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RESEARCH ARTICLE

Bivariate Gompertz generator of distributions: statistical properties and estimation with application to model football data

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Abstract: In this paper, the bivariate extension of the so called Gompertz-G family was introduced and studied in detail. Marshall and Olkin shock model was used to build the proposed bivariate family. The new family was constructed from three independent Gompertz-H families using a minimisation process. Some of its statistical properties such as joint probability density function, coefficient of median correlation, moments, product moment, covariance, conditional probability density function, joint reliability function, stress-strength reliability and joint reversed (hazard) rate function were derived. After introducing the general class, three special models of the new family were discussed. Maximum likelihood method was used to estimate the family parameters. A simulation study was carried out to examine the bias and mean square error of the maximum likelihood estimators. Finally, the importance of the proposed bivariate family was illustrated by means of real dataset, and it was found that the proposed model provides better fit than other well-known models in the statistical literature such as bivariate Gompertz, bivariate generalized Gompertz, bivariate Gumbel Gompertz, bivariate Burr X Gompertz and bivariate exponentiated Weibull-Gomperz

Keywords: Bivariate distributions, Gompertz-H family of distributions, Marshall-Olkin shock model, maximum likelihood method.

INTRODUCTION

Several classes of distributions have been developed and applied to describe various phenomena in different areas such as engineering, biological studies, economics, actuarial, environmental, lifetime analysis and Olympic games, among others.

However, in many applied areas such as lifetime analysis, describing the pattern of adult deaths, Olympic games and insurance, there is a clear need for extended forms of these classes to model such data. For this reason, many classes have been proposed and studied in statistical literature, for example, transformed-transformer (T-X) family by Alzaatreh *et al*. (2013); generating T-Y family by Aljarrah *et al*. (2014); exponentiated half-logistic family by Cordeiro *et al*. (2014); Kumaraswamy Marshall-Olkin family by Alizadeh *et al*. (2015); a new Weibull-G family by Tahir *et al*. (2016); Gompertz-G family by Alizadeh *et al*. (2017) and its discrete version by Eliwa *et al*. (2020a); exponentiated Gompertz generated family by Cordeiro *et al*. (2016); odd Chen-G family by El-Morshedy *et al.* (2020a); exponentiated odd Chen-G

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family by Eliwa *et al*. (2020b); odd flexible Weibull-H family by El-Morshedy and Eliwa (2019); odd log-logistic Lindley-G family by Alizadeh *et al*. (2020) and discrete Gompertz family by Eliwa *et al*. (2020a), among others.

In many practical situations, it is important to consider different bivariate families that could be used to model bivariate data. The bivariate data could be exchange rates in two time periods, strength components, results of two teams in Olympic games etc. Therefore, many bivariate distributions are proposed in literature, for example, bivariate generalised exponential distribution by Kundu and Gupta (2009); bivariate generalised linear failure rate distribution by Sarhan *et al*. (2011) Marshall-Olkin bivariate Weibull distribution by Kundu and Gupta (2013); bivariate Kumaraswamy distribution by Barreto-Souza and Lemonte (2013); bivariate exponential distribution by Balakrishnan and Shiji (2014); bivariate exponentiated generalised Weibull-Gompertz distribution by El-Bassiouny *et al*. (2016); bivariate exponentiated modified Weibull extension distribution by El-Gohary *et al*. (2016); bivariate exponentiated extended Weibull family by Roozegar and Jafari (2016); bivariate inverse Weibull distribution by Hiba (2016); bivariate exponentiated discrete Weibull distribution by El-Morshedy *et al*. (2020c); bivariate exponentiated generalised linear exponential distribution by Ibrahim *et al*. (2019); bivariate Gumbel-G family by Eliwa and El-Morshedy (2019); univarite and multivariate generalized slash student distribution by El-Bassiouny and El-Morshedy (2015), univariate and multivariate double slash distribution by El-Morshedy *et al*. (2020c), bivariate discrete inverse Weibull distribution by Eliwa and El-Morshedy (2020a), bivariate odd Weibull-G family by Eliwa and El-Morshedy (2020b), bivariate Burr X generator by El-Morshedy *et al*. (2020c), among others. However, in many practical situations, classical bivariate distributions do not provide adequate fits to real data. Therefore, there has been an increased interest in developing more flexible distributions. Thus, in this paper, we introduce a flexible bivariate family based on Marshall-Olkin shock model (Marshall & Olkin, 1967), in the so-called bivariate Gompertz-H (BGo-H) family.

Alizadeh *et al*. (2017) proposed and studied a flexible univariate family of distributions, in the so-called Gompertz-G (Go-H) family. The random variable Y is said to have Go-H family if its CDF is given by

$$V_{Go-H}\ (y;\theta,\alpha,\eta) = 1 - e^{\frac{\theta}{\alpha}\left\{1-\left[\overline{H}(y;\eta)\right]^{-\alpha}\right\}};\quad y \geq 0, \qquad ...(01)$$

where $\theta > 0$ and $\alpha > 0$ are two additional shape parameters, and η is a vector of parameters ($1\times\ k$; k = 1, 2, 3, ...). Also, the $H(y;\eta) = 1 - \overline{H}(y;\eta)$ is the baseline CDF depending on a parameter vector $\eta > 0$. The survival function of the random variable Y is given by

$$\overline{V}_{Go-H}\ (y;\theta,\alpha,\eta) = e^{\frac{\theta}{\alpha}\left\{1-\left[\overline{H}(y;\eta)\right]^{-\alpha}\right\}};\quad y \geq 0. \qquad ...(02)$$

The probability density function (PDF) corresponding to equation (1) is given by

$$v_{Go-H}\ (y;\theta,\alpha,\eta) = \theta h(y;\eta)\left[\overline{H}(y;\eta)\right]^{-(\alpha+1)} e^{\frac{\theta}{\alpha}\left\{1-\left[\overline{H}(y;\eta)\right]^{-\alpha}\right\}}; y \geq 0, \qquad ...(03)$$

where $h(y;\eta)$ is the baseline PDF. The main reasons for introducing this family are:

1. The joint CDFs and joint PDFs should preferably have a closed form representation; at least numerical evaluation should be possible.
2. This class of distributions is an important model that can be used in a variety of problems for modelling bivariate lifetime data.
3. This class contains several special bivariate models.

METHODOLOGY: BGO-H FAMILY

A random vector (Y_1, Y_2) follows the Marshall-Olkin shock model $\leftrightarrow$ there exist three independent random variables Z_1, Z_2 and Z_3 such that $(Y_1 = \max(Z_1, Z_3)$ and $Y_2 = \max(Z_2, Z_3))$ or $(Y_1 = \min(Z_1, Z_3)$ and $Y_2 = \min(Z_2, Z_3))$. The proposed BGo-H family is constructed from three independent Go-H families using a minimisation process. Assume three mutually independent random variables $Z_i \sim Go - H\ (\theta_i, \alpha, \eta)$, such that i = 1, 2, 3. Define $Y_1 = \min\{Z_1, Z_3\}$ and $Y_2 = \min\{Z_2, Z_3\}$. So, the bivariate vector (Y_1, Y_2) has the BGo-H family with parameter vector $\boldsymbol{\Upsilon} = (\theta_1, \theta_2, \theta_3, \alpha, \eta)$. The joint survival function of (Y_1, Y_2) is given as follows

$$\begin{aligned}\overline{V}_{Y_1,Y_2}(y_1,y_2) &= \mathbf{P}\left[\left(\min\{Z_1,Z_3\} > y_1\right), \left(\min\{Z_2,Z_3\} > y_2\right)\right] \\ &= \mathbf{P}\left[Z_1 > y_1, Z_2 > y_2, Z_3 > \max(y_1,y_2)\right] \\ &= \prod_{i=1}^{3} e^{\frac{\theta_i}{\alpha}\left\{1-\left[\overline{H}(y_i;\eta)\right]^{-\alpha}\right\}},\end{aligned} \qquad ...(04)$$

where $y_3 = \max\{y_1, y_2\}$. Equation (4) can be written as follows

$$\overline{V}_{Y_1,Y_2}(y_1,y_2)=\begin{cases}\overline{V}_{Go-H}\ (y_1;\theta_1,\alpha,\eta)\times\overline{V}_{Go-H}\ (y_2;\theta_2+\theta_3,\alpha,\eta)\ ;\ y_1\le y_2\\ \overline{V}_{Go-H}\ (y_1;\theta_1+\theta_3,\alpha,\eta)\times\overline{V}_{Go-H}\ (y_2;\theta_2,\alpha,\eta)\ ;\ y_1>y_2.\end{cases}$$

...(05)

Moreover, we can get the joint PDF of (Y_1,Y_2) as follows:

$$v_{Y_1,Y_2}(y_1,y_2)=\begin{cases}v_1(y_1,y_2) & ;\ 0<y_1<y_2<\infty\\ v_2(y_1,y_2) & ;\ 0<\ y_2<y_1<\infty\\ v_3(y) & ;\ 0<y_1=y_2=y<\infty,\end{cases}$$

...(06)

where

$$v_1(y_1,y_2)=v_{Go-H}\ (y_1;\theta_1,\alpha,\eta)\times$$
$$v_{Go-H}\ (y_2;\theta_2+\theta_3,\alpha,\eta),$$

$$v_2(y_1,y_2)=v_{Go-H}\ (y_1;\theta_1+\theta_3,\alpha,\eta)\times$$
$$v_{Go-H}\ (y_2;\theta_2,\alpha,\eta),$$

and

$$v_3(y)=\frac{\theta_3}{\theta_1+\theta_2+\theta_3}v_{Go-H}\ (y;\theta_1+\theta_2+\theta_3,\alpha,\eta).$$

