J.Natn.Sci.Foundation Sri Lanka 2024 52 (1): 113 - 123
DOI: http://dx.doi.org/10.4038/jnsfsr.v52i1.12015

RESEARCH ARTICLE

Graph Theory

All Ramsey critical graphs for large cycles vs a complete graph of order six

CJ Jayawardene[1*] WCW Navaratna[2] and JN Senadheera[2]
[1] *Department of Mathematics, Faculty of Sciences, University of Colombo, Colombo, Sri Lanka.*
[2] *Department of Mathematics, Faculty of Natural Sciences, The Open University of Sri Lanka, Nawala, Sri Lanka.*

Submitted: 17 May 2023; Revised: 06 February 2024; Accepted: 27 February 2024

Abstract : A new area of graph theory emerged in the last few decades is the calculation of star critical Ramsey numbers related to different classes of graphs. Formally, we will say that $K_n \rightarrow (G,H)$ if given any coloring of K_n there is a copy of G in the first color, red, or a copy of H in the second color, blue. The Ramsey number $r(G, H)$ is defined as the smallest positive integer n such that $K_n \rightarrow (G,H)$. A closely related concept of Ramsey number is the star-critical Ramsey number $r_*(G,H)$ defined as the largest value of k such that $K_{r(G,H)-1} \sqcup K_{1,k} \rightarrow (G,H)$. A two-coloring of $K_{r(G,H)-1}$ such that $K_{r(G,H)-1} \nrightarrow (G,H)$ is called a Ramsey critical coloring. A Ramsey critical $r(G,H)$ graph is a graph induced by the first color of a Ramsey critical coloring. Lower bounds for star critical Ramsey numbers are usually found with the aid on Ramsey critical graphs. The particular problem we handle in this paper, on star critical Ramsey numbers, is based on a conjecture posed in 1973 by Bondy and Erdos relating to Ramsey numbers for large cycles versus complete graphs. Based on certain lemmas we present with proof, furthermore we show that there exist exactly sixty eight non-isomorphic Ramsey critical $r(C_n, K_6)$ graphs, when $n \geq 15$.

Keywords: Graph theory, Ramsey theory, Ramsey critical graphs.

INTRODUCTION

For any two graphs G and H, the Ramsey number $r(G,H)$ is the smallest positive integer n such that $K_n \rightarrow (G,H)$. The Ramsey number $r(G,H)$ is defined as the smallest positive integer n such that $K_n \rightarrow (G,H)$. Many interesting variations of the basic problem of finding classical Ramsey numbers have emerged. One such variation is the calculation of the number of Ramsey critical (G,H) graphs, where (G,H) represents any pair of graphs (Yali *et al*., 2015; Hook & Isaak, 2011; Budden, 2023). Another variation is the size Ramsey Numbers (Zhang *et al*., 2023). In this paper, we show that there are exactly 68 Ramsey critical $r(C_n, K_6)$ graphs, for all n exceeding fourteen.

Notations

Given a graph G and a vertex $v \in V(G)$, we define the neighbourhood of v in G, $\Gamma(v)$, as the set of vertices adjacent to v in G. The degree of a vertex v, $d(v)$, is defined as the cardinality of $\Gamma(v)$, i.e. $d(v) = |\Gamma(v)|$. The minimum degree of a graph $G(V,E)$ denoted by $\delta(G)$ is defined as $\min\{d(v)|v \in V\}$. Given a graph G, we say $I \subseteq V(G)$ is an *independent set*, if no pair of vertices of I is adjacent to each other in G. Equivalently, I forms a clique in G^c. Given a graph $G = (V,E)$ we define the independence number, $\alpha(G)$, as the size of the largest independent set. Thus, $\alpha(G) = \max\{|I| : I$ is an independent set of $G\}$. In the special case of $H = K_m$, alternatively r(G,Km) can be viewed as the smallest positive integer n such that every graph complete of order n either contains G as a subgraph or else satisfies $\alpha(G) \geq m$. For two disjoint subgraphs H and K of G, we denote the set of edges between H and K by $E(H,K)$.

* Corresponding author (c_jayawardene@yahoo.com; https://orcid.org/0000-0002-6951-6983)

LEMMAS USED TO GENERATE ALL RAMSEY CRITICAL (C_n, K_6) GRAPHS FOR $n \geq 15$

In an attempt to prove Bondy and Erdös conjecture $r(C_n, K_m) = (n-1)(m-1)+1$, for all $(n,m) \neq (3,3)$ satisfying $n \geq m \geq 3$ under certain restrictions, Schiermeyer has proved that $r(C_n, K_6) = 5(n-1)+1$, for $n \geq 6$ (Bondy *et al.*, 1973; Schiermeyer 2003; Radziszowsk 2021). Characterizing all Ramsey critical (C_n, K_6) graphs boils down to finding all (red/blue) colorings of $K_{r(C_n,K_6)-1}$ such that there is no red C_n or a blue K_6. This is achieved by finding all C_n-free graphs G on $K_{r(C_n,K_6)-1}$ vertices such that $\alpha(G) < 6$. We first prove that any C_n-free graph (where $n \geq 15$) of order $5(n-1)$ with $\alpha(G) \leq 5$ contains a $5K_{n-1}$. To prove this, we use seven lemmas of which the first three are already proven results. For ease of reference, we reiterate Lemma 1 (Jayawardene & Samarasekara, 2017), Lemma 2 (Jayawardene, 2019) and Lemma 3 (Bollobás *et al.*, 2013).

Lemma 1: A C_n- free graph G of order N with independent number less than or equal to m has minimal degree greater than or equal to $N - r(C_n, K_m)$.

Lemma 2: A C_n- free graph (where $n \geq 7$) of order $4(n-1)$ with no independent set of 5 vertices contains a $4K_{n-1}$.

Lemma 3: Suppose G contains the cycle $(u_1, u_2, ..., u_{n-1}, u_1)$ of length $n-1$ but no cycle of length n. Let $Y = V(G) \setminus \{u_1, u_2, ..., u_{n-1}\}$. Then,

(a) No vertex $x \in Y$ is adjacent to two consecutive vertices on the cycle.

(b) If $x \in Y$ is adjacent to u_i and u_j then $u_{i+1}u_{j+1} \notin E(G)$.

