

# Justifying the WKB approximation in pure katabatic flows

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## ABSTRACT

Pure katabatic flow is studied with a Prandtl-type model allowing eddy diffusivity/conductivity to vary with height. Recently we obtained an asymptotic solution to the katabatic flow assuming the validity of the WKB method, which solves the fourth-order governing equation coupling the momentum and heat transfer. The WKB approximation requires that eddy diffusivity may vary only gradually compared to the calculated quantities, i.e., potential temperature and wind speed. This means that the scale height for eddy diffusivity must be higher than that for the calculated potential temperature and wind speed. The ratio between the maximum versus the mean eddy diffusivity should be less than that for the scale heights of the diffusivity versus the calculated quantities (temperature and wind). Here we justify (*a posteriori*) the WKB method independently based on two arguments: (*i*) a scaling argument and (*ii*) a philosophy behind a higher-order closure turbulence modeling. Both the eddy diffusivity maximum and the level of the relevant maximum turbulent kinetic energy are above the strongest part of the near-surface inversion and the low-level jet which is required for the WKB validity. Thus, the numerical modeling perspective lends credibility to the simple WKB modeling. This justification is important before other data sets are analyzed and a parameterization scheme written.

## 1. Introduction

Since katabatic flows are regular features of stable boundary layers (SBL) over inclined cold surfaces, it is desirable to understand their detailed structure (reviews in, e.g., Stull, 1988; Egger, 1990; Oerlemans, 1998). For instance, the turbulence parameters of katabatic flow enter the mass balance calculations for glaciers (Oerlemans and Fortuin, 1992; Van den Broeke, 1997; Oerlemans et al., 1999). At the same time, our basic understanding of katabatic flows is still limited (e.g., Mahrt, 1982; 1998; Denby and Smeets, 2000; Pawlak and Armi, 2000). In the SBL there is a multitude of processes that complicate the understanding and consequently our ability to parameterize the SBL (Zilitinkevich and Mironov, 1996). The latter is needed for developing further our weather and climate models that can deal with very stable atmospheric conditions.

Recent studies of katabatic flows show that the classic theory of Monin and Obukhov (MO) for the surface layer may not work (e.g., Mahrt, 1998; Van der Avoird and Duynkerke, 1999), a fact that was indicated earlier by Munro and Davies (1978). Although there are some significant advancements in MO theory (Zilitinkevich and Calanca, 2000), it is still questionable whether these can deal with katabatic flows having a well defined low-level jet at only several metres above the surface ( $z_j \sim 5$  m). Munro and Davies (1978) suggest that the problem with MO theory used over inclined surfaces is because of the incorrect assumption that the buoyant force acts in the vertical only. The fact is that a part of the buoyant force is directed along the slope, thus driving the katabatic wind. In very stable stratifications in general, the behavior of the SBL can be far from what MO theory suggests (e.g., Mahrt, 1998; Pahlow et al., 2001). In view of this, it is worthwhile to investigate further the katabatic model of Prandtl (1942), which couples the dynamics and thermodynamics in a simple yet efficient and apparently correct way (Defant, 1949; Egger, 1990; Oerlemans, 1998).

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One way to develop the theory of pure katabatic flow is by employing the WKB<sup>1</sup> method (Grisogono and Oerlemans 2001a,b; henceforward GOa, b). This method relies on the assumption that the background fields may vary only gradually with respect to the perturbation (calculated) fields. In other words, the background should not depart much from its constancy. More about the WKB techniques may be found in, e.g., Bender and Orszag (1978) or Gill (1982).

The aim of this study is to justify the WKB method starting with the eddy diffusivity,  $K$ , profile for pure katabatic flows. The WKB method, which implies that  $K(z) = K_{\text{WKB}}(z)$  varies only gradually, was first applied to katabatic flows in GOa. Furthermore, in GOb it was outlined why the WKB can be justified, but a proof was not given. Both papers also compare the method against a limited data set and numerical examples. In this paper we work on the ‘proof’, inspired by the questions of many reviewers and editors who dealt with the applicability of the WKB approach in the boundary-layer models addressed in Grisogono (1995a), GOa and GOb. It will be shown that the maximum of  $K(z)$  is around or above the principal maximum of the katabatic turbulent kinetic energy (TKE), which is always above  $z_j$  and the strongest part of the surface inversion  $z_{\text{inv}}$ . Hence, the  $K(z)$  maximum is also always above  $z_j$  and  $z_{\text{inv}}$ . The problem is sketched in Fig. 1 showing the typical katabatic profiles.

Previous applications of the WKB method in the SBL usually fail (Grisogono, 1994). There the WKB assumption was applied to the Taylor–Goldstein equation, requiring the wave-number function to be a gradually varying function of height. The method failed because the calculated buoyancy wave often has a longer vertical wavelength than the SBL depth, the latter being the background. This contradicts the WKB assumption, but the WKB, as other linear theories, does not know itself when it fails. But besides waves there are other important boundary layer features onto which one may exercise the WKB method. For example, the problem of baroclinic Ekman layer was treated in Berger and Grisogono (1998) applying the WKB approximation on  $K(z)$ . It was simply required that the vertical scale on which  $K(z)$  varies be longer than that of the calculated quantities, namely the wind field. A similar idea is used here as well but now addressing pure katabatic flows.