The expressions $v_i(y_1,y_2),\ i=1,2$ can be obtained by differentiating the equation (4) with respect to $y_i,\ i=1,2$. However, we can use the following fact to get $v_3(y)$

$$\int_0^\infty v_3(y)\ dy=1-\int_0^\infty\int_0^{y_2}v_1(y_1,y_2)dy_1dy_2$$
$$-\int_0^\infty\int_0^{y_1}v_2(y_1,y_2)dy_2dy_1.$$

...(07)

On the other hand, the marginal survival functions for the BGo-H family can be represented as follows

$$\overline{V}_{Y_i}(y_i)=\mathbf{P}\left(\min\{Z_i,Z_3\}>y_i\right)$$
$$=\overline{V}_{Go-H}\ (y_i;\theta_i+\theta_3,\alpha,\eta),i=1,2.$$

...(08)

So, we can get the marginal PDFs for the BGo-H family as follows:

$$v_{Y_i}(y_i)=v_{Go-H}\ (y_i;\theta_i+\theta_3,\alpha,\eta);i=1,2.$$

...(09)

Using the power series for the exponential function and the generalized binomial theorem, we nd the marginal PDFs for the BGo-H family can be expressed as an innite linear combination of exponentiated-H (exp-H) density functions as follows:

$$v_{Y_i}(y_i)=\sum_{l=0}^{\infty}\Psi_{l+1}^{(i)}P_{l+1}(y_i;\eta),$$

...(10)

where

$$P_{l+1}(y_i;\eta)=(l+1)h(y_i;\eta)H(y_i;\eta)^l,$$

...(11)

represents the PDF of the exp-H family with power parameter $(l+1)$, $\Psi_0^{(i)}=1-\psi_0$, $\Psi_l^{(i)}=-\psi_l^{(i)}$ for $(l=1,2,3,...)$, $\psi_l^{(i)}=\sum_{m=0}^{\infty}\sum_{j=0}^{m}\Omega_{m,j,l}^{(i)}$ and

$$\Omega_{m,j,l}^{(i)}=\frac{(-1)^{m+l}}{m!}\left(\frac{\theta_i+\theta_3}{\alpha}\right)^m\left(\begin{array}{c}-\alpha j\\ l\end{array}\right)\left(\begin{array}{c}m\\ j\end{array}\right).$$

...(12)

Assume (Y_1,Y_2) be a two dimensional random variable with joint survival function $\overline{V}_{Y_1,Y_2}(y_1,y_2)$, and the marginal survival functions are $\overline{V}_{Y_1}(y_1)$ and $\overline{V}_{Y_2}(y_2)$, then the joint CDF is given by

$$V_{Y_1,Y_2}(y_1,y_2)=1-\overline{V}_{Y_1}(y_1)\ -\overline{V}_{Y_2}(y_2)+\overline{V}_{Y_1,Y_2}(y_1,y_2)$$
$$=1-e^{\frac{(\theta_1+\theta_3)}{\alpha}\left\{1-\left[\overline{H}(y_1;\eta)\right]^{-\alpha}\right\}}$$
$$-e^{\frac{(\theta_2+\theta_3)}{\alpha}\left\{1-\left[\overline{H}(y_2;\eta)\right]^{-\alpha}\right\}}$$
$$+\prod_{i=1}^{3}e^{\frac{\theta_i}{\alpha}\left\{1-\left[\overline{H}(y_i;\eta)\right]^{-\alpha}\right\}}.$$

...(13)

If the bivariate vector (Y_1,Y_2) has the BGo-H family, then the distributions of $S=\max\{Y_1,Y_2\}$ and $T=\min\{Y_1,Y_2\}$ can be represented as

$$V_S(t)=\prod_{i=1}^{3}V_{Go-H}\ (t;\theta_i,\alpha,\eta)$$

...(14)

and

$$V_T(t)=1-\prod_{i=1}^{3}\overline{V}_{Go-H}\ (t;\theta_i,\alpha,\eta),$$

...(15)

respectively.

Different statistical properties

BGo-H family using Marshall-Olkin copula properties

In this section, we find that the BGo-H family has both a singular part along the line $y_1 = y_2$ with weight $\frac{\theta_3}{\theta_1+\theta_2+\theta_3}$ and an absolute continuous part on $0 < y_1 \neq y_2 < \infty$ with weight $\frac{\theta_1+\theta_2}{\theta_1+\theta_2+\theta_3}$, similar to Marshall and Olkin's bivariate exponential model. Moreover, the BGo-H family can be obtained by using the Marshall-Olkin copula with the marginals as the Go-H families as follows: for $B^* : [0,1] \times [0,1] \to [0,1]$, we get

$$\overline{V}_{Y_1,Y_2}(y_1,y_2) = B^*\left(\overline{V}_{Y_1}(y_1), \overline{V}_{Y2}(y_2)\right); \text{ for all } (y_1,y_2) \in R^2, \quad ...(16)$$

where

$$B^*_{\tau_1,\tau_2}(D_1^*, D_2^*) = D_1^{*1-\tau_1} D_2^{*1-\tau_2} \max\left(D_1^{*\tau_1}, D_2^{*\tau_2}\right), \quad \text{for } 0 < \tau_1, \tau_2 < 1,$$

$\tau_i = \frac{\theta_3}{\theta_i+\theta_3}$ and $D_i^* = \overline{V}_{Y_i}(y_i); i = 1, 2.$. For more details around Marshall-Olkin copula properties see, Nelsen, 1999. Also, we find that

$$B^*(D_1^*, D_2^*) \geq D_1^* D_2^* \text{ for all } D_1^*, D_2^* \in [0,1]^2. \quad ...(17)$$

So, if (Y_1, Y_2) follow the BGo-H family, then they are positive quadrant dependent.

Note: For every pair of increasing functions $f_{Y_1}(\cdot)$ and $f_{Y_2}(.)$, we get $\mathbf{Cov}\{f_{Y_1}(Y_1), f_{Y_2}(Y_2)\} \geq 0$ (Barlow & Proschan, 1975).

Coefficient of median correlation

Assume D_{Y_1} and D_{Y_2} denote the median of Y_1 and Y_2, respectively. If $Y_1 \sim Go-H\ (\theta_1+\theta_3, \alpha, \eta)$ and $Y_2 \sim Go-H\ (\theta_2+\theta_3, \alpha, \eta)$, then

$$D_{Y_i} = H^{-1}\left\{1 - \left(1 - \frac{\alpha}{\theta_i+\theta_3}\log\left[1-U\right]\right)^{\frac{-1}{\alpha}}\right\}; \quad i = 1, 2, \quad ...(18)$$

where U has a uniform $U(0, 1)$ distribution, and $H^{-1}(.)$ represents the baseline quantile function. Domma (2010) presented the median correlation coefficient D_{Y_1,Y_2} as a form $D_{Y_1,Y_2} = 4V_{Y_1,Y_2}(D_{Y_1}, D_{Y_2}) - 1$. So, the coefficient of median correlation between Y_1 and Y_2 is given as follows:

$$D_{X_1,X_2} = \begin{cases} 4V_{Go-H}\ (D_{Y_2}; \theta_2+\theta_3, \alpha, \eta) \times V_{Go-H}\ (D_{Y_1}; \theta_1, \alpha, \eta) - 1 & ;\ y_1 < y_2 \\ 4V_{Go-H}\ (D_{Y_1}; \theta_1+\theta_3, \alpha, \eta) \times V_{Go-H}\ (D_{Y_2}; \theta_2, \alpha, \eta) - 1 & ;\ y_1 > y_2. \end{cases} \quad ...(19)$$

Moments, product moment and covariance

In this section, we derive the *r*th moment, the *n*th central moment and the *s*th incomplete moment of Y_i when $Y_i \sim Go-H\ (\theta_i+\theta_3, \alpha, \eta)$, such that i = 1, 2. Also, we present the product moment, covariance and the $Var(Y_1+Y_2)$ of the bivariate distribution (Y_1, Y_2). The *r*th moment of Y_i, say $W_i^{(r)}$, can be expressed as follows $W_i^{(r)} = \mathbf{E}(Y_i^r) = \int_0^\infty y_i^r v_{Y_i}(y_i) dy_i$, using equation (10), we get

$$\begin{aligned} W_i^{(r)} &= \sum_{l=0}^{\infty} \Psi_{l+1}^{(i)} \int_0^\infty y_i^r\ P_{l+1}(y_i;\eta) dy_i \\ &= \sum_{l=0}^{\infty} \Psi_{l+1}^{(i)}\ \mathbf{E}(X_{i,l+1}^r), \end{aligned} \quad ...(20)$$

where $X_{i,l+1}^r; i = 1, 2$ be a random variables having the exp-H CDF with power parameter (l+1). The moments of the exp-H distributions are given by Nadarajah and Kotz (2006). Setting $r = 1, 2$ in equation (20), we get the mean $(W_i^{(1)})$ and the variance $(\sigma^2_{Y_i})$ of $Y_i, i = 1, 2$ as

$$W_i^{(1)} = \sum_{l=0}^{\infty} \Psi_{l+1}^{(i)}\ \mathbf{E}(X_{i,l+1}^1), \quad ...(21)$$

and

$$\sigma^2_{Y_i} = \sum_{l=0}^{\infty} \Psi_{l+1}^{(i)}\ \mathbf{E}(X_{i,l+1}^2) - \left[W_i^{(1)}\right]^2. \quad ...(22)$$