(c) If $x \in Y$ is adjacent to u_i and u_j then no vertex $x' \in Y$ is adjacent to both u_{i+1} and u_{j+2}.

(d) Suppose $\alpha(G) = m-1$ where $m \leq \frac{n+2}{2}$ and $\{x_1, x_2, ..., x_{m-1}\} \subseteq Y$ is an $(m-1)$-element independent set. Then, no member of this set is adjacent to $m-2$ or more vertices on the cycle (We have taken the liberty of making a slight correction to the inequality $m \leq \frac{n+2}{2}$ of the original [1], Lemma 5(d)).

The next lemma plays a pivotal role in proving the main results of this paper.

Lemma 4: A C_n -free graph (where $n \geq 15$) of order $5(n-1)$ with no independent set of 6 vertices contains a $5K_{n-1}$.
Proof. We shall assume that in each of the three cases $n = 15$, $n = 16$ and $n \geq 17$ we consider, G as a graph on $5(n-1)$ vertices satisfying $C_n \nsubseteq G$ and $\alpha(G) \leq 5$. Since $r(C_{n-1}, K_6) = 5n-9 \leq 5(n-1)$ (see [1, 6]), there exists a cycle $C = (u_1, u_2, ..., u_{n-1}, u_1)$ of length $n-1$ in G. In consistent with the notation of [1], define H as the induced subgraph of G not containing the vertices of the cycle C. Then, $|V(C)| = n-1$ and $|V(H)| = 4(n-1)$.

Suppose there exists an independent set $Y = \{y_1, y_2, y_3, y_4, y_5\}$ of size 5 in H, so that $\alpha(G) = 5$. From Lemma 3 (as $5 \leq \frac{n+2}{2}$), it follows that no vertex of Y is adjacent to four or more vertices of the C_{n-1}. Thus, $|E(Y, V(C))| \leq 15$. For ease of reference, we define such a graph structure as a **Standard Configuration** (n).

Case 1: $n \geq 17$

Now, $|E(Y, V(C))| \leq 15 < n - 1$. Thus, there exists a vertex $x \in V(C)$ adjacent to no vertex of Y. This gives, an independent set $Y \cup \{x\}$ of size 6, a contradiction.

Case 2: $n = 16$

In this case as $n - 1 = 15$, in order to avoid an independent set of size 6, each vertex of $V(C)$ must be adjacent to at least one vertex of Y. Thus, we get that for each $1 \leq i \leq 5$, $|\Gamma(y_i) \cap V(C)| = 3$ and for each $1 \leq j < j' \leq 5$, $\Gamma(y_j) \cap \Gamma(y_{j'}) \cap V(C) = \phi$.

By Lemma 1, as $\delta(G) \geq 14$, $|\Gamma(y_i) \cap V(H \setminus Y)| \geq 11$ for $i = 1, 2$. Since $r(P_3, K_6) = 11$ and $\alpha(G) < 6$, each of $G[\Gamma(y_1) \cap V(H \setminus Y)]$ and $G[\Gamma(y_2) \cap V(H \setminus Y)]$ contains a copy of P_3. Thus, $P_3 \subseteq \Gamma(y_1) \cap V(H \setminus Y)$, where the P_3 is induced by $\{x, y, z\}$ such that $(x, y), (y, z) \in E(G)$ and $P_3 \subseteq \Gamma(y_2) \cap V(H \setminus Y)$, where this P_3 is induced by $\{p, q, r\}$ such that $(p, q), (q, r) \in E(G)$.

Suppose that x is not adjacent to any vertex of $\{y_2, y_3, y_4, y_5\}$ and p is not adjacent to any vertex of $\{y_1, y_3, y_4, y_5\}$. Re-order the vertices of the cycle such that $y_1 \in Y$ is adjacent to u_1. In this ordering, let y_1 be also adjacent to u_t where $2 \leq t \leq 15$.

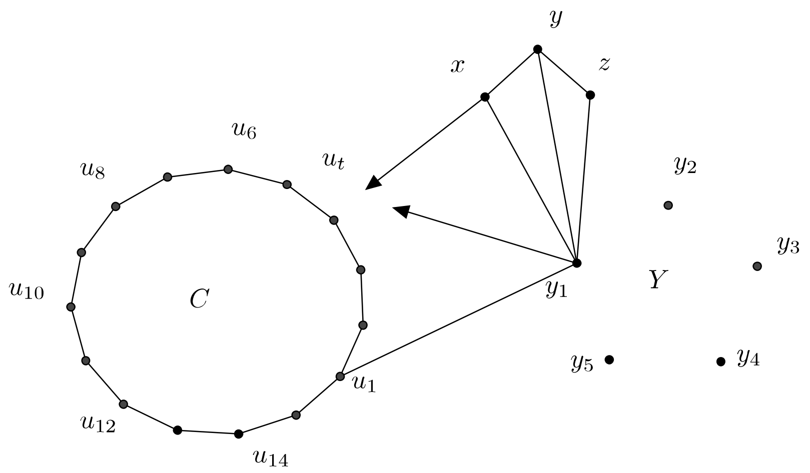


Figure 1: Configuration for $n = 16$

By Lemma 3(a), $t \neq 2$. In order to avoid an independent set of size 6, induced by $\{x, u_t, y_2, y_3, y_4, y_5\}$, we get that $(x, u_t) \in E(G)$. However, $t \neq 3$, in order to avoid a C_{16} comprising $(u_1, y_1, x, u_3, ..., u_{15}, u_1)$. Also, $t \neq 4$ in order to avoid a C_{16} comprising $(u_1, y_1, y, x, u_4, ..., u_{15}, u_1)$ and $t \neq 5$ in order to avoid a C_{16} comprising $(u_1, y_1, z, y, x, u_5, ..., u_{15}, u_1)$.

Thus, any pair of vertices adjacent to y_1 in C cannot be separated by a path of length 1, 2, 3 or 4 along C. Thus, $\Gamma(y_1) \cap C = \{u_1, u_6, u_{11}\}$. In this scenario, we use the prerogative that $(y_2, u_2) \in E(G)$. Then, by the previous argument $\Gamma(y_2) \cap C = \{u_2, u_7, u_{12}\}$. But by Lemma 3(b), $(u_2, u_7) \notin E(G)$. Henceforth, we will get that $\{u_2, u_7, y_1, y_3, y_4, y_6\}$ is an independent set of size 6, a contradiction.