<sup>1</sup>Wentzel, Kramers, Brillouin, Jeffreys and Rayleigh (e.g., Bender and Orszag, 1978).

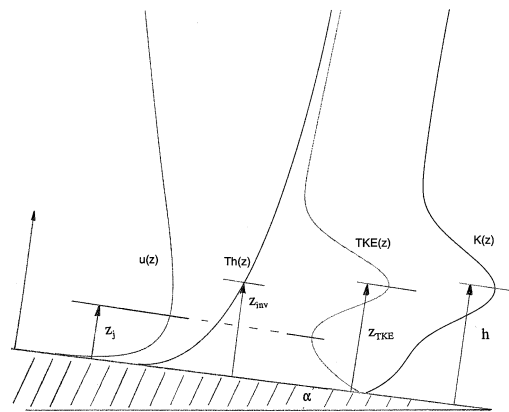


Fig. 1. A sketch illustrating the pure katabatic flow. It is similar to that in Stull (1988, p. 535) showing the wind speed  $u$  (thin), potential temperature deficit  $\Theta[=Th(z)]$  and turbulent kinetic energy TKE (very thin). Here a typical  $K(z)$  pertaining to the flow is displayed too. Various important heights are depicted: low-level jet,  $z_j$ , guessed near-surface inversion,  $z_{\text{inv}}$ , and heights of the TKE and  $K$  maxima. Note that  $\max(K)$  occurs slightly above or at the  $\max(\text{TKE})$ , but the latter is definitively above  $z_j$ . Also note that  $z_j$ , corresponding to the  $\max(u)$ , coincides to the  $\min(\text{TKE})$ .

## 2. The ‘WKB-ready’ governing equation

The pure katabatic model is one-dimensional, hydrostatic, irrotational and without a large-scale pressure gradient. It consists of the momentum and thermodynamics equations, the boundary conditions, and it is discussed in a number of papers: Defant (1949), Nappo and Rao (1987), Egger (1990), Oerlemans (1998), Denby (1999), Smeets et al. (2000), etc. Thus, the overall discussion is not repeated: only the single, governing equation for the potential temperature perturbation  $\Theta$  (the equivalent can be done for the wind speed  $u$ ) will be recalled as the starting point here after the governing set is shown first. The katabatic flow model with rotation can be found in, e.g., Denby (1999); neglecting the rotation as unimportant for the pure katabatic flow and using  $K$ -theory, the governing set is:

$$\partial u / \partial t = g(\Theta / \Theta_0) \sin(\alpha) + \partial(K Pr \partial u / \partial z) / \partial z \quad (1a)$$

$$\partial \Theta / \partial t = -u \gamma \sin(\alpha) + \partial(K \partial \Theta / \partial z) / \partial z, \quad (1b)$$

where  $Pr$  is the turbulent Prandtl number,  $\gamma$  is the potential temperature lapse rate,  $K$  is the turbulent heat conductivity,  $\alpha$  is the surface slope and the other symbols have their usual meaning. Here we are concerned

with the pure steady katabatic flow addressing relationships between  $K(z)$  and  $[\Theta(z), u(z)]$ .

First, if  $K$  and  $Pr$  are constants, then the steady form of eqs. (1a,b) gives the classic model of Prandtl (1942):

$$d^4\Theta/dz^4 + (\sigma/K)^2\Theta = 0, \tag{1c}$$

where

$$\sigma^2 \equiv g\gamma \sin^2(\alpha)/(\Theta_0 Pr) \tag{2}$$

is a suitable buoyancy frequency squared. Note that  $\sigma$  is proportional to the common buoyancy frequency,  $N \equiv (g\gamma/\Theta_0)^{1/2}$ , divided by  $Pr^{1/2}$ , the proportionality factor being  $\sin(\alpha)$ . Equation (1c) is mathematically equivalent to that for the Ekman-layer problem (e.g., Holton, 1992; Stull, 1988), having exponentially decaying sine and cosine functions as its solution. The SBL depth,  $H$ , can be ideally assumed to be governed solely by the katabatic flow. Then  $H$  follows from the flow solution depending on how it is defined. If we define it as the level where  $u = 0$ , namely  $H_{u=0}$ , then a moderate overestimation for the katabatic flow thickness is attained, which is probably closer the SBL depth itself. We adopt the definition as in Oerlemans (1998):  $H$  is the height where both  $u$  and  $\Theta$  decrease by e-fold,  $H_e$  determining the katabatic flow depth. As it will be shown later, our main result does not crucially depend on this definition. To summarize, eqs. (1c) and (2) set the stage for the classic problem of Prandtl (1942).