, respectively. Furthermore, the *n*th central moment of Y_i, say $L_i^{(n)}$, is given by

$$L_i^{(n)} = \sum_{r=0}^{n}\sum_{l=0}^{\infty} \binom{n}{r} \left[-W_i^{(1)}\right]^{n-r} \Psi_{l+1}^{(i)}\ \mathbf{E}(X_{i,l+1}^r); i = 1, 2. \quad ...(23)$$

On the other hand, the incomplete moments are very important, which the main applications of the first incomplete moment are related to the mean deviations, Bonferroni and Lorenz curves. These curves are very useful in demography, economics, medicine, insurance and reliability. The *s*th incomplete moment of Y_i, say

$\Lambda_i^{(s)}(t_i)$ can be expressed as follows:

$$\Lambda_i^{(s)}(t_i) = \int_0^{t_i} y_i^s \; v(y_i) \; dy_i$$
$$= \sum_{l=0}^{\infty} \Psi_{l+1}^{(i)} \; \Lambda_i^{(s)*}(t_i) \;,\; i = 1, 2, \qquad ...(24)$$

where $\Lambda_i^{(s)*}(t_i) = \int_0^{t_i} y_i^s \; P_{l+1}(y_i;\eta) \; dy_i$. So, the mean deviations about the mean and the median are given by $\lambda_i^* = 2W_i^{(1)} V(W_i^{(1)}) - 2\Lambda_i^{(1)}(W_i^{(1)})$ and $\omega_i^* = W_i^{(1)} - 2\Lambda_i^{(1)}(D_{Y_i})$, respectively. Moreover, the product moment, say $\mathbf{E}(Y_1^r Y_2^r)$, can be represented as

$$\mathbf{E}(Y_1^r Y_2^r) = \int_0^{\infty}\int_0^{y_2} y_1^r y_2^r v_1(y_1, y_2) dy_1 dy_2 +$$
$$\int_0^{\infty}\int_0^{y_1} y_1^r y_2^r v_2(y_1, y_2) dy_2 dy_1 + \int_0^{\infty} y^{2r} v_3(y) \; dy$$
$$= \sum_{l,l^*=0}^{\infty} \left[\Psi_{l+1}^{(2)} \Psi_{l^*+1}(\theta_1)\Delta_{(2)}^{(r)} + \Psi_{l+1}^{(1)} \Psi_{l^*+1}(\theta_2)\Delta_{(1)}^{(r)}\right]$$
$$+ \frac{\theta_3}{\theta_1+\theta_2+\theta_3} \times \sum_{l^*=0}^{\infty} \Psi_{l^*+1}(\theta_1+\theta_2+\theta_3)\Phi_{l^*+1}^{*}, \qquad ...(25)$$

where $\Psi_{l^*}(\xi) = -\sum_{m=0}^{\infty}\sum_{j=0}^{m} \Omega_{m,j,l^*} \; (\xi)$ for $(l^* = 1, 2, 3, ...)$,

$$\Omega_{m,j,l^*}(\xi) = \frac{(-1)^{m+l}}{m!}\left(\frac{\xi}{\alpha}\right)^m \begin{pmatrix} -\alpha j \\ l^* \end{pmatrix}\begin{pmatrix} m \\ j \end{pmatrix},$$

$$\Delta_{(i)}^{(r)} = \int_0^{\infty} y_i^r \Phi_{l+1}(y_i) P_{l^*+1}(y_i;\eta) dy_i \;;\, i = 1, 2,$$

$$\Phi_{l+1}(y_i) = \int_0^{y_i} y_{3-i}^r \; P_{l+1}(y_{3-i};\eta) dy_{3-i} \;;\, i = 1, 2,$$

and

$$\Phi_{l^*+1}^{*} = \int_0^{\infty} y^{2r} \; P_{l^*+1}(y;\eta) dy \;.$$

So, by using equations (20) and (25) when $r = 1$, we get

$$\mathbf{Cov}(Y_1, Y_2) = \sum_{l,l^*=0}^{\infty} \left[\Psi_{l+1}^{(2)} \Psi_{l^*+1}(\theta_1)\Delta_{(2)}^{(1)} + \Psi_{l+1}^{(1)} \Psi_{l^*+1}(\theta_2)\Delta_{(1)}^{(1)}\right]$$
$$+ \frac{\theta_3}{\theta_1+\theta_2+\theta_3} \times \sum_{l^*=0}^{\infty} \Psi_{l^*+1}(\theta_1+\theta_2+\theta_3)\Phi_{l^*+1}^{*}$$
$$- \sum_{l=0}^{\infty}\Psi_{l+1}^{(1)} \; \mathbf{E}(Y_{1,l+1}^{1})\sum_{l=0}^{\infty}\Psi_{l+1}^{(2)} \; \mathbf{E}(Y_{2,l+1}^{1}),$$

where $\mathbf{Cov}(Y_1, Y_2) = \mathbf{E}(Y_1 Y_2) - \mathbf{E}(Y_1)\mathbf{E}(Y_2)$. Thus, we can compute the variance of $(Y_1 + Y_2)$ as follows $\mathbf{Var}\,(Y_1 + Y_2) = \sum_{i=1}^{2} \sigma_{Y_i}^2 + \mathbf{Cov}(Y_1, Y_2)$. If the two random variables Y_1 and Y_2 are uncorrelated $\rightarrow \mathbf{Cov}(Y_1, Y_2) = 0$ and $\mathbf{Var}\,(Y_1 \pm Y_2) = \sum_{i=1}^{2} \sigma_{Y_i}^2$

Conditional PDFs

The conditional PDF of Y_i given $Y_j = y_j$, $(i, j = 1, 2,$ $i \neq j)$, is given by

$$v_{Y_i|Y_j}(y_i \mid y_j) = \begin{cases} v_{Y_i|Y_j}^{(1)}(y_i \mid y_j) & ;\; 0 < y_i < y_j < \infty \\ v_{Y_i|Y_j}^{(2)}(y_i \mid y_j) & ;\; 0 < y_j < y_i < \infty \\ v_{Y_i|Y_j}^{(3)}(y_i \mid y_j) & ;\; 0 < y_i = y_j < \infty, \end{cases} \qquad ...(26)$$

where

$$v_{Y_i|Y_j}^{(1)}(y_i \mid y_j) =$$
$$\theta_i h(y_i;\eta)\left[\overline{H}(y_i;\eta)\right]^{-(\alpha+1)} e^{\frac{\theta_i}{\alpha}\left\{1-\left[\overline{H}(y_i;\eta)\right]^{-\alpha}\right\}},$$

$$v_{Y_i|Y_j}^{(2)}(y_i \mid y_j) =$$

$$\frac{\theta_j(\theta_i+\theta_3)h(y_i;\eta)\left[\overline{H}(y_i;\eta)\right]^{-(\alpha+1)}}{(\theta_j+\theta_3)e^{\frac{\theta_3}{\alpha}\left\{1-\left[\overline{H}(y_j;\eta)\right]^{-\alpha}\right\}}} e^{\frac{(\theta_i+\theta_3)}{\alpha}\left\{1-\left[\overline{H}(y_i;\eta)\right]^{-\alpha}\right\}},$$

and

$$v_{Y_i|Y_j}^{(3)}(y_i \mid y_j) = \frac{\theta_3}{\theta_j+\theta_3} e^{\frac{\theta_i}{\alpha}\left\{1-\left[\overline{H}(y_i;\eta)\right]^{-\alpha}\right\}}.$$

Equation (26) can be obtained by substituting from equations (6) and (9) in the relation $v_{Y_i|Y_j}(y_i \mid y_j) = \frac{v_{Y_i,Y_j}(y_i,y_j)}{v_{Y_j}(y_j)}$, $(i \neq j = 1, 2)$.

Stress-strength reliability function

There are appliances, which survive due to their strength. These appliances receive a certain level of stress (load). The load may be defined as temperature, environment, mechanical load, and electric current, etc. However, if a higher level of load is applied, then their strength is unable to sustain and they break down. Let $Y_1 \sim Go-H\;(\theta_1+\theta_3,\alpha,\eta)$ be a random variable representing the stress, and $Y_2 \sim Go-H\;(\theta_2+\theta_3,\alpha,\eta)$ be a random variable representing the strength, then the reliability function R_* is given as follows:

$$R_* = \mathbf{P}[Y_1 < Y_2] = \frac{\theta_2+\theta_3}{\theta_1+\theta_2+2\theta_3}. \qquad ...(27)$$

Similarly, if Y_2 is a random variable representing the stress, and Y_1 is a random variable representing the strength, then the reliability function R^* is

$$R^* = \mathbf{P}[Y_2 < Y_1] = \frac{\theta_1+\theta_3}{\theta_1+\theta_2+2\theta_3}. \quad ...(28)$$

It is clear that the stress-strength model does not depend on the baseline function $H(y_i;\eta)$.