This implies that there is a vertex of $X = \{x, y, z\}$ adjacent to some vertex of $\{y_2, y_3, y_4, y_5\}$ or there is a vertex of $\{p, q, r\}$ adjacent to some vertex of $\{y_1, y_3, y_4, y_5\}$. Therefore, without loss of generality, we may assume that y_1 is adjacent to $X = \{x, y, z\} \subseteq V(H \setminus Y)$ and y_1 is adjacent to $X' = \{x', y', z'\} \subseteq V(C)$ where X' induces a P_3 and y_2 is adjacent to x. Next since C has 15 points without loss of generality, $\{x', y', z'\} = \{u_1, u_3, u_5\}$ or $\{x', y', z'\} = \{u_1, u_3, u_6\}$ or $\{x', y', z'\} = \{u_1, u_3, u_7\}$ or $\{x', y', z'\} = \{u_1, u_3, u_8\}$ or $\{x', y', z'\} = \{u_1, u_3, u_9\}$ or $\{x', y', z'\} = \{u_1, u_4, u_7\}$ or

$\{x', y', z'\} = \{u_1, u_4, u_8\}$ or $\{x', y', z'\} = \{u_1, u_4, u_9\}$ or $\{x', y', z'\} = \{u_1, u_4, u_{10}\}$ or $\{x', y', z'\} = \{u_1, u_5, u_9\}$ or $\{x', y', z'\} = \{u_1, u_5, u_{10}\}$ or $\{x', y', z'\} = \{u_1, u_6, u_{11}\}$. Moreover, as y_1 and y_2 are connected by paths of lengths 2, 3 and 4 in H, no pair of vertices selected from $\Gamma(y_1) \cap V(C)$ and $\Gamma(y_2) \cap V(C)$ can be separated by a path of length 3, 4 or 5 along the cycle C. Using this we argue that when $\{x', y', z'\} = \{u_1, u_3, u_5\}$, $\Gamma(y_2) \cap V(C) = \phi$, when $\{x', y', z'\} = \{u_1, u_3, u_6\}$, $\Gamma(y_2) \cap V(C) = \phi$ and when $\{x', y', z'\} = \{u_1, u_3, u_7\}$, $\Gamma(y_2) \cap V(C) \subseteq \{u_9\}$. Similarly, when $\{x', y', z'\} = \{u_1, u_3, u_8\}$, $\Gamma(y_2) \cap V(C) \subseteq \{u_2, u_9, u_{10}\}$, when $\{x', y', z'\} = \{u_1, u_3, u_9\}$, $\Gamma(y_2) \cap V(C) \subseteq \{u_2, u_{10}\}$, when $\{x', y', z'\} = \{u_1, u_4, u_7\}$, $\Gamma(y_2) \cap V(C) = \phi$ and when $\{x', y', z'\} = \{u_1, u_4, u_8\}$, $\Gamma(y_2) \cap V(C) \subseteq \{u_2, u_{10}\}$. When $\{x', y', z'\} = \{u_1, u_4, u_9\}$, $\Gamma(y_2) \cap V(C) \subseteq \{u_2, u_3, u_{10}\}$, when $\{x', y', z'\} = \{u_1, u_4, u_{10}\}$, $\Gamma(y_2) \cap V(C) \subseteq \{u_2, u_3\}$, when $\{x', y', z'\} = \{u_1, u_5, u_9\}$, $\Gamma(y_2) \cap V(C) \subseteq \{u_3, u_7\}$ and when $\{x', y', z'\} = \{u_1, u_5, u_{10}\}$, $\Gamma(y_2) \cap V(C) \subseteq \{u_3\}$ and when $\{x', y', z'\} = \{u_1, u_6, u_{11}\}$, $\Gamma(y_2) \cap V(C) = \phi$. Since none of these give a viable configuration, we get a contradiction.

Case 3: $n = 15$

To deal with the case $n = 15$, we first prove three Lemmas. Lemma 5, deals with the possible scenarios generated by the Standard Configuration (15). Lemmas 6 and 7 deal with showing that none of the scenarios generated by Lemma 5 give viable configurations.

Lemma 5: In the Standard Configuration ($n = 15$), one of the following three scenarios (a), (b) and (c) will occur:

(a) $y_1 \in Y$ is a vertex of the subgraph K_4 (see Figure 2(a)) in H.

(b) $y_1, y_2 \in Y$ are vertices of the subgraph K (see Figure 2(b1)) or subgraph K' (see Figure 2(b2)) in H.

(c) $y_1, y_2 \in Y$ are vertices of the subgraph L (see Figure 2(c)) in H.

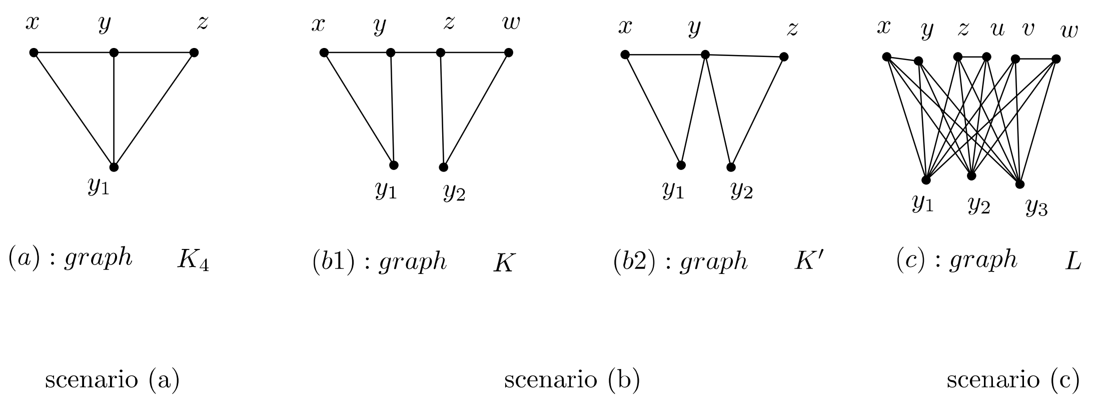


Figure 2: (a),(b1), (b2) and (c). The first three scenarios, scenario (a), scenario (b) and scenario (c) of the Standard Configuration ($n = 15$).