Second, if  $K = K(z)$ , then eqs. (1a) and (1b) yield

$$d^4\Theta/dz^4 + 4(dK/dz)/K d^3\Theta/dz^3 + f_2 d^2\Theta/dz^2 + f_1 d\Theta/dz + (\sigma/K)^2\Theta = 0 \tag{3}$$

where

$$f_2 \equiv 3d^2K/dz^2/K + 2(dK/dz/K)^2, \tag{4}$$

$$f_1 \equiv d^3K/dz^3/K + (dK/dz)d^2K/dz^2/K^2,$$

as in GOa. For  $K(z)$  that is smooth everywhere, growing from an arbitrarily small value at  $z = 0$ , reaching its maximum  $K_{MAX}$  somewhere in the SBL and then decreasing to an arbitrarily small value sufficiently high above the surface,  $z > H$ , one expects that the  $f_1$  and  $f_2$  terms can be sufficiently small that the first and the last terms in eq. (3) dominate. In other words, if  $K(z)$  is gradual enough, eq. (1c) still remains a reasonable zeroth-order approximation to eq. (3). A first-order correction is the inclusion of the second-term on the LHS of eq. (3) containing  $dK_{WKB}/dz$  (also see Grisogono, 1995a; GOa). A second-order correction

would include the  $f_2$ -term, etc. The WKB expansion goes equivalently: a zeroth-order WKB approximation solves eq. (1c) but still allowing for  $K_{WKB}(z)$ , a first-order WKB approximation solves eq. (3) with the  $K_{WKB}(z)$  and  $(dK_{WKB}/dz)$  terms but without the  $f_1$  and  $f_2$  terms, etc. Figure 2 illustrates both the constant  $K$  and WKB solutions. The WKB solution shows sharper gradients near the surface while approaching the upper boundary conditions more gradually. That is exactly what is needed from an improved theory as implied by Defant (1949) and Egger (1990, e.g., his Fig. 3.5 adapted from Defant).

Going back to eq. (1), it follows that the classic model of Prandtl holds where the following is fulfilled:

$$|dK/dz du/dz| < |K d^2u/dz^2| \tag{5a}$$

and

$$|dK/dz d\Theta/dz| < |K d^2\Theta/dz^2| \tag{5b}$$

because eq. (1c) results from  $d/dz[K d()/dz] \rightarrow K d^2()/dz^2$  in eqs. (1a,b). Obviously  $K(z)$  must be gradual and without zeros inside the domain for eq. (5) to hold. However, eq. (5) cannot be fulfilled everywhere, especially not at and very near the surface, because  $K$  should vanish there unless we adopt the roughness concept, or some other approach, with  $z \geq z_0$ , so that  $K(z_0) > 0$  (done in a weak sense for the fluxes in GOb). Instead it is better to require eq. (5) to hold only for a certain part of the domain. Namely, let eq. (5) hold around the low-level jet, because it is the jet that determines the pure katabatic flow (Oerlemans, 1998; Van der Avoird and Duynkerke, 1999). The rest of the flow will adjust to the jet governing the flow. Of course, the larger the flow domain satisfying eq. (5), the more accurate the WKB approximation. In short, it is assumed that eq. (5) should be fulfilled only in the most dynamically active part of the flow, which is around the low-level jet.

Figure 3 depicts typical profiles of the terms in eq. (5) as calculated from a WKB example in GOB using a gradual  $K_{WKB}(z)$  as discussed later. While eq. (5a) is fulfilled above 1.17 m, eq. (5b) is fulfilled only for  $z \geq z_j$ . The WKB solution, not fulfilling eq. (5b) below  $z_j$ , is shown to be significantly better than the constant- $K$  solution when compared against the data set and numerical solution (GOa; b) throughout the flow. Thus, relaxing eq. (5) away from  $z_j$  allows for sharper flow gradients that are poorly resolved with  $K = \text{const}$ . If eq. (5) were applied everywhere, it would be locally too restrictive. However, if eq. (5)

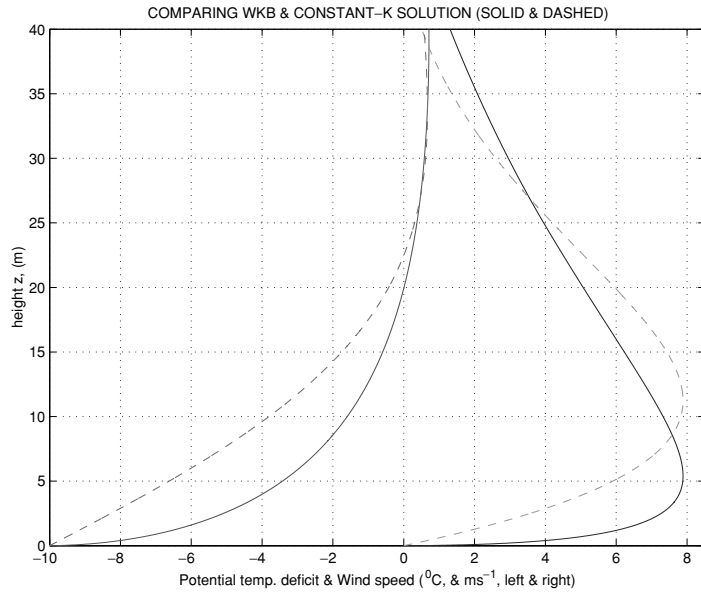


Fig. 2. Typical profiles of a pure katabatic flow with constant  $K$  (the classic solution of Prandtl, dashed) and the WKB solution using eqs. (7a–c) and (8a), solid. The particular input consists of  $(\alpha, \gamma, Pr, C, K_{MAX}, h, K_{CONST}) = (-0.1 \text{ rad}, 4 \text{ K km}^{-1}, 1.5, -10 \text{ K}, 0.25 \text{ m}^2 \text{ s}^{-1}, 25 \text{ m}, 0.1 \text{ m}^2 \text{ s}^{-1})$ . There is no input to the constant  $K$  model that would produce both the low-level gradients and the smooth transition above the low-level jet obtained by the WKB model.