Joint hazard rate function and its marginal functions

Let (Y_1, Y_2) be a two dimensional random variable with joint PDF $v_{Y_1,Y_2}(y_1,y_2)$ and joint reliability function $\overline{V}_{Y_1,Y_2}(y_1,y_2)$. Basu (1971) defined the bivariate hazard rate (BHR) function, say $hz_{Y_1,Y_2}(y_1,y_2)$, as follows: $hz_{Y_1,Y_2}(y_1,y_2) = \frac{v_{Y_1,Y_2}(y_1,y_2)}{\overline{V}_{Y_1,Y_2}(y_1,y_2)}$. So, if the random vector (Y_1, Y_2) has the BGo-H family, then the BHR function is given by

$$hz_{Y_1,Y_2}(y_1,y_2) = \begin{cases} hz_1(y_1,y_2) & ; \quad 0<y_1<y_2<\infty \\ hz_2(y_1,y_2) & ; \quad 0<y_2<y_1<\infty \\ hz_3(y) & ; \quad 0<y_1=y_2=y<\infty, \end{cases} \quad ...(29)$$

where

$$hz_1(y_1,y_2) = \theta_1(\theta_2+\theta_3)h(y_1;\eta)h(y_2;\eta)\left[\{\overline{H}(y_1;\eta)\}\{\overline{H}(y_2;\eta)\}\right]^{-(\alpha+1)},$$

$$hz_2(y_1,y_2) = \theta_2(\theta_1+\theta_3)h(y_1;\eta)h(y_2;\eta)\left[\{\overline{H}(y_1;\eta)\}\{\overline{H}(y_2;\eta)\}\right]^{-(\alpha+1)},$$

and

$$hz_3(y) = \theta_3 h(y;\eta)\left\{\overline{H}(y;\eta)\right\}^{-(\alpha+1)}.$$

The marginal hazard rate (HR) functions $hz_i(y_i), i=1,2$ of the BGo-H family can be represented by

$$hz_i(y_i) = (\theta_i+\theta_3)h(y_i;\eta)\left\{\overline{H}(y_i;\eta)\right\}^{-(\alpha+1)}; \; i=1,2. \quad ...(30)$$

Furthermore, the joint reliability function of (Y_1, Y_2) can be represented in terms of the HR functions as follows:

$$\overline{V}_{Y_1,Y_2}(y_1,y_2) = \exp\left\{-\int_0^{y_1}\zeta_1(v,\infty)dv - \int_0^{y_2}\zeta_2(y_1,v)dv\right\}$$

or

$$\overline{V}_{Y_1,Y_2}(y_1,y_2) = \exp\left\{-\int_0^{y_1}\zeta_1(v,y_2)dv - \int_0^{y_2}\zeta_2(\infty,v)dv\right\}.$$

where $\zeta_1(y_1,\infty) = hz_1(y_1)$ and $\zeta_2(\infty,y_2) = hz_2(y_2)$ are the marginal HR functions of Y_1 and Y_2, respectively. Further, if $hz_2(y_1,y_2) = hz_1(y_1)\times hz_2(y_2) \longleftrightarrow$ the variables Y_1 and Y_2 are independent.

Joint reversed hazard rate function and its marginal functions

Bismi (2005) defined the bivariate reversed hazard rate (BRHR) function as a scalar, given by $rhz_{Y_1,Y_2}(y_1,y_2) = \frac{v_{Y_1,Y_2}(y_1,y_2)}{V_{Y_1,Y_2}(y_1,y_2)}$. So, if the random vector (Y_1, Y_2) has the BGo-H family, then

$$rhz_{Y_1,Y_2}(y_1,y_2) = \begin{cases} rhz_1(y_1,y_2); & 0<y_1<y_2<\infty \\ rhz_2(y_1,y_2); & 0<y_2<y_1<\infty \\ rhz_3(y) & ; \; 0<y_1=y_2=y<\infty, \end{cases} \quad ...(31)$$

$$rhz_1(y_1,y_2) = v_{Go-H}(y_1;\theta_1,\alpha,\eta)\times v_{Go-H}(y_2;\theta_2+\theta_3,\alpha,\eta)\times[1-\overline{V}_{Y_1}(y_1;\theta_1+\theta_3,\alpha,\eta) - \overline{V}_{Y_2}(y_2;\theta_2+\theta_3,\alpha,\eta) + \overline{V}_{Y_1}(y_1;\theta_1,\alpha,\eta)\times\overline{V}_{Y_2}(y_2;\theta_2+\theta_3,\alpha,\eta)]^{-1},$$

$$rhz_2(y_1,y_2) = v_{Go-H}(y_1;\theta_1+\theta_3,\alpha,\eta)\times v_{Go-H}(y_2;\theta_2,\alpha,\eta)\times[1-\overline{V}_{Y_1}(y_1;\theta_1+\theta_3,\alpha,\eta) - \overline{V}_{Y_2}(y_2;\theta_2+\theta_3,\alpha,\eta) + \overline{V}_{Y_1}(y_1;\theta_1+\theta_3,\alpha,\eta)\times\overline{V}_{Y_2}(y_1;\theta_2,\alpha,\eta)]^{-1},$$

and

$$rhz_3(y) = \frac{\theta_3}{\theta_1+\theta_2+\theta_3}v_{Go-H}(y;\theta_1+\theta_2+\theta_3,\alpha,\eta)\times[1-\overline{V}_Y(y;\theta_1+\theta_3,\alpha,\eta) - \overline{V}_Y(y;\theta_2+\theta_3,\alpha,\eta) + \overline{V}_Y(y;\theta_1+\theta_2+\theta_3,\alpha,\eta)]^{-1}.$$

The marginal reversed hazard rate (RHR) functions $rhz_i(y_i), i=1,2$ of the BGo-H family can be expressed as follows:

$$rhz_i(y_i) = \frac{(\theta_i+\theta_3)h(y_i;\eta)\left\{\overline{H}(y_i;\eta)\right\}^{-(\alpha+1)}}{e^{\frac{-(\theta_i+\theta_3)}{\alpha}\left\{1-\left[\overline{H}(y_i;\eta)\right]^{-\alpha}\right\}}-1}; \quad i=1,2. \qquad ...(32)$$

Special models

Bivariate Gompertz-log-logistic distribution (BGoLLD)

Let $H(y;a,b) = \frac{y^b}{y^b+a^b}$, for $a,b,y>0$, for $a,b,y>0$, be the CDF of the log-logistic distribution, then the joint survival of the BGoLLD is given by

$$\overline{V}_{Y_1,Y_2}(y_1,y_2) = \prod_{i=1}^{3} e^{\frac{\theta_i}{\alpha}\left\{1-\left[1-\frac{y_i^b}{y_i^b+a^b}\right]^{-\alpha}\right\}}. \qquad ...(33)$$

Figure 1 shows the joint PDF, BHR function and the BRHR function of the BGoLLD for the parameters $\theta_1 = \theta_2 = \theta_3 = 5, \alpha = b = 3$ and $a = 50$.

Bivariate Gompertz-Frechet distribution (BGoFD)

Let $H(y;a,b) = e^{-\left(\frac{a}{y}\right)^b}$, for $a,b,y>0$, be the CDF of the Frechet distribution, then the joint survival of the BGoFD is given by

$$\overline{V}_{Y_1,Y_2}(y_1,y_2) = \prod_{i=1}^{3} e^{\frac{\theta_i}{\alpha}\left\{1-\left[1-e^{-\left(\frac{a}{y_i}\right)^b}\right]^{-\alpha}\right\}}. \qquad ...(34)$$

Figure 2 shows the joint PDF, BHR function and the BRHR function of the BGoFD for the parameters $\theta_1 = \theta_2 = \theta_3 = 3, \alpha = 2, b = 0.5$ and $a = 50$.

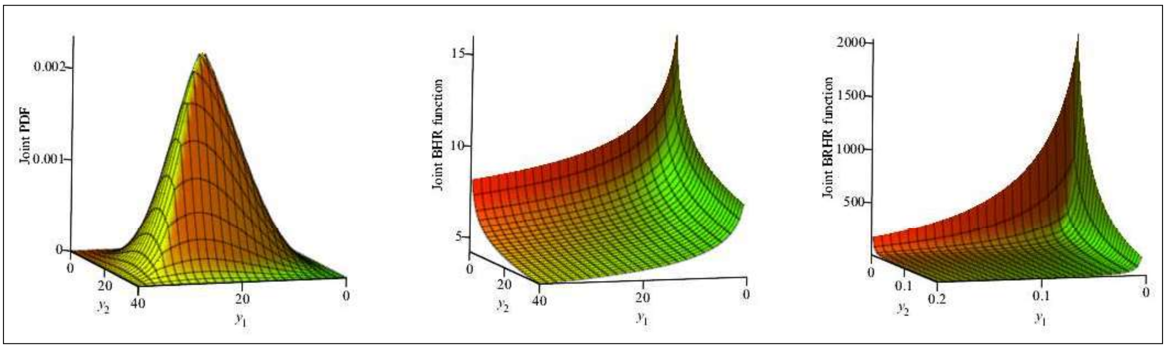

Figure 1: The surface plots of the joint PDF (left panel), BHR function (middle panel) and the BRHR function (right panel) of the BGoLLD for $\theta_1 = \theta_2 = \theta_3 = 5, \alpha = b = 3$ and $a = 50$.