Proof. As in the case of $n = 16$, we get that without loss of generality $1 \leq i \leq 4$, $|\Gamma(y_i) \cap V(C)| = 3$ and for each $1 \leq j < j' \leq 4$, $\Gamma(y_j) \cap \Gamma(y_{j'}) \cap V(C) = \phi$. Also $|\Gamma(y_5) \cap V(C)| \in \{2, 3\}$. In particular, if $|\Gamma(y_5) \cap V(C)| = 2$ then, for each $1 \leq j < j' \leq 5$, $\Gamma(y_j) \cap \Gamma(y_{j'}) \cap V(C) = \phi$ and if $|\Gamma(y_5) \cap V(C)| = 3$

then, for each $1 \leq j < j' \leq 5$,

$$|\Gamma(y_j) \cap \Gamma(y_{j'}) \cap V(C)| = \begin{cases} 1 & \text{if } j, j' \in \{4,5\}, \\ \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 1, as $\delta(G) \geq 13$, we get that $|\Gamma(y_i) \cap V(H \setminus Y)| \geq 10$. Suppose that there is some $1 \leq i \leq 3$ (say $i = 1$) such that $|\Gamma(y_i) \cap V(H \setminus Y)| \geq 11$. Then as $r(P_3, K_6) = 11$ we get scenario (a). Next, assume that for all $1 \leq i \leq 3$, $|\Gamma(y_i) \cap V(H \setminus Y)| = 10$. By the classification of the Ramsey critical (P_3, K_6) graphs, we get that for all $1 \leq i \leq 3$, $G[\Gamma(y_i) \cap V(H \setminus Y)] \supseteq 5K_2$.

This gives two possibilities. The first possibility is $\left|\cup_{i=1}^{3} \Gamma(y_i) \cap V(H \setminus Y)\right| = 10$. In this case, as for all $1 \leq i \leq 3$, $5K_2 \subseteq G[\Gamma(y_i) \cap V(H \setminus Y)]$ we get scenario (c). The second possibility if $\left|\cup_{i=1}^{3} \Gamma(y_i) \cup V(H \setminus Y)\right| \geq 11$.
Without loss of generality, we may assume that $|\Gamma(y_1) \cup \Gamma(y_2) \cup V(H \setminus Y)| \geq 11$. Let x_{11} be any vertex of $\Gamma(y_2) \cap (\Gamma(y_1))^c \cup V(H \setminus Y)$. Since $r(P_3, K_6) = 11$, we get that $G[\Gamma(y_1) \cup V(H \setminus Y) \cup \{x_{11}\}]$ contains a subgraph P isomorphic to a P_3. If P is contained in $G[\Gamma(y_1) \cup V(H \setminus Y)]$ we get scenario (a). Otherwise, $x_{11} \in P$. However, x_{11} is an element of $5K_2 \subseteq G[\Gamma(y_2) \cap V(H \setminus Y)]$ and therefore, x_{11} is adjacent to some other vertex say w in $G[\Gamma(y_2) \cap V(H \setminus Y)]$. Depending on whether or not w belongs to $V(P)$, we get scenarios $(b2)$ or $(b1)$ respectively. Hence the Lemma.

Lemma 6: In the Standard Configuration ($n = 15$), $y_1 \in Y$ can not be a vertex of a K_4 in H (see Figure 2(a)).
Proof. As indicated in Figure 2(a), let $x \in \Gamma(y_1) \cap V(H \setminus Y)$. Then we get two possibilities depending on whether or not x is adjacent to a vertex of $\{y_2, y_3, y_4, y_5\}$. In the first possibility, x is adjacent to some vertex of Y (say y_2). Then, as y_1 and y_2 are connected by paths of lengths 2, 3 and 4 in H, no pair of vertices selected from of $\Gamma(y_1) \cap V(C)$ and $\Gamma(y_2) \cap V(C)$ can be separated by a path of length 3, 4 or 5 along the cycle C. However, as argued in $n = 16$, we get that $|\Gamma(y_2) \cap V(C)| \leq 2$ and $|\Gamma(y_2) \cap V(C)| = 2$ only when $\{x', y', z'\} = \{u_1, u_3, u_8\}$, $\{x', y', z'\} = \{u_1, u_4, u_9\}$ or $\{x', y', z'\} = \{u_1, u_5, u_9\}$. This gives a contradiction. In the second possibility, re order the vertices of the cycle such that $y_1 \in X$ is adjacent to u_1. In this ordering, suppose further that y_1 is also adjacent to u_t where $2 \leq t \leq 14$. By the argument used in $n = 16$, we get that any pair of vertices adjacent to y_1 in C cannot be separated by a path of length 1, 2, 3 or 4 along C. However, this again leads to a contradiction.

Lemma 7: In the Standard Configuration ($n = 15$), the vertices $y_1, y_2 \in Y$ can not be vertices of the subgraph K, K' or L in H (see Figure 2).
Proof. In the case $y_1, y_2 \in K$, since y_1 and y_2 are connected by paths of lengths 3, 4 and 5 in H, no pair of vertices selected from of $\Gamma(y_1) \cap V(C)$ and $\Gamma(y_2) \cap V(C)$ can be separated by a path of length 4, 5 or 6 along the cycle C. The cardinality of the possible vertex sets of $\Gamma(y_i) \cap V(C)$ $(i = 1, 2)$, subject to this condition, are presented in Table 1.