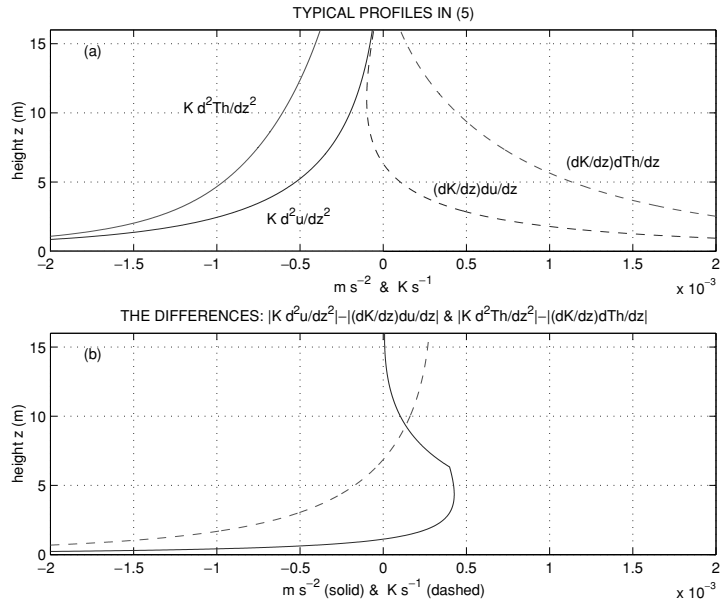


Fig. 3. Typical profiles for the terms in eq. (5). The curvature terms (solid), those involving  $d^2/dz^2$ , are on the left, and the gradient terms (dashed) are on the right (a). While the temperature-curvature term,  $|K d^2\Theta/dz^2| (= |K d^2Th/dz^2|)$ , is larger than its gradient counterpart only from  $z_j$  upward, the wind-curvature term is already so from about 1.113 m upward (b). The latter dominates the dynamics of pure katabatic flows.

holds only around  $z_j$ , then WKB captures the main features of the pure katabatic flow, and it can be said that the WKB is a valid approximation.

The gradual behavior of  $K(z)$ , i.e.,  $K(z) = K_{\text{WKB}}(z)$ , is essential and also explains the nature of the WKB technique. For instance, if  $K = \text{const}$ , then eq. (5) is trivially fulfilled, and this was the starting point for the WKB employment in GOa. The main point is that eq. (5) is valid only around  $z_j$ , because it is the low-level jet that determines the dynamics of this type of flow (Van der Avoird and Duynkerke, 1999). Away from  $z_j$ , we allow for the implicit influence of  $K$ -variations, which in turn results in a better resolution of the flow gradients. A scale analysis of eq. (5) using representative values for  $K$ ,  $u$ ,  $\Theta$  and their gradients (the maxima divided by the heights where they occur) simplifies the criterion:

$$(K_{\text{MAX}}/h)u_{\text{MAX}}/z_j \leq \bar{K} u_{\text{MAX}}/z_j^2 \quad (6a)$$

and

$$(K_{\text{MAX}}/h)\Theta_{\text{MAX}}/z_{\text{inv}} \leq \bar{K} \Theta_{\text{MAX}}/z_{\text{inv}}^2 \quad (6b)$$

where  $\bar{K}$  denotes suitable averaging of  $K(z)$  in the vicinity of the low-level jet  $z_j$ ; the other parameters are the maximum wind speed,  $u_{\text{MAX}}$ , the maximum potential temperature perturbation  $\Theta_{\text{MAX}}$  and the surface inversion height,  $z_{\text{inv}}$ . To unify the spatial averaging in eq. (6), the averaging is performed between the surface and the height where  $K$  reaches its maximum,  $h$ , since  $h$  must embrace both the jet and inversion, i.e., the WKB requirement, and it is the relevant scale-height for  $K(z)$ . Thus,  $\bar{K}$  and the averaging in eq. (6) are obtained over  $h$ ,  $h > \max(2z_j, z_{\text{inv}})$ . Note that this procedure “softens” eq. (5) because now  $\bar{K}$  is sufficiently beyond zero. For all tests performed with gradual  $K(z)$ , i.e., with  $K_{\text{WKB}}(z)$ , eq. (6) was satisfied (not shown). We use interchangeably  $K(z)$  and  $K_{\text{WKB}}(z)$  henceforth. Now, as one of the reviewers pointed out, eq. (6) results in

$$K_{\text{MAX}}/\bar{K} \leq \max(h/z_j, h/z_{\text{inv}}). \quad (6c)$$

In GOa eq. (3) was solved upward from the height  $h$ ,  $h \equiv h(K_{\text{MAX}}) = h[\max(K)]$ , with the first-order WKB method where  $K(z)$  gradually decreases, and below  $h$  with the zeroth-order WKB method where  $K(z)$  increases with height. However, for the purpose of this paper it suffices to use only the zeroth-order WKB solution, thus omitting the solution patching at  $h$ . This effectively means solving eq. (1c) but with

