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Research Article

Regular and Normal Spaces for certain Generalized Open Sets in Ideal Topological Spaces

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Abstract: In this paper, we study four types of generalized open sets: Pre-I-open sets, Semi-I-open sets, α-I-open sets, and b-I-open sets, in ideal topological spaces and define the regular and normal spaces for these generalized open sets. We establish some of their properties and compare these regular and normal spaces.

Keywords: Generalized topological spaces, Generalized open sets, Ideal topological spaces, Regular spaces, Normal spaces.

1. Introduction

Ideals in topological spaces have been considered since 1930. The subject of ideals in topological spaces was studied by Kuratowski (1966) and Vaidyanathaswamy (1944) almost half a century ago, which motivated the research in applying topological ideals to generalize the most basic properties in general topology. Janković and Hamlett (1990) obtained new topologies using old ones and introduced the notion of ideal topological spaces. It was the works of Newcomp (1967), Rancin (1972), Samuels (1975), Hamlett (1990a), Hamlett (1990b), Hamlett and Jankovic (1990), and Hamlett and Jankovic (1992) which motivated the research in applying topological ideals to generalize the most basic properties in general topology. Certain generalized open sets, such as pre-I-open sets, were defined by Dontchev (1999), semi-I-open sets and α-I-open sets were defined by Harit and Noiri (2002). A similar type of generalized open set called b-I-open set was defined and studied by Guler and Aslam (2005). The separation axioms: T_0,T_1,T_2 spaces for b-I-open set were defined and studied by the same authors (Guler and Aslam, 2005).

Here, we define regular and normal spaces for the above four types of generalized open sets and develop some of their properties. Finally, we established the relationships between the generalized open sets, relationships between some regular spaces, and relationships between some normal spaces in an ideal topological spaces.

2. Preliminaries

An ideal I on a topological space (X,τ) is defined as a non-empty collection I of subsets of X satisfying the following two conditions:

(i) If $A \in I$ and $B \subset A$ then $B \in I$.

(ii) If $A \in I$ and $B \in I$, then $A \cup B \in I$.

Given a topological space (X,τ) with an ideal I on the set X and if $P(X)$ is the set of all subsets of X, a set operator $(.)^* : P(X) \longrightarrow P(X)$ called the local function (Vaidyanathaswamy, 1944) of A with respect to τ and I, is defined as follows : for $A \subset X$, $A^*(\tau,I) = \{ x \in X \,/\, U \cap A \notin I$ for every open neighbourhood U of x $\}$. A Kuratowski closure operator Cl*(.) for a topology $\tau^*(\tau,I)$ called the *-topology, finer than τ is defined by Cl*$(A) = A \cup A^*(\tau,I)$, where there is no chance of confusion, $A^*(I)$ is denoted by A^*.

Definition 2.1. (Ekici and Noiri, 2009) An ideal topological space (X,τ,I) is said to be $\star$-extremally disconnected if the $\star$-closure of every open subset A of X is open.

Theorem 2.2. *(Ekici and Noiri, 2009) For an ideal topological space (X,τ,I), the following properties are eqivalent.*

(i) X is $\star$-extremally disconnected.

(ii) cl(int(A)) $\subseteq$ int(cl*(A)) for every subset A of X.*

Definition 2.3. A subset A of an ideal topological space (X,τ,I) is said to be

(i) pre-I-open set (Maragathavalli and Vinodhini, 2014) if $A \subseteq$ int(cl*(A)),

(ii) semi-I-open set (Dontchev, 1999) if $A \subseteq$ cl*(int(A)),

(iii) α-I-open set (Dontchev, 1999) if $A \subseteq$ int(cl*(int(A))),

(iv) b-I-open set (Harit and Noiri, 2002) if $A \subseteq$ int(cl*(A)) $\cup$ cl*(int(A)).

Definition 2.4. (Balaji and Rajesh, 2014) An ideal topological space (X,τ,I) is said to be b-I-T_1 space if for each pair of distinct points x, y of X, there exist a pair of b-I-open sets one containing x but not y and the other containing y but not x.

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Definition 2.5. An ideal topological space (X,τ,I) is said to be pre-I-T_1(semi-I-T_1,α-I-T_1) space if for each pair of distinct points x, y of X, there exist a pair of pre-I-open(semi-I-open,α-I-open) sets one containing x but not y and the other containing y but not x.

