	1.8	1.8	1.8	6.96	3.09	4.3	2.9	4.5	4.9	3	2.2
Year	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009
EPS	7.4	7.3	10.3	10.8	11.2	12.4	16.9	26.7	30.2	52.8	61.4	73.96	22.37
Year	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019			
EPS	37.81	88.16	34.68	33.96	12.03	46.38	96.14	100.61	22.14	108.7			

Hence, this shows that the proposed EEWMA control chart under PE has a more remarkable ability to see more minor changes earlier than the EEWMA control chart under MMLM.

Real life application

Real-life data for earnings per share (EPS) of the National Refinery Ltd. were taken from the State Bank of Pakistan (SBP) report for non-financial companies from the year 1984-2019. The data follows the RPFD and are plotted for EEWMA control charts under MMLM and PE, as shown in Figure 11 and Figure 12.

We have constructed EEWMA control charts on real life data under MMLE and PE. In Figure 11 and Figure 12, we see that EEWMA under PE detects the process shift early as compare to EEWMA under MMLE, which shows that EEWMA under PE is better to be used in real life when the distribution of underlying process is RPFD.

CONCLUSION

The current study explained the application of RPFD functions in management sciences and reliability engineering fields. It introduces control charts based on PE and MMLM estimators under the assumption that the proposed distribution follows RPFD. We constructed a memory-based control chart, *i.e.*, HEWMA and EEWMA control charts under MMLM and PE estimators. The findings indicate that the performance of PE remains consistently good in all control charts, while EEWMA based PE is proved to be the best among all the proposed control charts. Finally, it is expected that these findings will be helpful for the scholars and practitioners in different field of applied sciences. We can use different estimators for reflected power function distribution to see their performance in statistical process control.

Acknowledgement

The authors extend their appreciation to Taif University, Saudi Arabia, for supporting this work through project number (TU-DSPP-2024-162).

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