$K(z)$ . On the other hand, it implies that  $K$ -derivatives do not affect  $(\Theta, u)$  explicitly. Thus, the leading behavior of the WKB solution to eq. (3) taken from GOa or GOb is

$$F \approx \exp \left[ - (1+i)(\sigma/2)^{1/2} \int_0^z K^{-1/2} dz \right]. \quad (7a)$$

Here  $F$  is a dimensionless complex function,  $F \equiv (\Theta, u)_{\text{D-LESS}}$ . The  $\Theta$  and  $u$  are

$$\Theta(z) = C \text{Re}[F(z)], \text{ and } u(z) = C \mu \text{Im}[F(z)], \quad (7b)$$

$$\mu \equiv [g/(\Theta_0 Pr \gamma)]^{1/2}, \quad (7c)$$

where  $C$  is the surface potential temperature deficit,  $C < 0$  (the lower boundary condition together with no-slip for  $u$ ). Compared to the case with constant  $K$  (i.e., the classic solution of Prandtl), there is a distinctly different behavior, e.g., near the surface: here  $F$  grows rapidly for  $0 < z \ll h$ , etc. Figure 2 emphasizes some of these differences. Numerical comparisons presented and discussed in GOa, b are in the favor of the WKB solution (not shown). Note that the classical  $K$ -constant solution emerges from eq. (7) as

$$\begin{aligned} (\Theta, u) &\sim \exp\{-(1+i)[\sigma/(2K)]^{1/2}z\} \\ &= \exp[-(1+i)z/H_e], \end{aligned} \quad (7d)$$

$$H_e \equiv (4\Theta_0[\max(K)]^2 Pr / \{\gamma g[\sin(\alpha)]^2\})^{1/4}.$$

The weakness of the whole method is the prescription of  $K(z)$  instead of  $K$  being an interactive function of the flow. In the latter case, we do not know how to solve the resulting nonlinear partial differential equation analytically, and to find explicit relations between  $K$  and  $(\Theta, u)$ . Thus, we are bound to use the linear approach symbolized by eqs. (3)–(5). However, one may remedy this in a numerical iterative procedure solving eq. (1a,b) by parameterizing  $K$  in terms of the flow variables. Next we concentrate on the choice and justification of  $K(z)$ ; furthermore, a linearized link between the flow and  $K$  will be given.

### 3. The choice of $K(z)$

#### 3.1. Input parameters

Besides the realizability criterion,  $K \geq 0$ ,  $K(z)$  must satisfy the following conditions in pure katabatic flows:  $K(z \rightarrow 0) \rightarrow 0$  and again  $K(z \ll \infty) \rightarrow 0$

while reaching its maximum,  $K_{\text{MAX}}$ , at  $h$  within the SBL. Furthermore, the class of  $K(z)$  considered here must also comply with the WKB criterion that the scale height of  $K$  is larger than that of  $u$  and  $\Theta$ , namely  $h > \max(2z_j, z_{\text{inv}})$ , where  $2z_j$  and  $z_{\text{inv}}$  are the respective scale heights for  $u$  and  $\Theta$  (see below).

For modeling studies of stable and near-neutral boundary layers, the profile of  $K$  due to O'Brien (1970) is often used, see e.g., Pielke (1984) or Stull (1988). O'Brien's  $K$  is a third-order polynomial,  $K_3(z)$ . It is generalized into a linear-exponential function, in GOa, i.e.,

$$K(z) = (K_{\text{MAX}} e^{1/2}/h)z \exp[-0.5(z/h)^2]. \quad (8a)$$

With its convenient analytic properties, this  $K_{\text{WKB}}(z)$  from eq. (8a) is used instead of a  $K_3(z)$ ; its dimensionless form appears in the next section as eq. (8b). Note that if eq. (8a) is expanded for small  $z$ , it will approximate a  $K_3(z)$ . An additional advantage of eq. (8a) with respect to  $K_3(z)$  is that it has only two parameters ( $K_{\text{MAX}}, h$ ), while any  $K_3(z)$  generally needs four input parameters. This is relevant not only for analytical studies, but also for data analyses. It is possible to show that eq. (8a) can be broadened into a class of functions  $(z/h)^p$  multiplied by the exponential factor, thus yielding a broader class of  $(\Theta, u)$  profiles. For example,  $p = 2$  gives the log-profile; also, certain correction functions can be constructed in this way.

From eqs. (5), (6) and (8a) one obtains

$$\max(2z_j, z_{\text{inv}}) \leq (e^{1/2} - 1)h \quad (9a)$$

if the average  $\bar{K}$  in eq. (6) is done over  $h$ . Because numerical experiences with the WKB tells that the method is still valid even when the scale heights become closer, i.e., going toward " $\leq$ " in eq. (9a), see e.g., Bender and Orszag (1978), Laprise (1993), or Grisogono (1995a), one is able to estimate  $h$  as

$$\begin{aligned} h &\approx \text{const}_0 \max(2z_j, z_{\text{inv}})(e^{1/2} - 1)^{-1} \\ &= \text{const} \max(2z_j, z_{\text{inv}}) \end{aligned} \quad (9b)$$