Definition 2.6. A pre-I-regular(semi-I-regular, α-I-regular, b-I-regular) space is a pre-I-T_1(semi-I-T_1, α-I-T_1, b-I-T_1) space X such that for each closed subset C of X and each point a not in C, there exist disjoint pre-I-open(semi-I-open, α-I-open, b-I-open) sets U and V in X such that $a \in U$ and $C \subset V$.

Definition 2.7. A pre-I-T_1-space X is a pre-I-normal(semi-I-normal, α-I-normal, b-I-normal) space provided that for each pair A, B of disjoint closed sets in X there exist disjoint pre-I-open(semi-I-open, α-I-open, b-I-open) sets U and V such that A is contained in U and B is contained in V.

3. Results and discussion

Proposition 3.1. *Let (X,τ,I) be an ideal topological space. Then, every α-I-open set is a pre-I-open set.*

Proof. Let A be a subset of X. Since, int(A) $\subseteq$ A we get, cl*(int(A)) $\subseteq$ cl*(A). Then, int(cl*(int(A))) $\subseteq$ int(cl*(A)). Thus, $A \subseteq$ int(cl*(int(A))) $\subseteq$ int(cl*(A)). Therefore, every α-I-open set is a pre-I-open set. □

Proposition 3.2. *Let (X,τ,I) be an ideal topological space. Then, every α-I-open set is a semi-I-open set.*

Proof. Let A be a subset of X. Since, int(cl*(int(A))) $\subseteq$ cl*(int(A)) we get A $\subseteq$ int(cl*(int(A))) $\subseteq$ cl*(int(A)). Therefore, every α-I-open set is a semi-I-open set. □

Proposition 3.3. *Let (X,τ,I) be an ideal topological space. Then, every pre-I-open set and every semi-I-open set is a b-I-open set.*

Proof. Let A be a subset of X. If A is a pre-I-open set, then $A \subseteq$ int(cl*(A)). But, int(cl*(A)) $\subseteq$ int(cl*(A)) $\cup$ cl*(int(A)). Then, $A \subseteq$ int(cl*(A)) $\subseteq$ int(cl*(A)) $\cup$ cl*(int(A)). Therefore, every pre-I-open set is a b-I-open sets.
If A is a semi-I-open set, then $A \subseteq$ cl*(int(A)). But, cl*(int(A)) $\subseteq$ int(cl*(A)) $\cup$ cl*(int(A)). Then, $A \subseteq$ cl*(int(A)) $\subseteq$ int(cl*(A)) $\cup$ cl*(int(A)). Therefore, every semi-I-open set is a b-I-open sets. □

Proposition 3.4. *Let (X,τ,I) be an ideal topological space such that X is $\star$-extremally disconnected. Then, every semi-I-open set is a pre-I-open set.*

Proof. Let A be a subset of X. Assume that, X is $\star$-extremally disconnected. Then by theorem 2.2, $A \subseteq$ cl*(int(A)) $\subseteq$ int(cl*(A)) $\subseteq$. Therefore, every semi-I-open set is a pre-I-open set. □

Lemma 3.5. *Let Y be a subspace of an ideal topological space X. Then the intersection of pre-I-open set in X and Y is pre-I-open set in Y.*

Proof. let I be the ideal on X and U be a pre-I-open set in X. Since U is pre-I-open in X, its complement $X \setminus U$ belongs to the ideal I. Now, consider the intersection $U \cap Y$ in the subspace Y. To show that $U \cap Y$ is pre-I-open in Y, we need to show that its complement $Y \setminus (U \cap Y)$ belongs to the ideal I. Note that $Y \setminus (U \cap Y)$ $= (Y \setminus U) \cup (Y \setminus Y) = (Y \setminus U)$, which is the complement of U in Y. Since $X \setminus U$ belongs to the ideal I, it follows that $(Y \setminus U) = Y \setminus (U \cap Y)$ also belongs to the ideal I. Therefore, the intersection of a pre-I-open set U in X with the subspace Y, is a pre-I-open set in Y. □

Proposition 3.6. *Every subspace of a pre-I-regular space is a pre-I-regular space.*

Proof. Let X be a pre-I-regular space and let Y be a subspace of$X$. We have to show that Y is a Pre-I-regular space. Let C be a closed set in Y and $p \in Y$, such that $p \notin C$. Since Y is a subspace $C = Y \cap K$, K is closed in X. Since $Y \subset X$, $p \in X$. As $p \notin C$, $p \notin A \cap K$, K is closed in X $p \notin K$. Since X is pre-I-regular space, $\exists$ pre-I-open sets G and H such that $p \in G$, $K \subset H$ and $G \cap H = \emptyset$. Therefore $p \in Y \cap G$, $Y \cap K \subset Y \cap H$. Since Y is a subspace of X, $Y \cap G$ and $Y \cap H$ are Pre-I-open in Y and $(Y \cap G) \cap (Y \cap H)$ $= \emptyset$. Thus Y is a Pre-I-regular space.