Bivariate Gompertz-Weibull distribution (BGoWD)

Let $H(y;a,b) = 1 - e^{-ay^b}$, for $a,b,y>0$, be the CDF of the Weibull distribution, then the joint survival of the BGoWD is given by

$$\overline{V}_{Y_1,Y_2}(y_1,y_2) = \prod_{i=1}^{3} e^{\frac{\theta_i}{\alpha}\left\{1-\left[e^{-ay_i^b}\right]^{-\alpha}\right\}}. \qquad ...(35)$$

Figure 3 shows the joint PDF, BHR function and the BRHR function of the BGoWD for the parameters $\theta_1 = \theta_2 = \theta_3 = 3, \alpha = 2, b = 0.5$ and $a = 0.8$.

From Figures 1, 2 and 3, we note the BGo-H family presents different shapes of the joint PDF, BHR function and the BRHR function for different baseline CDF $H(y;\eta)$.

Maximum likelihood estimation (MLE)

In this section, we estimate the unknown parameters of the BGo-H family using the maximum likelihood method. Suppose that $(y_{11}, y_{21}), (y_{12}, y_{22}),..., (y_{1n}, y_{2n})$ is a sample of size n from the BGo-H family. We use the following notation $I_1 = \{y_{1i} < y_{2i}\}$, $I_2 = \{y_{1i} > y_{2i}\}$, $I_3 = \{y_{1i} = y_{2i} = y_i\}$, $I = I_1 \cup I_2 \cup I_3$, $|I_1| = n_1$, $|I_2| = n_2$, $|I_3| = n_3$, and $|I| = n_1 + n_2 + n_3 = n$.

Based on the observations, the likelihood function $l(\Upsilon)$ of this sample is

$$l(\Upsilon) = \prod_{i\in I_1} v_1(y_{1i}, y_{2i}) \prod_{i\in I_2} v_2(y_{1i}, y_{2i}) \prod_{i\in I_3} v_3(y_i). \qquad ...(36)$$

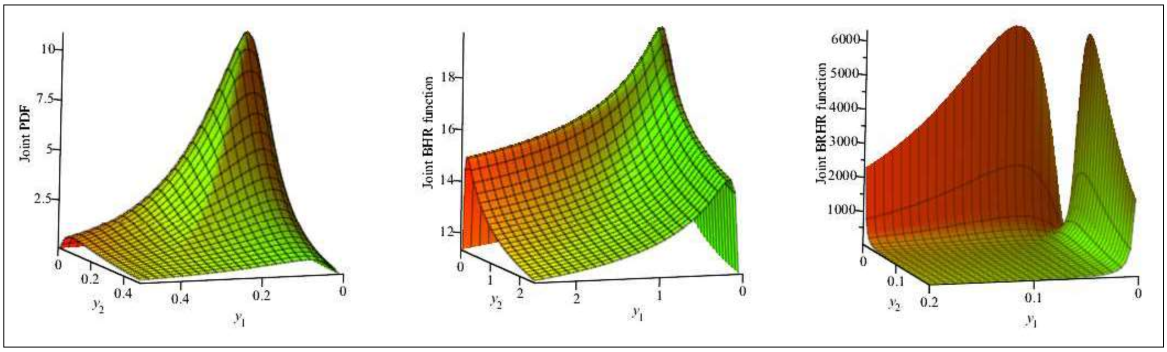

Figure 2: The surface plots of the joint PDF (left panel), BHR function (middle panel) and the BRHR function (right panel) of the BGoFD for $\theta_1 = \theta_2 = \theta_3 = 3, \alpha = 2, b = 0.5$ and $a = 50$

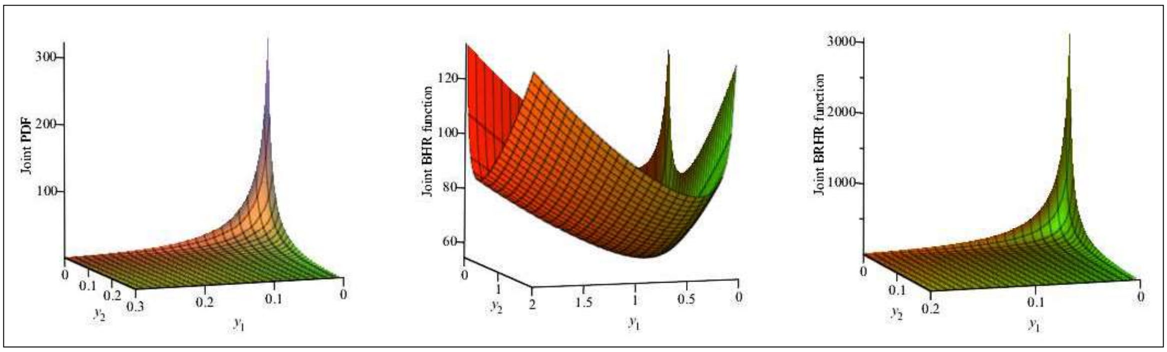

Figure 3: The surface plots of the joint PDF (left panel), BHR function (middle panel) and the BRHR function (right panel) of the BGoWD for $\theta_1 = \theta_2 = \theta_3 = 3, \alpha = 2, b = 0.5$

Substituting equation (6) into equation (36), the log-likelihood function $L(\boldsymbol{\Upsilon})$ can be written as

$$L(\boldsymbol{\Upsilon}) = n_1 \ln\left[\theta_1\left(\theta_2+\theta_3\right)\right] + \sum_{i\in I_1} \ln\left[h(y_{1i};\eta)\right] -$$

$$(\alpha+1)\sum_{i\in I_1} \ln\left[\overline{H}(y_{1i};\eta)\right] + \frac{\theta_1}{\alpha}\sum_{i\in I_1}\left[1-\left(\overline{H}(y_{1i};\eta)\right)^{-\alpha}\right]$$

$$-(\alpha+1)\sum_{i\in I_1} \ln\left[\overline{H}(y_{2i};\eta)\right] + \sum_{i\in I_1} \ln\left[h(y_{2i};\eta)\right]$$

$$+\frac{\theta_2+\theta_3}{\alpha}\sum_{i\in I_1}\left[1-\left(\overline{H}(y_{2i};\eta)\right)^{-\alpha}\right]$$

$$+n_2 \ln\left[\theta_2\left(\theta_1+\theta_3\right)\right] + \sum_{i\in I_2} \ln\left[h(y_{1i};\eta)\right]$$

$$-(\alpha+1)\sum_{i\in I_2} \ln\left[\overline{H}(y_{1i};\eta)\right]$$

$$+\frac{\theta_1+\theta_3}{\alpha}\sum_{i\in I_2}\left[1-\left(\overline{H}(y_{1i};\eta)\right)^{-\alpha}\right] + \sum_{i\in I_2} \ln\left[h(y_{2i};\eta)\right]$$

$$-(\alpha+1)\sum_{i\in I_2} \ln\left[\overline{H}(y_{2i};\eta)\right] + \frac{\theta_2}{\alpha}\sum_{i\in I_2}\left[1-\left(\overline{H}(y_{2i};\eta)\right)^{-\alpha}\right]$$

$$+n_3 \ln\theta_3 + \sum_{i\in I_3} \ln\left[h(y_i;\eta)\right] - (\alpha+1)\sum_{i\in I_3} \ln\left[\overline{H}(y_i;\eta)\right]$$

$$+\frac{\theta_1+\theta_2+\theta_3}{\alpha}\sum_{i\in I_3}\left[1-\left(\overline{H}(y_i;\eta)\right)^{-\alpha}\right]. \quad \text{...(37)}$$

The first partial derivatives of equation (37) with respect to $\theta_1, \theta_2, \theta_3, \alpha$ and η_k $(k = 1, 2, 3, ...)$ are

$$\frac{\partial L}{\partial \theta_1} = \frac{n_1}{\theta_1} + \frac{1}{\alpha}\sum_{i\in I_1}\left[1-\left(\overline{H}(y_{1i};\eta)\right)^{-\alpha}\right] + \frac{n_2}{\theta_1+\theta_3} +$$

$$\frac{1}{\alpha}\sum_{i\in I_2}\left[1-\left(\overline{H}(y_{1i};\eta)\right)^{-\alpha}\right] + \frac{1}{\alpha}\sum_{i\in I_3}\left[1-\left(\overline{H}(y_i;\eta)\right)^{-\alpha}\right], \quad \text{...(38)}$$

$$\frac{\partial L}{\partial \theta_2}=\frac{n_1}{\theta_2+\theta_3}+\frac{1}{\alpha}\sum_{i\in I_1}\left[1-\left(\overline{H}(y_{2i};\eta\right)^{-\alpha}\right]+\frac{n_2}{\theta_2}+$$

$$\frac{1}{\alpha}\sum_{i\in I_2}\left[1-\left(\overline{H}(y_{2i};\eta\right)^{-\alpha}\right]+\frac{1}{\alpha}\sum_{i\in I_3}\left[1-\left(\overline{H}(y_i;\eta\right)^{-\alpha}\right],$$

$$\frac{\partial L}{\partial \theta_3}=\frac{n_1}{\theta_2+\theta_3}+\frac{1}{\alpha}\sum_{i\in I_1}\left[1-\left(\overline{H}(y_{2i};\eta\right)^{-\alpha}\right]+\frac{n_2}{\theta_1+\theta_3}+$$

$$\frac{1}{\alpha}\sum_{i\in I_2}\left[1-\left(\overline{H}(y_{1i};\eta\right)^{-\alpha}\right]+\frac{n_2}{\theta_3}+\frac{1}{\alpha}\sum_{i\in I_3}\left[1-\left(\overline{H}(y_i;\eta\right)^{-\alpha}\right],$$