Because $\Gamma(y_2) \cap V(C) = 3$, we are only left to deal with the last two possibilities of Table 1 for $\Gamma(y_2) \cap V(C)$. In both possibilities, $\Gamma(y_1) \cap V(C)$ will induce a C_3 by Lemma 3(b). In the first possibility, $\Gamma(y_1) \cap V(C) = \{u_1, u_5, u_9\}$ gives rise to the 15-cycle given by $(u_1, u_5, ..., u_8, y_2, y, x, y_1, u_9, ..., u_{14}, u_1)$, a contradiction. In the second possibility, $\Gamma(y_1) \cap V(C) = \{u_1, u_5, u_{10}\}$ gives rise to the 15-cycle given by $(u_1, u_5, u_6, ..., u_8, y_2, z, y, x, y_1, u_{10}, ..., u_{14}, u_1)$, a contradiction.
In the case $y_1, y_2 \in K'$, since y_1 and y_2 are connected by paths of lengths 2, 3 or 4 in H, no pair of vertices selected from of $\Gamma(y_1) \cap V(C)$ and $\Gamma(y_2) \cap V(C)$ can be separated by a path of length 3, 4 or 5 along the cycle C.

Table 1: Cardinality of $\Gamma(y_2) \cap V(C)$: Graph K.

$\{x', y', z'\}$ equals	$\Gamma(y_2) \cap V(C)$ is contained in	Cardinality of $\Gamma(y_2) \cap V(C)$
$\{u_1, u_3, u_5\}$	$\{u_2, u_4\}$	$\|\Gamma(y_2) \cap V(C)\| \leq 2$
$\{u_1, u_3, u_6\}$	$\{u_4\}$	$\|\Gamma(y_2) \cap V(C)\| \leq 1$
$\{u_1, u_3, u_7\}$	$\{u_4, u_{14}\}$	$\|\Gamma(y_2) \cap V(C)\| \leq 2$
$\{u_1, u_3, u_8\}$	ϕ	$\|\Gamma(y_2) \cap V(C)\| = 0$
$\{u_1, u_3, u_9\}$	$\{u_2\}$	$\|\Gamma(y_2) \cap V(C)\| \leq 1$
$\{u_1, u_4, u_7\}$	ϕ	$\|\Gamma(y_2) \cap V(C)\| = 0$
$\{u_1, u_4, u_8\}$	ϕ	$\|\Gamma(y_2) \cap V(C)\| = 0$
$\{u_1, u_4, u_9\}$	$\{u_2\}$	$\|\Gamma(y_2) \cap V(C)\| \leq 1$
$\{u_1, u_5, u_9\}$	$\{u_2, u_8, u_{12}\}$	$\|\Gamma(y_2) \cap V(C)\| \leq 3$
$\{u_1, u_5, u_{10}\}$	$\{u_3, u_8, u_{12}\}$	$\|\Gamma(y_2) \cap V(C)\| \leq 3$

The cardinality of the possible vertex set of $\Gamma(y_1) \cap V(C)$ is presented in Table 2 and each of these leads to a contradiction as $\Gamma(y_2) \cap V(C) < 3$.
In the case $y_1, y_2 \in K'$, since y_1 and y_2 are connected by paths of lengths 2, 3, 4 and 5 in H, no pair of vertices selected from of $\Gamma(y_1) \cap V(C)$ and $\Gamma(y_2) \cap V(C)$ can be separated by paths of length 3, 4, 5 or 6 along the cycle C. Thus, Table 2 will give us the required contradiction.

Table 2: Cardinality of $\Gamma(y_2) \cap V(C)$: Graph K'

$\{x', y', z'\}$ equals	$\Gamma(y_2) \cap V(C)$ is contained in	Cardinality of $\Gamma(y_2) \cap V(C)$
$\{u_1, u_3, u_5\}$	ϕ	$\|\Gamma(y_2) \cap V(C)\| = 0$
$\{u_1, u_3, u_6\}$	ϕ	$\|\Gamma(y_2) \cap V(C)\| = 0$
$\{u_1, u_3, u_7\}$	$\{u_9\}$	$\|\Gamma(y_2) \cap V(C)\| \leq 1$
$\{u_1, u_3, u_8\}$	$\{u_2, u_9\}$	$\|\Gamma(y_2) \cap V(C)\| \leq 2$
$\{u_1, u_3, u_9\}$	$\{u_2\}$	$\|\Gamma(y_2) \cap V(C)\| \leq 1$
$\{u_1, u_4, u_7\}$	ϕ	$\|\Gamma(y_2) \cap V(C)\| = 0$
$\{u_1, u_4, u_8\}$	$\{u_2\}$	$\|\Gamma(y_2) \cap V(C)\| \leq 1$
$\{u_1, u_4, u_9\}$	$\{u_2, u_3\}$	$\|\Gamma(y_2) \cap V(C)\| \leq 2$
$\{u_1, u_5, u_9\}$	$\{u_3, u_7\}$	$\|\Gamma(y_2) \cap V(C)\| \leq 2$
$\{u_1, u_5, u_{10}\}$	$\{u_3\}$	$\|\Gamma(y_2) \cap V(C)\| \leq 1$

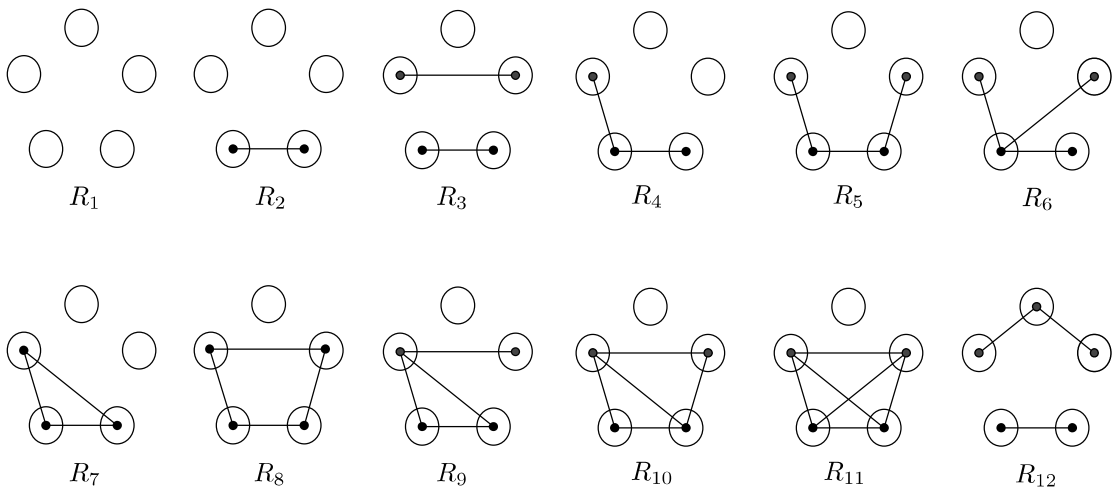
Similarly, in the case $y_1, y_2 \in L$, since y_1 and y_2 are connected by paths of lengths 2, 3, 4, 5 and 6 in H, no pair of vertices selected from of $\Gamma(y_1) \cap V(C)$ and $\Gamma(y_2) \cap V(C)$ can be separated by paths of length 3, 4, 5, 6 or 7 along the cycle C. As before, for all possibilities $|\Gamma(y_2) \cap V(C)| < 3$, a contradiction. Thus, lemmas 5, 6 and 7 imply that H cannot have an independent set of size 5.
Having proved that H cannot have an independent set of size 5 in all three cases $n = 15, 16$ and 17, we next continue with the main proof. Since, H satisfies all conditions of Lemma 2, H contains a $4K_{n-1}$.