□

Proposition 3.7. *Every closed subspace of a pre-I-normal space is a pre-I-normal space.*

Proof. Let X be a pre-I-normal space and let Y be a closed subspace of X. For each A, B of disjoint closed set in X such that $Y \cap A$ and $Y \cap B$ are disjoint closed set in Y. Since X is pre-I-normal space there are a pair U, V of disjoint pre-I-open sets of X such that $A \subseteq U$ and $B \subseteq V$. Thus, there are a pair $Y \cap U$, $Y \cap V$ of disjoint pre-I-open sets of Y such that $Y \cap A \subseteq Y \cap U$ and $Y \cap B \subseteq Y \cap V$. Therefore, every closed subspace of a pre-I-normal space is a pre-I-normal space. □

Proposition 3.8. *Let (X,τ,I) be an ideal topological space. Then, every α-I-Regular space is a pre-I-regular space.*

Proof. Assume that, (X,τ,I) be a α-I-regular space. Then, for each closed subset C of X and each point a not in C, there exist disjoint α-I-open sets U and V in X such that $a \in U$ and $C \subset V$. Since, every α-I-open set is a pre-I-open set (X,τ,I) is a pre-I-regular space. □

Proposition 3.9. *Let (X,τ,I) be an ideal topological space. Then, every α-I-regular space is a semi-I-regular space.*

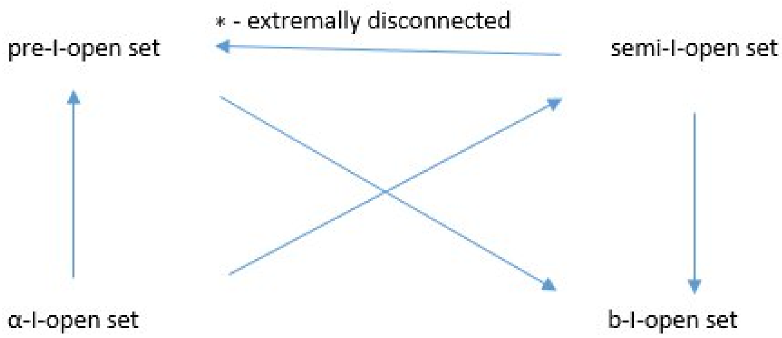

Figure 1: Relationship between the generalized open sets in an ideal topological spaces

Proof. Assume that, (X,τ,I) be a α-I-regular space. Then, for each closed subset C of X and each point a not in C, there exist disjoint α-I-open sets U and V in X such that $a \in U$ and $C \subset V$. Since, every α-I-open set is a semi-I-open set (X,τ,I) is a semi-I-regular space. □

Proposition 3.10. *Let (X,τ,I) be an ideal topological space. Then, every pre-I-regular space is a b-I-regular space.*

Proof. Assume that, (X,τ,I) be a pre-I-regular space. Then, for each closed subset C of X and each point a not in C, there exist disjoint pre-I-open sets U and V in X such that $a \in U$ and $C \subset V$. Since, every pre-I-open set is a b-I-open set (X,τ,I) is a b-I-regular space. □

Proposition 3.11. *Let (X,τ,I) be an ideal topological space. such that X is $\star$-extremally disconnected. Then, every semi-I-regular space is a pre-I-regular space.*

Proof. Assume that, (X,τ,I) be a semi-I-regular space. Then, for each closed subset C of X and each point a not in C, there exist disjoint semi-I-open sets U and V in X such that $a \in U$ and $C \subset V$. If X is $\star$-extremally disconnected. Then, every semi-I-open set is a pre-I-open set. Hence, (X,τ,I) is a pre-I-regular space. □

Proposition 3.12. *Let (X,τ,I) be an ideal topological space. Then, every semi-I-regular space is a b-I-regular space.*

Proof. Assume that, (X,τ,I) be a semi-I-Regular space. Then, for each closed subset C of X and each point a not in C, there exist disjoint semi-I-open sets U and V in X such that $a \in U$ and $C \subset V$. Then, every semi-I-open set is a b-I-open set. Hence, (X,τ,I) is a b-I-regular space. □

Proposition 3.13. *Let (X,τ,I) be an ideal topological space. Then, every α-I-normal space is a pre-I-normal space.*