...(40)

$$\frac{\partial L}{\partial \alpha}=-\sum_{i\in I_1}\ln\left(\overline{H}(y_{1i};\eta)\right)+$$

$$\frac{\theta_1}{\alpha^2}\sum_{i\in I_1}\left[\left(\overline{H}(y_{1i};\eta)\right)^{-\alpha}\left\{\alpha\ln\left(\overline{H}(y_{1i};\eta)\right)+1\right\}-1\right]$$

$$-\sum_{i\in I_1}\ln\left(\overline{H}(y_{2i};\eta)\right)+\frac{\theta_2+\theta_3}{\alpha^2}\sum_{i\in I_1}\left[\left(\overline{H}(y_{2i};\eta)\right)^{-\alpha}\right.$$

$$\left.\left\{\alpha\ln\left(\overline{H}(y_{2i};\eta)\right)+1\right\}-1\right]-\sum_{i\in I_2}\ln\left(\overline{H}(y_{1i};\eta)\right)+$$

$$\frac{\theta_1+\theta_3}{\alpha^2}\sum_{i\in I_2}\left[\left(\overline{H}(y_{1i};\eta)\right)^{-\alpha}\left\{\alpha\ln\left(\overline{H}(y_{1i};\eta)\right)+1\right\}-1\right]$$

$$-\sum_{i\in I_2}\ln\left(\overline{H}(y_{2i};\eta)\right)+\frac{\theta_2}{\alpha^2}\sum_{i\in I_2}\left[\left(\overline{H}(y_{2i};\eta)\right)^{-\alpha}\right.$$

$$\left.\left\{\alpha\ln\left(\overline{H}(y_{2i};\eta)\right)+1\right\}-1\right]-\sum_{i\in I_3}\ln\left(\overline{H}(y_i;\eta)\right)+$$

$$\frac{\theta_1+\theta_2+\theta_3}{\alpha^2}\sum_{i\in I_2}\left[\left(\overline{H}(y_i;\eta)\right)^{-\alpha}\left\{\alpha\ln\left(\overline{H}(y_i;\eta)\right)+1\right\}-1\right],$$

...(41)

and

$$\frac{\partial L}{\partial \eta_k}=\sum_{i\in I_1}\frac{[h(y_{1i};\eta)]^{(\eta_k)}}{h(y_{1i};\eta)}+\sum_{i\in I_1}\frac{[h(y_{2i};\eta)]^{(\eta_k)}}{h(y_{2i};\eta)}$$

$$-\theta_1\sum_{i\in I_1}\left(\overline{H}(y_{1i};\eta)\right)^{-\alpha-1}[H(y_{1i};\eta)]^{(\eta_k)}$$

$$-(\theta_2+\theta_3)\sum_{i\in I_1}\left(\overline{H}(y_{2i};\eta)\right)^{-\alpha-1}[H(y_{2i};\eta)]^{(\eta_k)}$$

$$+(\alpha+1)\sum_{i\in I_1}\frac{[H(y_{1i};\eta)]^{(\eta_k)}}{\overline{H}(y_{1i};\eta)}+(\alpha+1)\sum_{i\in I_1}\frac{[H(y_{2i};\eta)]^{(\eta_k)}}{\overline{H}(y_{2i};\eta)}$$

$$+\sum_{i\in I_2}\frac{[h(y_{1i};\eta)]^{(\eta_k)}}{h(y_{1i};\eta)}+\sum_{i\in I_2}\frac{[h(y_{2i};\eta)]^{(\eta_k)}}{h(y_{2i};\eta)}$$

$$-(\theta_1+\theta_3)\sum_{i\in I_2}\left(\overline{H}(y_{1i};\eta)\right)^{-\alpha-1}$$

$$[H(y_{1i};\eta)]^{(\eta_k)}+(\alpha+1)\sum_{i\in I_2}\frac{[H(y_{1i};\eta)]^{(\eta_k)}}{\overline{H}(y_{1i};\eta)}$$

$$-\theta_2\sum_{i\in I_2}\left(\overline{H}(y_{2i};\eta)\right)^{-\alpha-1}[H(y_{2i};\eta)]^{(\eta_k)}$$

$$+(\alpha+1)\sum_{i\in I_2}\frac{[H(y_{2i};\eta)]^{(\eta_k)}}{\overline{H}(y_{2i};\eta)}+\sum_{i\in I_3}\frac{[h(y_i;\eta)]^{(\eta_k)}}{h(y_i;\eta)}$$

$$-(\theta_1+\theta_2+\theta_3)\sum_{i\in I_3}\left(\overline{H}(y_i;\eta)\right)^{-\alpha-1}[H(y_i;\eta)]^{(\eta_k)}$$

$$+(\alpha+1)\sum_{i\in I_3}\frac{[H(y_i;\eta)]^{(\eta_k)}}{\overline{H}(y_i;\eta)}$$

...(42)

where $[A(.)]^{(\eta_k)}$ means the derivative of the function A(.) with respect to η_k. By equating the equations (38 – 42) by zeros, we get the non-linear normal equations. So, the solution has to be obtained numerically.

Simulation results

In this section, the MLE method is used to estimate the parameters $a, b, \alpha, \theta_1, \theta_2$ and θ_3 of the BGoLLD. The population parameters are generated using software **R** package. The sampling distributions are obtained for different sample sizes *n* = [50; 250; 600; 1000] from *N* = 1000 replications. This study presents an assessment of the properties of the MLE for the parameters in terms of variance (Var) and mean square error (MSE). The following algorithm shows how to generate data from the BGoLLD:

1. Generate U_1, U_2 and U_3 from $U(0,1)$.

2. Compute $Z_k = H^{-1}\left\{1-\left(1-\frac{\alpha}{\theta_k}\log\left[1-U_k\right]\right)^{\frac{-1}{\alpha}}\right\}$; k = 1, 2, 3.

3. Obtain $Y_j = \min\{Z_j, Z_3\}$; $j = 1, 2$.

Table 1: MLEs, Var and MSE values for the BGoLLD

n	Parameter	Estimate	Var	MSE
50	$a = 0.7$	0.7654	0.07694	0.08121
	$b = 0.9$	0.9877	0.10317	0.11086
	$\alpha = 1.5$	1.8015	0.35470	0.44560
	$\theta_1 = 1.6$	1.7218	0.14329	0.15812
	$\theta_2 = 1.7$	1.7598	0.07035	0.07392
	$\theta_3 = 1.8$	1.8659	0.07752	0.08187
250	$a = 0.7$	0.7501	0.05894	0.03644
	$b = 0.9$	0.9701	0.08247	0.06593
	$\alpha = 1.5$	1.8001	0.35305	0.36738
	$\theta_1 = 1.6$	1.7048	0.12329	0.12957
	$\theta_2 = 1.7$	1.7265	0.03117	0.00697
	$\theta_3 = 1.8$	1.8425	0.05001	0.04853
600	$a = 0.7$	0.7200	0.02354	0.02394
	$b = 0.9$	0.9308	0.03623	0.03718
	$\alpha = 1.5$	1.6187	0.13964	0.15373
	$\theta_1 = 1.6$	1.6784	0.09223	0.09838
	$\theta_2 = 1.7$	1.7015	0.00176	0.00176
	$\theta_3 = 1.8$	1.8288	0.03388	0.03471
1000	$a = 0.7$	0.7002	0.0002	0.00023
	$b = 0.9$	0.9126	0.01480	0.01495
	$\alpha = 1.5$	1.6014	0.11929	0.12957
	$\theta_1 = 1.6$	1.6358	0.04211	0.04339
	$\theta_2 = 1.7$	1.7004	0.00047	0.00048
	$\theta_3 = 1.8$	1.8015	0.00175	0.00177

Table 1 lists the MLEs, Var and MSE values for the BGoLLD.

From Table 1, we note that the Var and the MSE are reduced as the sample size is increased. These results indicate that the BGoLLD works well under the situation where no censoring occurs, and the MLE is a good method to estimate the model parameters.

RESULTS AND DISCUSSION: REAL DATA ANALYSIS

This data represents football (soccer) data of the UEFA Champion's League (Meintanis, 2007). We consider the BGoLLD to analyse this data comparing with other famous bivariate models, such as Marshall-Olkin bivariate exponential (MOBE); bivariate generalised exponential (BGE); bivariate Gumbel exponential (BGuE); bivariate Burr X exponential (BBUXE); bivariate Weibull exponential (BWE); bivariate generalised linear failure rate (BGLFR); bivariate Weibull (BW); bivariate exponentiated Weibull (BEW); bivariate generalised power Weibull (BGPW); bivariate Gompertz (BGo); bivariate generalised Gompertz (BGGo); bivariate Gumbel Gompertz (BGuGo); bivariate Burr X Gompertz (BBUXGo); bivariate exponentiated Weibull Gomperz (BEWGo); bivariate exponetiated modied Weibull extension (BEMWEx), bivariate expone- tiated log-logistic (BELL); and bivariate Kumaraswamy log-logistic (BKwLL) models. To make this comparison, we will use the log-likelihood values (L), Bayesian information criterion (BIC), Akaike information criterion (AIC), correct Akaike information criterion (CAIC) and Hannan-Quinn information criterion (HQIC). Figure 4 shows the data representation.