where 'const' is better to be determined from data; it should be roughly  $O(1)$ . Note that eq. (9a) suggests a value close to 1.5. Two criteria are inter-related here. First, the general WKB criterion that the background quantity, here  $K_{\text{WKB}}(z)$ , varies on the scale larger than those of the calculated perturbations. That resulted in  $h > \max(2z_j, z_{\text{inv}})$  because  $K$  varies over  $h$ ,  $u$  varies over  $2z_j$  and  $\Theta$  changes over  $z_{\text{inv}}$ . Second, from eq. (6)  $z_j/h < K_{\text{average}}/K_{\text{MAX}}$  relating the former

and more general length-scale inequality to the particular gradient of  $K(z)$ . The final relation including the  $(e^{1/2} - 1)$  factor comes from the chosen form of  $K(z)$ , namely eq. (8a), the linear-exponential height dependence. Clearly, another  $K_{\text{WKB}}(z)$  will give another factor there.

Since  $du/dz = 0$  at  $z_j$ , the imaginary part of eq. (7a) maximizes at  $z_j$ . This gives another integral constraint for  $K$  having integrated  $K^{-1/2}$  up to  $z_j$  as in GOB [their eq. (4.2)], so that eq. (9), together with the approximate relation

$$K_{\text{MAX}} \approx 32N_\alpha h z_j \pi^{-2} (ePr)^{-1/2} \quad (10a)$$

derived in GOB [see their eq. (4.5)], will help estimating the profile of  $K_{\text{WKB}}$ , yielding its maximum for pure katabatic flows;  $N_\alpha \equiv \sigma Pr^{-1/2} = N \sin(\alpha)$ . One could have intuitively expected from the gradual variability of  $K_{\text{WKB}}$  that its  $K_{\text{MAX}} \propto h$  as in eq. (10a). From the dynamics point of view, it is more important that  $z_j$  enters directly in  $K_{\text{MAX}}$ . For instance, if  $(z_j, z_{\text{inv}}) = (6, 13)$  m,  $N_\alpha = 0.0015 \text{ s}^{-1}$ ,  $Pr \approx 1.5$ , then a first guess for  $h$  might be around 20 m, yielding  $K_{\text{MAX}} \approx 0.29 \text{ m}^2 \text{ s}^{-1}$ . This is a reasonable value, close to that found in GOa by fitting to a data set from the Pasterze glacier, Austria. Based on eqs. (8)–(10) and guessing a reasonable  $Pr \geq 1$ , the WKB katabatic model input is nearly completed. One way of making the procedure weakly nonlinear is by an iterative adjustment of  $h[K(K_{\text{MAX}}, h)]$  letting the new value of  $h$  tend to  $H_e$  and assigning a new value to  $K$  (and also recalculating  $z_j$ ).

Because the katabatic jet is the key dynamic feature here (e.g., Van der Avoird and Duynkerke 1999; GOB), it makes sense to also require as in Oerlemans [1998, see his eq. (19)] that  $K$  is proportional to the flow depth  $H_e$  and maximum wind speed:

$$K_{\text{MAX}} = C_1 H_e \max(u) = c_1 e^{-\pi/4} \sin(\pi/4) H_e (-C\mu). \quad (10b)$$

The thicker and the faster the katabatic flow, the larger  $K_{\text{MAX}}$ . It is beyond this analytical study to find the coefficient  $c_1 [ < O(10^{-2}) ]$ , because this requires more data to be processed. Equating eq. (10a), based on the WKB method, and eq. (10b), from the underlying dynamics, we obtain  $z_j \sim -C/[\gamma \sin^{1/2}(\alpha)]$ . Now even the scale-height of the essential dynamics, namely the height of the low-level jet, is related to the input parameters; thus, everything becomes calculable up to a constant. A similar expression was obtained by Oerlemans [1998, his eq. (24)]. The last relation

$$z_j \propto -C/[\gamma \sin^{1/2}(\alpha)], \quad (11)$$

tying in the background parameters determining  $z_j$ , together with eqs. (8)–(10) completes the estimation of  $K(z)$  that senses the pure katabatic flow. In other words, a set of constraints for  $\max(K)$  and its position is found, which is important. Now the input consisting of  $(\alpha, \gamma, C, Pr)$  determine the flow, in particular the low-level jet intensity and position. Of course, generally  $K_{\text{WKB}}(z)$  profiles are not limited to eq. (8a), provided they satisfy the WKB criterion requiring that  $K(z)$  varies gradually as stated explicitly in eq. (5) or (6). In short, we have combined here our previous findings about the katabatic flow and  $K_{\text{WKB}}$  (Oerlemans, 1998; GOa,b).

### 3.2. Spatio-temporal scales

The dimensionless integral in the exponent of eq. (7a) determines the katabatic flow depth  $H_e$  (which ideally can be related to the SBL depth, since  $H_e \leq H$ ). If the integral is considered only, it defines the square root of an important time scale related to turbulence. Consequently, turbulent-diffusion time scales are

$$\tau(z) \sim \left( \int_0^{-z} K^{-1/2} dz \right)^2.$$

If the integration goes up to  $z = H_e$ , it will correspond to the natural time scale for the steady katabatic flow defined by (or at least proportional to)  $\sigma^{-1}$ . More about  $\tau$ ,  $H$ ,  $z_j$  and their relations to turbulent fluxes can be found in GOb. Needless to say, the MO length does not appear here, but seems to be close to  $h$ . Next we want to see how eq. (8a) relates to state-of-the-art numerical models.