Proof. Assume that, (X,τ,I) be a α-I-normal space. Then, for each pair A, B of disjoint closed sets in X, there exist disjoint α-I-open sets U and V such that A is contained in U and B is contained in V. Since, every α-I-open set is a pre-I-open set (X,τ,I) is a pre-I-normal space. □

Proposition 3.14. *Let (X,τ,I) be an ideal topological space. Then, every α-I-normal space is a semi-I-normal space.*

Proof. Assume that, (X,τ,I) be a α-I-normal space. Then, for each pair A, B of disjoint closed sets in X, there exist disjoint α-I-open sets U and V such that A is contained in U and B is contained in V. Since, every α-I-open set is a semi-I-open set (X,τ,I) is a semi-I-normal space. □

Proposition 3.15. *Let (X,τ,I) be an ideal topological space. Then, every pre-I-normal space is a b-I-normal space.*

Proof. Assume that, (X,τ,I) be a pre-I-normal space. Then, for each pair A, B of disjoint closed sets in X, there exist disjoint pre-I-open sets U and V such that A is contained in U and B is contained in V. Since, every pre-I-open set is a b-I-open set (X,τ,I) is a b-I-normal space. □

Proposition 3.16. *Let (X,τ,I) be an ideal topological space. such that X is $\star$-extremally disconnected. Then, every semi-I-normal space is a pre-I-normal space.*

Proof. Assume that, (X,τ,I) be a semi-I-normal space. Then, for each pair A, B of disjoint closed sets in X, there exist disjoint semi-I-open sets U and V such that A is contained in U and B is contained in V. If X is $\star$-extremally disconnected. Then, every semi-I-open set is a pre-I-open set. Hence, (X,τ,I) is a pre-I-normal space. □

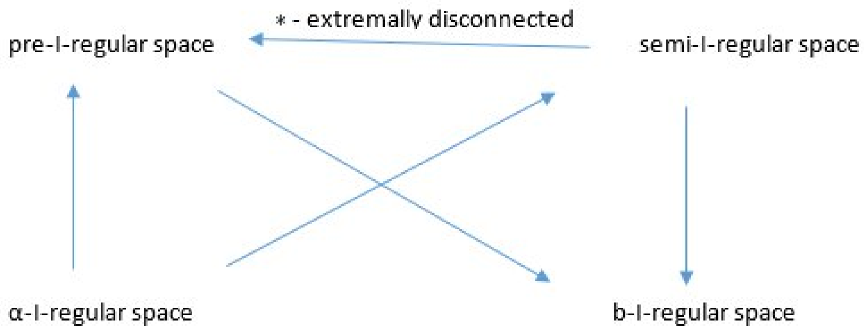

Figure 2: Relationship between some regular spaces in an ideal topological spaces

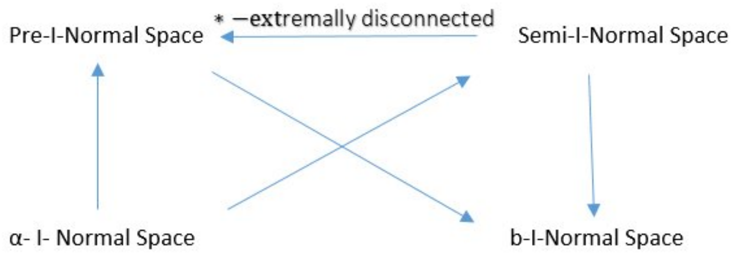

Figure 3: Relationship between some normal spaces in an ideal topological spaces

Proposition 3.17. *Let (X,τ,I) be an ideal topological space. Then, every semi-I-normal space is a b-I-normal space.*

Proof. Assume that, (X,τ,I) be a semi-I-normal space. Then, for each pair A, B of disjoint closed sets in X, there exist disjoint semi-I-open sets U and V such that A is contained in U and B is contained in V.Then, every semi-I-open set is a b-I-open set. Hence, (X,τ,I) is a b-I-normal space. □

The relationships between the generalized open sets in an ideal topological spaces are illustrated in Figure 1, the relationships between some regular spaces in an ideal topological space are illustrated in Figure 2, and the relationships between some normal spaces in ideal topological spaces are illustrated in Figure 3. The converse parts of Figure 1, Figure 2, and Figure 3 will be discussed in future work.

4. Conclusion

In this paper, we studied four types of generalized open sets: Pre-I-open sets, Semi-I-open sets, α-I-open sets and b-I-open sets in a given ideal topological space. Using these generalized open sets, we defined the regular and normal spaces and established some of their properties, and also compared all these properties of Regular and normal Spaces.

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