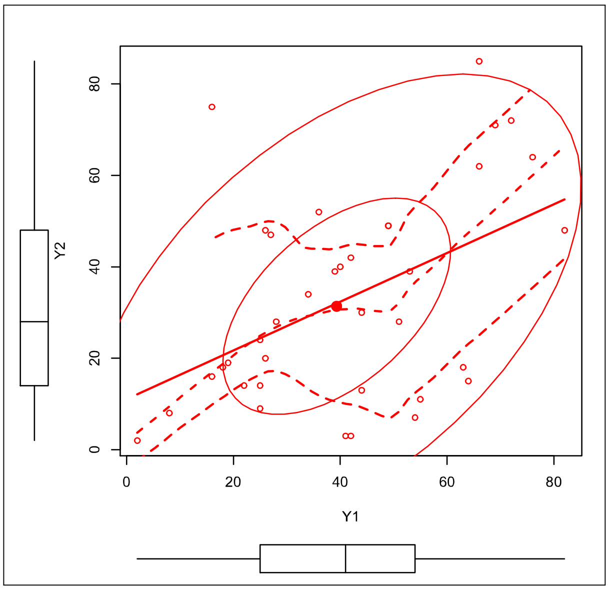

Figure 4: The scatter plot for football data

Table 2: The $-L$, K-S and p values for Y_1, Y_2 and min(Y_1, Y_2)

	Y_1			Y_2			min(Y_1, Y_2)		
Model	$-L$	K-S	p value	$-L$	K-S	p value	$-L$	K-S	p value
GoLL	161.9	0.095	0.889	162.6	0.099	0.864	157.808	0.082	0.966

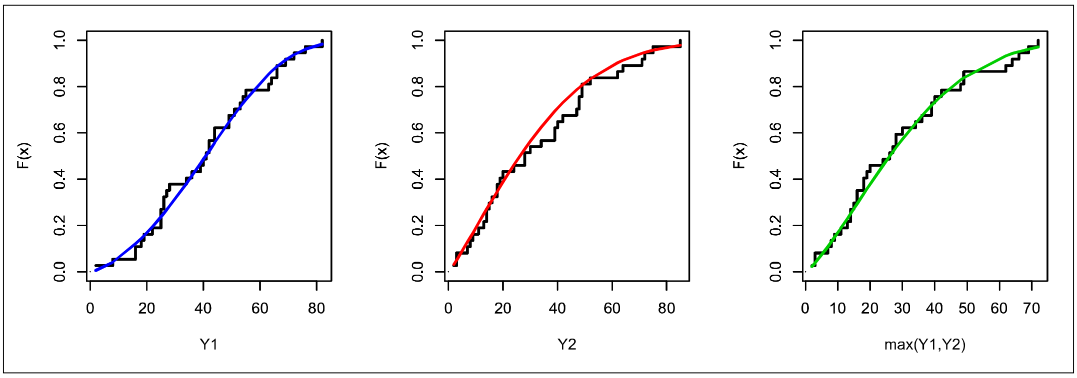

Figure 5: Estimated CDFs for the marginal distributions

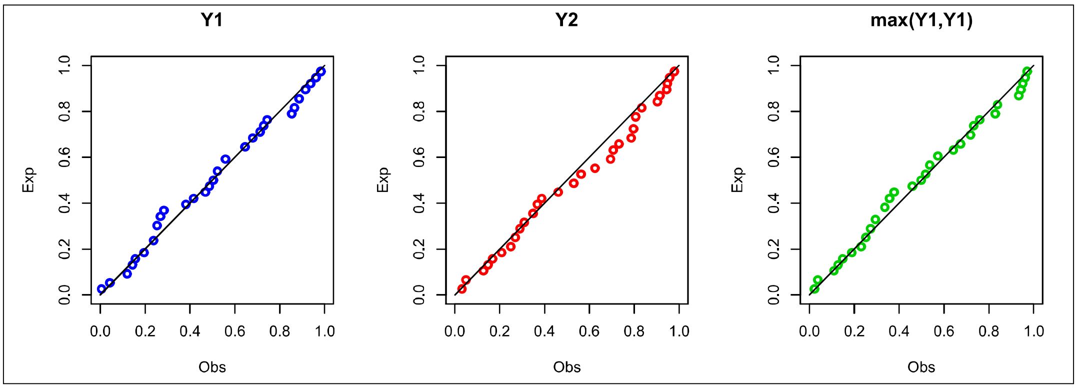

Figure 6: PP plots for the marginal distributions

Table 3: MLEs for models using football data

Model	$\widehat{\theta}_1$	$\widehat{\theta}_2$	$\widehat{\theta}_3$	$\widehat{a}$	$\widehat{b}$	$\widehat{\alpha}$	$\widehat{\gamma}$
MOBE	0.012	0.014	0.022	–	–	–	–
BGE	1.553	0.499	1.156	0.039	–	–	–
BGuE	2.678	0.962	2.065	5.011	4.081	–	–
BBUXE	0.385	0.136	0.310	0.012	–	–	–
BWE	0.135	0.302	0.265	0.025	–	–	–
BGLFR	0.452	0.156	0.360	0.0002	0.0008	–	–
BW	0.397	0.274	0.339	0.083	–	–	–
BEW	1.227	0.382	0.661	0.012	1.268	–	–
BGPW	3.229	1.983	4.084	0.037	–	–	–
BGo	0.003	0.002	0.021	0.040	–	–	–
BGGo	0.742	0.262	0.598	0.011	0.029	–	–
BGuGo	0.578	0.204	0.475	0.009	0.047	2.278	–
BBUXGo	0.132	0.187	0.201	0.006	0.015	–	–
BEWGo	0.547	0.192	0.444	0.411	0.079	0.005	1.358
BEMWEx	0.167	0.061	0.139	85.918	4.505	0.025	–
BELL	0.038	0.039	0.092	72.08	11.59	–	–
BKwLL	24.39	21.17	46.06	396.0	11.21	0.156	–
BGoLL	15.881	9.379	44.143	2903.789	1.076	144.350	–

Table 4: *L*, AIC, CAIC, HQIC and BIC values for the models using football data

Model	*L*	AIC	CAIC	BIC	HQIC
MOBE	–298.9	607.9	609.8	615.9	610.7
BGE	–299.9	607.7	608.9	614.2	609.9
BGuE	–297.8	605.6	607.5	613.6	608.4
BBUXE	–294.8	597.6	598.9	604.0	599.9
BWE	–291.1	592.3	594.2	600.3	595.1
BGLFR	–296.8	603.7	605.6	611.7	606.5
BW	–346.0	700.0	701.3	706.4	702.3
BEW	–298.9	607.9	609.8	615.9	610.7
BGPW	–344.8	697.5	698.8	703.9	699.8
BGo	–303.5	614.9	616.2	621.4	617.2
BGGo	–294.9	599.8	601.7	607.9	602.7
BGuGo	–294.2	600.5	603.3	610.1	603.9
BBUXGo	–301.2	612.4	614.3	620.5	615.2
BEWGo	–294.6	603.2	607.1	614.5	607.2
BEMWEx	–294.1	600.3	603.1	609.9	603.7
BELL	–284.9	579.8	581.8	587.9	582.7
BKwLL	–283.9	579.9	582.7	589.6	583.3
BGoLL	–272.8	557.6	560.4	567.3	561.0

Before trying to analyze the data using the BGoLLD, we fit at first the marginals Y_1 and Y_2 separately and the $\min(Y_1, Y_2)$ on the UEFA Champion's League data. The MLEs of the parameters (θ, a, b, α) of the corresponding Gompertz-log-logistic distribution (GoLLD) for Y_1, Y_2 and $\min(Y_1, Y_2)$ are (2.675, 137.3, 1.466, 6.059) (2.874, 127.4, 1.142, 3.631) and (3.469, 102.650, 2.674, 1.278), respectively. Table 2 reports –*L*, Kolmogorov-Smirnov (K-S) distance and p values for Y_1, Y_2 and $\min(Y_1,Y_2)$. Based on the p values, it is clear that the GoLLD fits the data for the marginals. Figures 5 and 6 show the estimated CDF and PP plots for real data, which support our results in Table 2.

Now, we fit the BGoLLD on this data. Tables 3 and 4 list the MLEs, *L*, AIC, CAIC, HQIC and BIC values for the competitive models based on football data.

From Table 4, it is clear that the BGoLLD provides a better fit than the other competitive models, because it has the smallest value among –*L*, AIC, CAIC, HQIC and BIC.

CONCLUSIONS

In this paper, we have presented a new flexible bivariate generator of distributions, in the so-called bivariate Gompertz-H (BGo-H) family, whose marginal distributions are Gompertz-H families. The joint CDF and joint PDF of the BGo-H family have simple forms; therefore, this new model can be easily used in practice for modelling bivariate data restricted in the interval $(0, \infty)$. Some statistical and mathematical properties of the new family have been studied. The simulation results have indicated that the MLE works quite satisfactorily and it can be used to compute the model parameters. Also, we have analysed a real dataset and showed through goodness-of-fit tests that the proposed family can be used for modelling the data considered herein.