## 4. A numerical modeling perspective

### 4.1. Background

Here we indicate that our  $K$  is consistent with turbulence schemes in mesoscale numerical models. Besides the scaling argument in the former section relating the input  $K$ -parameters to the flow, this section provides an additional and independent view of the WKB validity.

We will relate  $K$  to TKE via turbulent length scales. First we show that  $K_{\text{WKB}}$  has its maximum above or at the  $\max(\text{TKE})$  that is numerically modeled. Knowing turbulent length scales and  $K(z)$ , one can calculate

the  $\text{TKE}(z)$ , e.g., from the steady-state balance among shear production, buoyancy loss, transport and dissipation of the TKE, e.g., Pielke (1984), Stull (1988). In a numerical model with a higher-order closure turbulence parameterization scheme, typically  $K \sim L (\text{TKE})^{1/2}$ , where  $L$  is the master turbulent length scale and TKE is a prognostic variable (Pielke, 1984; Arritt and Pielke, 1986; Nappo and Rao, 1987; Stull, 1988; Enger, 1990). The modeled katabatic TKE profile looks like that more described in Arritt and Pielke (1986), Nappo and Rao (1987), Stull (1988), Denby (1999), or as simply sketched in Fig. 1. In short, the principal  $\max(\text{TKE})$  is above  $z_j$ , since  $\min(\text{TKE})$  occurs at  $z_j$ , where TKE shear production is lacking. Above  $z_j$ , the shear reactivates, usually under a relatively weaker stratification than below  $z_j$ , and consequently the absolute  $\max(\text{TKE})$  appears.

Suppose that  $L$  is either of the Blackadar type,  $L_B$ , or the ‘z-less stratification’ type,  $L_S$  in a considered mesoscale model (e.g., Pielke, 1984; Enger, 1990; Grisogono, 1995b). The latter is governed by the local buoyancy frequency or shear (Hunt et al., 1988; Tjernström, 1993; Schumann and Gerz, 1995). The former has its upper approximate value  $L_\infty$ , which will be also used here for normalizing height, thus  $\zeta \equiv z/L_\infty$ ,  $0 \leq \zeta < \infty$ . This summarized reasoning concerning  $K$ , TKE and  $L$  is fairly general and accepted in the modeling community, though with numerous details of various complexity; it is our starting point for relating the heights of the  $K$  and TKE maxima via the appropriate  $L$ . Normalizing eq. (8a) by, e.g.,  $e^{1/2} K_{\text{MAX}}$ , we obtain

$$K(\zeta) = a\zeta \exp(-b\zeta^2), \quad (8b)$$

where  $a$  and  $b$  are positive constants obtained straightforwardly from eq. (8). While the constant  $a$  in eq. (8b) is unimportant for the discussion,  $b$  is an important parameter,  $b \equiv 0.5(L_\infty/h)^2 < O(1)$ , typically  $0 < b < 0.5$ . The smallness of  $b$  is not arbitrary, but it results from the fact that  $L_\infty \leq z_j < h$ . In very stable boundary layers, as considered here, it is only conceivable that  $L_\infty < h$ . If opposite, then the characteristic eddy size would be allowed to mix the SBL thoroughly because  $h \sim H$  ( $h \leq H$ ). However, this is impossible in the very stable and steady boundary layer (e.g., Mahrt, 1998). The maximum of eq. (8b) is at  $\zeta_K = (2b)^{-1/2}$ , corresponding to the maximum of eq. (8a) at  $z = h$ . The normalized TKE is

$$\text{TKE}(\zeta) = (K/L)^2. \quad (12)$$

Thus, dimensionless  $K$  and TKE are related via turbulent length scales. [In eq. (12)  $Pr$ , which is constant in this study, is already absorbed.] Numerical mesoscale models usually choose smoothly the minimum between  $L_B$  and  $L_S$ . Now, we address separately the choice of  $L$  in eq. (12), namely  $L_B$ , and  $L_S$ .

#### 4.2. The length scale of Blackadar type, $L_B$

From the definition  $L_B(z) \equiv kzL_\infty/(kz + L_\infty)$  the dimensionless form becomes

$$L_B(\zeta) = k\zeta/(1 + k\zeta), \quad (13)$$

where  $k$  is the von Karman constant. Combining eqs. (8b), (12) and (13), we find

$$\text{TKE}(\zeta) = (a/k)^2 \exp(-2b\zeta^2)(1 + k\zeta)^2. \quad (14)$$

The  $\max[\text{TKE}(\zeta)]$  is obtained by equating  $d(\text{TKE})/d\zeta = 0$  from eq. (14) and solving for  $\zeta$  to find the height of  $\max(\text{TKE})$ ,  $\zeta_{\text{TKE}}$ . One ends up with a quadratic expression for  $\zeta_{\text{TKE}}$ , where only one root is positive and has a physical meaning:

$$\zeta_{\text{TKE}} = [(2b)^{-1} + (2k)^{-2}]^{1/2} - (2k)^{-1}. \quad (15)$$

Now, for all values of  $b$  that are realistic for the considered SBL,  $\zeta_{\text{TKE}} < \zeta_K = (2b)^{-1/2}$ , which is an important and not previously stated result. It should be mentioned that Nappo and Rao (1987), using only a version of eq. (13), obtained a set of realistic katabatic profiles and the related TKE [see their eqs. (4)–(6) and Fig. 6]. From their work it implies a  $K$  that corresponds to our  $K_{\text{WKB}}$  having its maximum above  $z_j$ . A similar conclusion follows from Arritt and Pielke (1986), where also  $K \sim \text{TKE}^{1/2}$ , and the elevated  $\max(\text{TKE})$  is above  $z_j$ . It is interesting to add that Arritt and Pielke (1986) use the O'Brien (1970) profile, the one that is generalized in GOa and here, for their model initialization.

#### 4.3. A “z-less” type of length scale, $L_S$

Typically, here in the dimensional form

$$L_S = D \min[(\text{TKE})^{1/2}/N, (\text{TKE})^{1/2}/S], \quad 0 < D \leq 1, \quad (16)$$

where the choice is determined locally by the gradient Richardson number,  $Ri = N/S$ ,  $N$  and  $S$  being the local buoyancy frequency and absolute shear, respectively. Note that using “min” form in eq. (16) is similar to  $1/L_S \sim (S/\text{TKE} + N/\text{TKE})$  which has the same mathematical structure as  $1/L_B$ . For  $0 < Ri < 1$ , it is

$S$  that determines the turbulence frequency scale and the corresponding  $L_S$ , while  $N$  does the same for  $Ri \geq 1$ ; for details see Hunt et al. (1988) and Schumann and Gerz (1995). This  $L_S$  does not sense the height, namely  $L_S \neq L_S(z)$ . Hence, from eq. (12) TKE behaves as

$$\text{TKE}(\zeta) = \text{const } K^2, \quad (17)$$

where “const” is a physical parameter not containing the height; eq. (17) has its maximum at  $\zeta_{\text{TKE}} = \zeta_K = (2b)^{-1/2}$ . It is interesting to note the implication of  $L_S \neq L_S(z)$  in a more general context. A tedious calculation, based on  $N$  and  $S$  from eq. (7) entering eqs. (12) and (16), results in an infinite number of zeros identifying the TKE extremes (i.e., infinitely many solutions to a transcendental equation for the height where TKE reaches extreme values). This is the mathematical manifestation of the “z-less” physical concept for the length scale. Since there is no preference to which zero solution of  $d(\text{TKE})/d\zeta = 0$  to choose, any idea of  $L_S(z)$  has to be abandoned, and the name of “z-less” length scale appears naturally (as it has been used in the modeling community). The conclusion of this section considering a numerical modeling perspective is that  $K$  and TKE are related, so that  $\zeta_{\text{TKE}} \leq \zeta_K$  for the assumed  $K$ , eqs. (8a) and (8b), used to describe pure katabatic flows with the WKB method. This is in agreement with the WKB requirement addressed in the former two sections.

## 5. Conclusions

An analytic model of Prandtl type (Prandtl, 1942; Defant, 1949; Egger, 1990) for katabatic flows was formulated and solved for almost any gradually varying eddy diffusivity  $K(z) = K_{\text{WKB}}(z)$ . The asymptotic solution employed the WKB method to solve the fourth-order governing equation, but the method was not fully justified. The justification is provided here as an extension of GOa and GOB, where the steady problem was solved, and the flux calculation was presented, respectively. The WKB justification, presented here, is based on two independent arguments: a scale analysis and a philosophy of the higher-order closure for turbulence parameterization in mesoscale numerical modeling.

Simultaneously with the WKB justification, the main parameters for eddy diffusivity and conductivity  $K(z)$  in pure katabatic flows, namely  $\max(K)$  and its height  $h$ , are estimated and related to the flow



variables, potential temperature deficit and katabatic wind  $(\Theta, u)$ . It is shown here that because the  $\max(\text{TKE})$  occurs above the low-level jet,  $z_j$ , where the  $\min(\text{TKE})$  appears, the  $\max(K)$  should also be above both  $z_j$ , and the strongest part of the near-surface inversion. Hence,  $K(z)$  must vary more gradually than the calculated  $(\Theta, u)$ , see Fig. 1 (which is exactly the WKB requirement for the pure katabatic flow), an assumption made *a priori* in GOa and GOb and justified here. Auxiliary work is done here, in Oerlemans (1998) and in GOb relating the two  $K$  parameters,  $\max(K)$  and its height  $h$ , to the calculated  $(\Theta, u)$ .

In principle, the same method could be used for the Ekman layer, which is often employed in theoretical studies of large-scale dynamics highly parametrizing boundary-layer effects. Such a study, combining

results from Grisogono (1995a) and GOa, is in progress. More field data are needed addressing interplays among the low-level jet, the near-surface inversion, and the upper part of the SBL. This is also on its way using data from another glacier on Iceland. We know next to nothing about entrainment processes and intermittent wave-breaking that may take place around the SBL top, say a few tens of metres above the ground. Along these lines, there is much need for more turbulence measurements above the low-level jet.

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