Next we show that $V(C_{n-1})$ induced a K_{n-1}. Suppose that there exists two vertices of $V(C)$, say v and w, such that $(v, w) \notin E(G)$. In order to avoid a C_n both v and w will have to be adjacent to at most one vertex of each of the four copies of K_{n-1} in H. Moreover, any vertex of any copy of K_{n-1} in H will have to be adjacent to at most one vertex of another copy of a K_{n-1} in H. Thus, each copy of a K_{n-1} will have at most 5 vertices adjacent to some vertex outside that of K_{n-1}, in $V(H) \cup \{v, w\}$. Since $(n-1)-5 \geq 1$, we can select x_1 in the first K_{n-1}, x_2 in the second K_{n-1}, x_3 in t he third K_{n-1} and x_4 in the fourth K_{n-1} such that $\{x_1, x_2, x_3, x_4\}$ is an independent set of size four a nd no vertex of $\{x_1, x_2, x_3, x_4\}$ is adjacent to any vertex of $\{v, w\}$. Hence $\{x_1, x_2, x_3, x_4, v, w\}$ is an in dependent set of size 6, a contradiction. Therefore, we get that any two pair of vertices of $V(C)$ a re connected by an edge. Hence, $G[V(C_{n-1})] = K_{n-1}$ as required. This K_{n-1} along with the $4K_{n-1}$ contained in H gives the required result.

ALL RAMSEY (C_n, K_6) CRITICAL GRAPHS FOR $n \geq 15$

We have already observed that any Ramsey (C_n, K_6) critical graph will consist of a red graph containing $5K_{n-1}$, with respect to the red/blue coloring. Let $\{V_i : i \in \{1, 2, ..., 5\}\}$ be the vertex set of the five K_{n-1} graphs. We notice that there are two types of Ramsey (C_n, K_6) critical graphs. The first type (Type1) of Ramsey (C_n, K_6) critical graphs will satisfy the condition that at most one vertex of each V_i is adjacent to any other vertex in V_i^c. The second type (Type2) of Ramsey (C_n, K_6) critical graphs will satisfy the condition that there exists a V_k for some $1 \leq k \leq 5$ such that at least two vertices of V_k have neighbors in V_k^c. Moreover, it is worth noting that a Type1 critical graph is completely determined by the structure of the external edges between V_i's and not by the $\binom{n}{2}$ edges inside each of the five V_i's. This fact is taken into consideration when representing the Ramsey (C_n, K_6) critical graphs.

Each subgraph of K_5 generates a unique Ramsey (C_n, K_6) critical graph of Type1. Thus, as illustrated in the following figure, there are 34 critical graphs (R_i, $1 \leq i \leq 34$) of Type1 generated by the 34 subgraphs of K_5.