A multivariate extension of the Gompertz-H family is presented as conclusion. Assume $Z_1, Z_2, ..., Z_{n+1}$ be independent random variables with $Z_i \sim Go - H\ (\theta_i, \alpha, \eta)$, such that $i = 1, 2, ..., n+1$. Define $Y_j = \min\{Z_j, Z_{n+1}\}$ for $j = 1, 2, ..., n$. Hence, the joint survival function of $Y_1, Y_2, ..., Y_n$ is given by

$$\overline{V}_{Y_1,Y_2,...Y_n}(y_1, y_2, ..., y_n)$$

$$= P\left[Z_1 > y_1, Z_2 > y_2, ..., Z_n > y_n, Z_{n+1} > y\right]$$

$$= \overline{V}(y, \theta_{n+1}, \alpha, \eta) \prod_{j=1}^{n} \overline{V}(y_j, \theta_j, \alpha, \eta),$$

for $(y_1, y_2, ..., y_n) \in (0, \infty)^n$, where $y = max\{y_1, y_2, ..., y_n\}$. Clearly, the BGo-H family arises from this multivariate Gompertz-H family by taking *n* = 2. In the future, we will discuss in detail the multivariate extension of the Gompertz-H family, because it has many applications in lifetime analysis, environmental, economics, engineering and medical sciences.

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REFERENCES

Alizadeh M., Afify A.Z., Eliwa M.S. & Ali S. (2020). The odd log-logistic Lindley-G family of distributions: properties, Bayesian and non-Bayesian estimation with applications. *Computational Statistics* **35**(1): 281–308. DOI: https://doi.org/10.1007/s00180-019-00932-9

Alizadeh M., Cordeiro G.M., Pinh L.G.B. & Ghosh I. (2017). The Gompertz-G family of distributions. *Journal of Statistical Theory and Practice* **11**(1): 179–207. DOI: https://doi.org/10.1080/15598608.2016.1267668

Alizadeh M., Tahir M.H., Cordeiro G.M., Mansoor M., Zubair M. & Hamedani G.G. (2015). The Kumaraswamy Marshal-Olkin family of distributions. *Journal of the Egyptian*

Mathematical Society **23**(3): 546–557.
DOI: http- s://doi.org/10.1016/j.joems.2014.12.002

Aljarrah M.A., Lee C. & Famoye F. (2014). On generating T-Y family of distributions using quantile functions. *Journal of Statistical Distributions and Applications* **1**(1): 1–17.
DOI: https://doi.org/10.1186/2195-5832-1-2

Alzaatreh A., Lee C. & Famoye F. (2013). A new method for generating families of continuous distributions. *Metron* **71**: 63–79.
DOI: https://doi.org/10.1007/s40300-013-0007-y

Balakrishnan N. & Shiji K. (2014). On a class of bivariate exponential distributions. *Statistics and Probability Letters* **85**: 153–160.
DOI: https://doi.org/10.1016/j.spl.2013.11.009

Barlow R.E. & Proschan F. (1975). *Statistical Theory of Reliability And Life Testing: Probability Models*. Florida State University, USA.

Basu A.P. (1971). Bivariate failure rate. *American Statistics Association* **66**: 103–104.

Barreto-Souza W. & Lemonte A.J. (2013). Bivariate Kumaraswamy distribution: properties and a new method to generate bivariate classes. *Statistics* **47**(6): 1321–1342.
DOI: https://doi.org/10.1080/02331888.2012.694446

Bismi G. (2005). Bivariate Burr distributions. *PhD thesis,* Cochin University of Science And Technology, India.

Cordeiro G.M., Alizadeh M. & Ortega E.M. (2014). The exponentiated half-logistic family of distributions: properties and applications. *Journal of Probability and Statistics* **2014**: Article ID 864396.

Cordeiro M.C., Alizadeh M., Abraao D.C. & Mahdi R. (2016). The exponentiated Gompertz generated family of distributions: properties and applications. *Chilean Journal of Statistics* **7**(2): 29–50.

Domma F. (2010). Some properties of the bivariate Burr type III distribution. *Statistics* **44**(2): 203–215.
DOI: http- s://doi.org/10.1080/02331880902986547

El-Bassiouny A.H., EL-Damcese M., Abdelfattah M. & Eliwa M.S. (2016). Bivariate exponentaited generalized Weibull-Gompertz distribution. *Journal of Applied Probability and Statistics* **11**(1): 25–46.

El-Bassiouny A.H. & El-Morshedy M. (2015). The univarite and multivariate generalized slash student distribution. *International Journal of Mathematics and its Applications* **3**(3-B): 35–47.

El-Gohary A., El-Bassiouny A.H. & El-Morshedy M. (2016). Bivariate exponentiated modified Weibull extension distribution. *Journal of Statistics Applications and Probability* **5**(1): 67–78.
DOI: http://dx.doi.org/10.18576/jsap/050106

El-Morshedy M., El-Bassiouny A.H., Tahir M.H. & Eliwa M.S. (2020c). Univariate and multivariate double slash distribution: properties and application. *Journal of Statistics Applications and Probability* **9**(3): 1–14.

El-Morshedy M. & Eliwa M.S. (2019). The odd flexible Weibull-H family of distributions: properties and estimation with applications to complete and upper record data. *Filomat* **33**(9): 2635–2652.
DOI: https://doi.org/10.2298/FIL1909635E

El-Morshedy M., Eliwa M.S. & Afify A.Z. (2020a). The odd Chen generator of distributions: properties and estimation methods with applications in medicine and engineering. *Journal of the National Science Foundation of Sri Lanka.* **48**(2): 113–130 .

El-Morshedy M., Eliwa M.S., El-Gohary A. & Khalil A.A. (2020b). Bivariate exponentiated discrete Weibull distribution: statistical properties, estimation, simulation and applications. *Mathematical Sciences* **14**(1): 29–42.
DOI: https://doi.org/10.1007/s40096-019-00313-9

El-Morshedy M., Alhussain Z.A., Atta D., Almetwally E.M. & Eliwa M.S. (2020d). Bivariate Burr X generator of distributions: properties and estimation methods with applications to complete and type-II censored samples. *Mathematics* **8**(2): 264.
DOI: https://doi.org/10.3390/math8020264

Eliwa M.S. & El-Morshedy M. (2019). Bivariate Gumbel-G family of distributions: statistical properties, Bayesian and non-Bayesian estimation with application. *Annals of Data Science* **6**(1): 39–60.
DOI: https://doi.org/10.1007/s40745-018- 00190-4

Eliwa M.S. & El-Morshedy M. (2020a). Bayesian and non-Bayesian estimation of four-parameter of bivariate discrete inverse Weibull distribution with applications to model failure times, football and biological data. *Filomat* **34**: 30–49.

Eliwa M.S. & El-Morshedy M. (2020b). Bivariate odd Weibull-G family of distributions: properties, Bayesian and non- Bayesian estimation with bootstrap confidence intervals and application. *Journal of Taibah University for Science* **14**(1): 331–345.
DOI: https://doi.org/10.1080/16583655.2020.1741919

Eliwa M.S., Alhussain Z.A. & El-Morshedy M. (2020b). Discrete Gompertz-G family of distributions for over- and under-dispersed data with properties, estimation, and applications. *Mathematics* **8**(3): 358.
DOI: http- s://doi.org/10.3390/math8030358

Eliwa M.S., El-Morshedy M. & Ali S. (2020a). Exponentiated odd Chen-G family of distributions: statistical properties, Bayesian and non-Bayesian estimation with applications. *Journal of Applied Statistics* **49**.
DOI: https://doi.org/10.1080/02664763.2020.1783520

Hiba Z.M. (2016). Bivariate inverse Weibull distribution. *Journal of Statistical Computation and Simulation* **86**(12): 2335–2345.
DOI: https://doi.org/10.1080/00949655.2015.1110585

Ibrahim M., Eliwa M.S. & El-Morshedy M. (2019). Bivariate exponentiated generalized linear exponential distribution: properties, inference and applications. *Journal of Applied Probability and Statistics* **14**(2): 1–23.

Kundu D. & Gupta K. (2013). Bayes estimation for the Marshall-Olkin bivariate Weibull distribution. *Journal of Computational Statistics and Data Analysis* **57**(1): 271–281.
DOI: https://doi.org/10.1016/j.csda.2012.06.002

Kundu D. & Gupta R.D. (2009). Bivariate generalized exponential distribution. *Journal of Multivariate Analysis* **100**: 581–593.

DOI: https://doi.org/10.1016/j.jmva.2008.06.012

Lehmann E.L. (1966). Some concepts of dependence. *Annals of Mathematical Statistics* **37**: 1137–1153

Marshall A.W. & Olkin I. (1967). A multivariate exponential model. *Journal of the American Statistical Association* 62: 30–44.

Meintanis S.G. (2007). Test of fit for Marshall-Olkin distributions with applications. *Journal of Statistical Planning and Inference* **137**: 3954–3963. DOI: https://doi.org/10.1016/j.jspi.2007.04.013

Nelsen R.B. (1999). *An Introduction to Copulas*, 2nd edition. Springer, New York, USA.

Nadarajah S. & Kotz S. (2006). The exponentiated type distributions. *Acta Applicandae Mathematicae* **92**: 97–111. DOI: https://doi.org/10.1007/s10440-006-9055-0

Roozegar R. & Jafari A.A. (2016). On bivariate exponentiated extended Weibull family of distributions. *Ciência e Natura* **38**(2): 564–576. DOI:10.5902/2179-460X19496

Sarhan A., Hamilton D.C., Smith B. & Kundu D. (2011). The bivariate generalized linear failure rate distribution and its multivariate extension. *Computational Statistics and Data Analysis* **55**(1): 644–654. DOI: https://doi.org/10.1016/j.csda.2010.06.006

Tahir M.H., Zubair M., Mansoor M., Cordeiro G.M., Alizadeh M. & Hamedani G.G. (2016). A new Weibull-G family of distributions. *Hacettepe Journal of Mathematics and Statistics* **45**(2): 629–647.