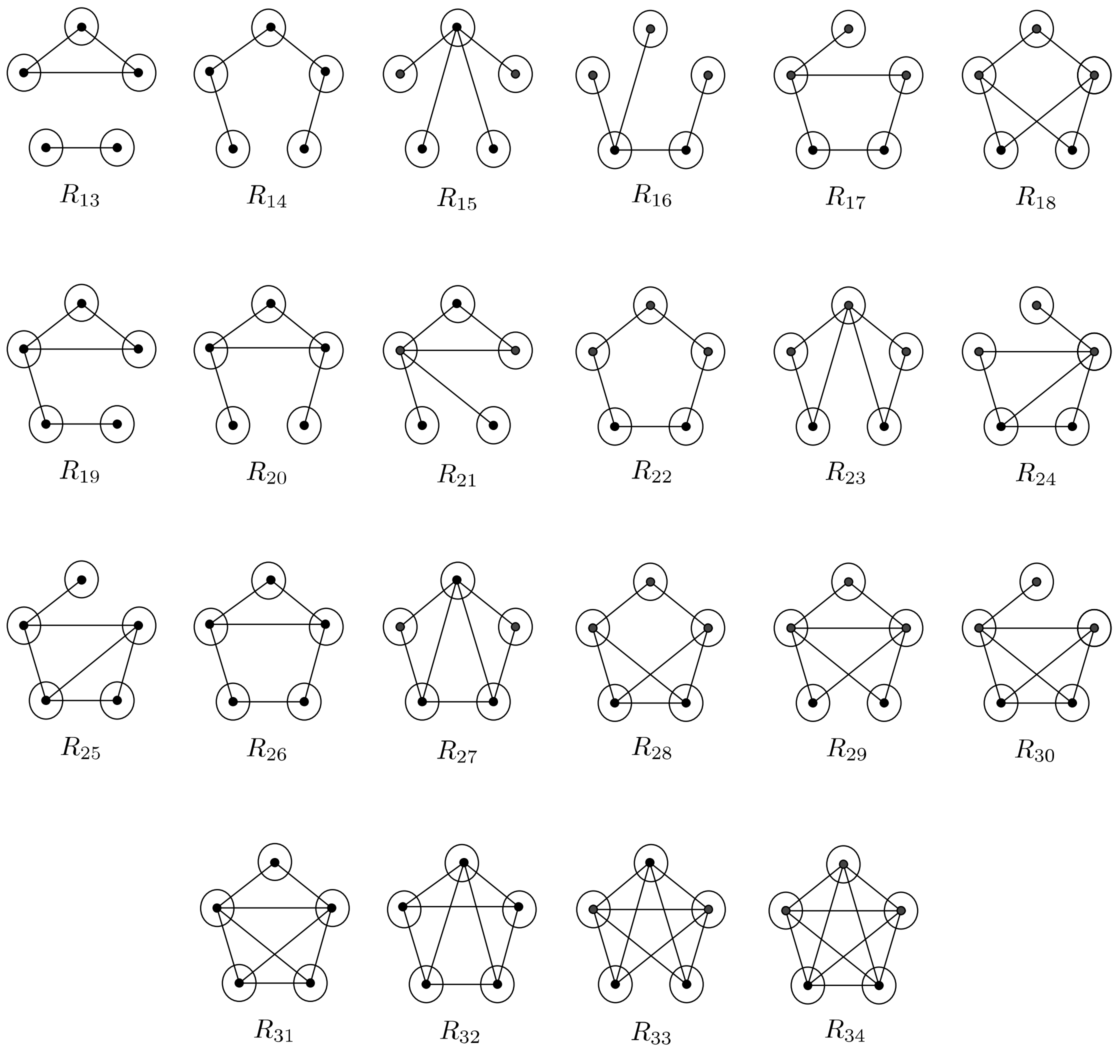
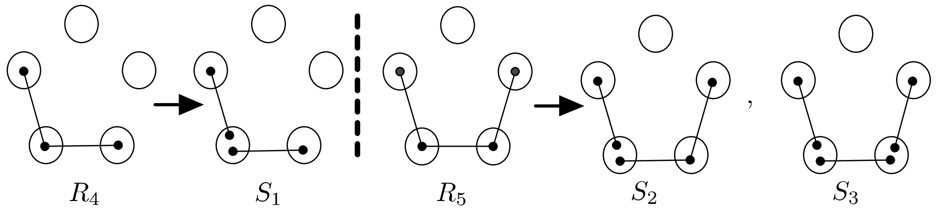
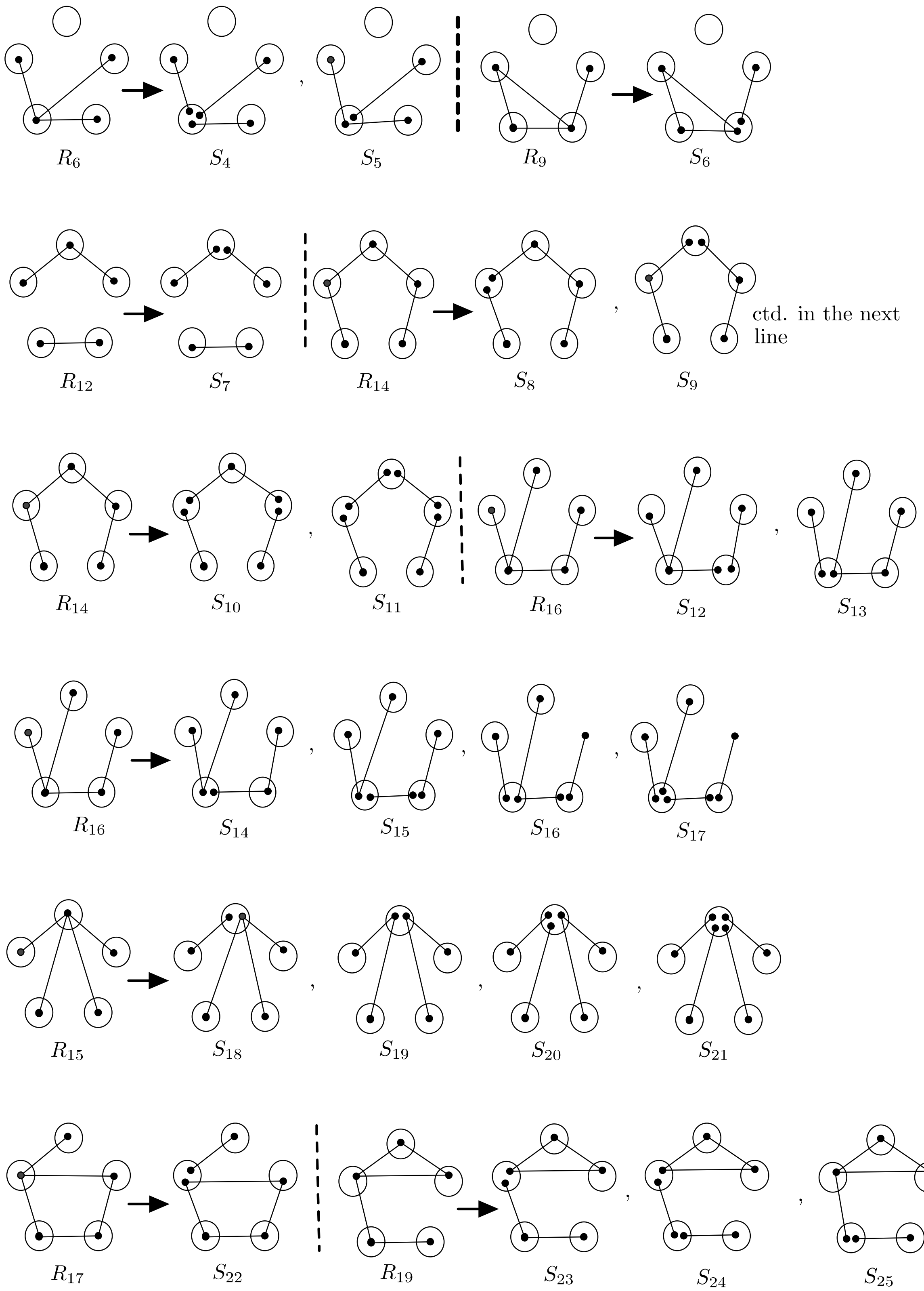


Figure 3: Ramsey (C_n, K_6) critical graphs of Type1, R_i $(1 \leq i \leq 34)$

First note that each and every Type2 critical graph is obtained by an appropriate vertex splitting of some Type1 critical graph. As illustrated in the Figure 4, there are exactly 34 Type2 critical graphs (labeled S_i where $1 \leq i \leq 34$) generated by 18 critical graphs of Type1, since exactly sixteen Type1 critical graphs do not generate Type2 critical graphs.




R_6
S_4
S_5
R_9
S_6
R_{12}
S_7
R_{14}
S_8
S_9
ctd. in the next line
R_{14}
S_{10}
S_{11}
R_{16}
S_{12}
S_{13}
R_{16}
S_{14}
S_{15}
S_{16}
S_{17}
R_{15}
S_{18}
S_{19}
S_{20}
S_{21}
R_{17}
S_{22}
R_{19}
S_{23}
S_{24}
S_{25}

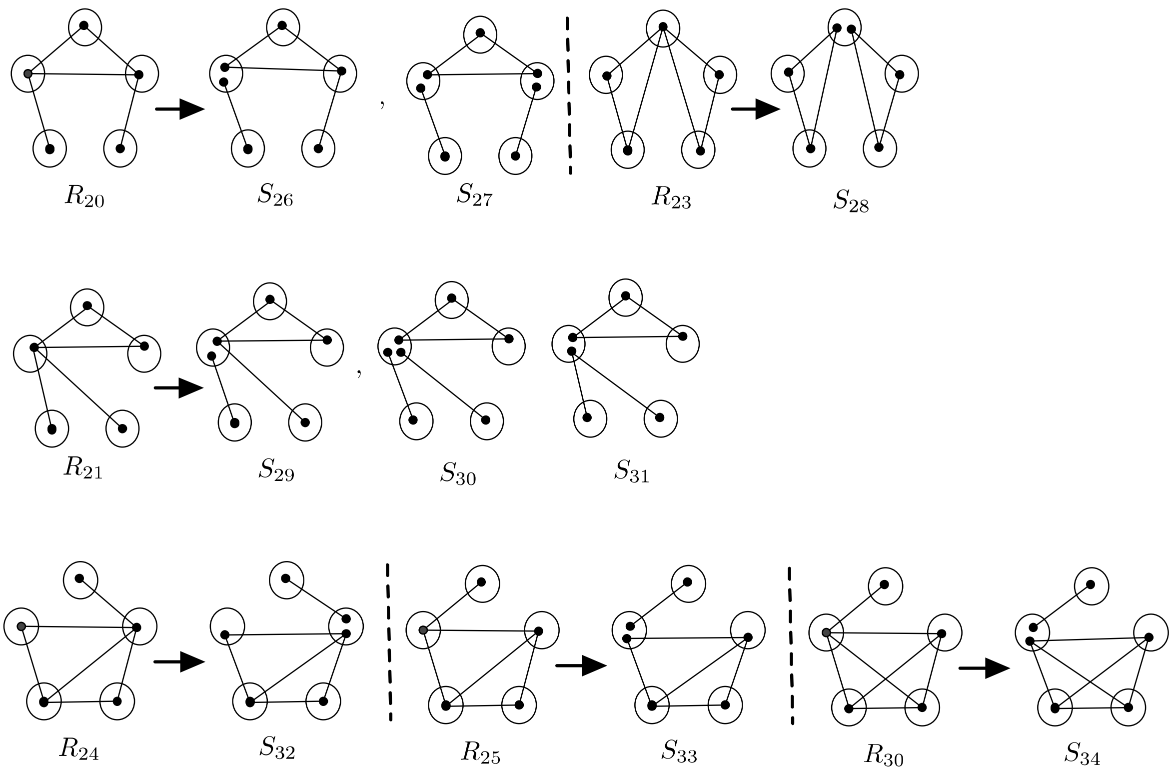


Figure 4: Ramsey (C_n, K_6) critical graphs of Type2 (S_i, $1 \leq i \leq 34$)

Henceforth, we conclude that there are exactly 68 Ramsey (C_n, K_6) critical graphs out of which 34 are categorized as Type1 critical graphs (labeled R_i, $1 \leq i \leq 34$) and the balance 34 are categorized as Type2 critical graphs (labeled S_i, $1 \leq i \leq 34$).

CONCLUSION

From this paper we see that the number of Ramsey Critical graphs of (C_n, K_6) is 68 and that the number of Ramsey Critical graphs of (C_n, K_m) is growing steadily in number as m increases up to 6. Moreover, this paper investigates the relationship between Ramsey Critial graphs of $m = 5$ and $m = 6$. This technique should play a pivotal role in finding all Ramsey Critical graphs for (C_n, K_m)f or $m \geq 7$.

REFERENCES

Bollobas B., Jayawardene C.J., Sheng Y.J., Ru H.Y., Rousseau C.C. & Min Z.K. (2000). On a conjecture involving cycle-complete graph Ramsey numbers. *The Australasian Journal of Combinatorics* **22**: 63–71.

Budden M.R. (2023). *Star-Critical Ramsey Numbers for Graphs*. Springer International Publishing, Cham, Switzerland.

Hook J. & Isaak G. (2011). Star-critical Ramsey numbers. *Discrete Applied Mathematics* **159**:328–334. DOI: https://doi.org/10.1016/j.dam.2010.11.007

Jayawardene C.J. (2019). The Star-critical Ramsey Number for any Cycle vs. a K5.arXiv:1901.04802.

Jayawardene C.J. & Samarasekara B.L. (2017). The size multipartite Ramsey numbers for stars vs cycles. *Journal of the national Science Foundation of Sri Lanka* **45**(1): 67–72. DOI: http://dx.doi.org/10.4038/jnsfsr.v45i1.8039

Radziszowski S.P. (2021). Small Ramsey numbers. *Electronic Journal of Combinatorics* **16**:DS1. DOI: https://doi.org/10.37236/21

Schiermeyer I. (2003). All cycle-complete graph Ramsey number r(Cm,K6). *Journal of Graph Theory* **44**: 251–260. DOI: https://doi.org/10.1002/jgt.10145

Yali W., Yongqi S. & Radziszowski S.P. (2015). Wheel and star-critical Ramsey numbers for quadrilaterals. *Discrete Applied Mathematics* **185**: 260–271. DOI: https://doi.org/10.1016/j.dam.2015.01.003

Zhang Y., Zhang Y. & Zhi H. (2023) A proof of a conjecture on matching. *AIMS Mathematics* **8**(4): 8027 –8033 DOI: https://doi.org/10.3934/math.2023406