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Sensitivity analysis of the climate of a chaotic system

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ABSTRACT

This paper addresses some fundamental methodological issues concerning the sensitivity analysis of chaotic geophysical systems. We show, using the Lorenz system as an example, that a naïve approach to variational ("adjoint") sensitivity analysis is of limited utility. Applied to trajectories which are long relative to the predictability time scales of the system, cumulative error growth means that adjoint results diverge exponentially from the "macroscopic climate sensitivity" (that is, the sensitivity of time-averaged properties of the system to finite-amplitude perturbations). This problem occurs even for time-averaged quantities and given infinite computing resources. Alternatively, applied to very short trajectories, the adjoint provides an incorrect estimate of the sensitivity, even if averaged over large numbers of initial conditions, because a finite time scale is required for the model climate to respond fully to certain perturbations. In the Lorenz (1963) system, an intermediate time scale is found on which an ensemble of adjoint gradients can give a reasonably accurate (O(10%)) estimate of the macroscopic climate sensitivity. While this ensemble-adjoint approach is unlikely to be reliable for more complex systems, it may provide useful guidance in identifying important parameter-combinations to be explored further through direct finite-amplitude perturbations.

1. Introduction

Much of our understanding of geophysical systems is based on computational modeling. A wide range of problems, from data assimilation to the analysis of uncertainty in model-based predictions, require an estimate of the sensitivity of model-simulated quantities to changes in initial conditions, boundary conditions, external forcing or the representation of unresolved processes. Sensitivity analysis achieves this by evaluating the change in model outputs arising from specified changes in these inputs. Here we evaluate two common approaches to sensitivity analysis, applied to the "climate" (time-averaged properties) of a chaotic system; the direct method and the adjoint method.

Under the direct method (Dickinson and Gelinas, 1976) the sensitivity of the model output to a change, say, in a model parameter is found by comparing one model integration with another, which differs only by a finite perturbation to the parameter of interest. The key disadvantage of the direct method is computational cost. If the sensitivity to a number of model parameters is required, then a separate model integration must be performed for each one.

An alternative method of sensitivity analysis is to use the adjoint to the model (Errico and Vukicevic, 1992; Errico, 1997; Talagrand, 1997; Giering and Kaminski, 1998). The adjoint gives the sensitivity of a single component of the model

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output to infinitesimal changes in all the model input variables simultaneously. In contrast, the direct method yields the full perturbed model output to a non-vanishing change in a single model input. The adjoint method is therefore attractive when the important features of the model output are known a priori and we wish to discover those model inputs that are most important in determining them (Hall, 1986).

We wish to compare the direct and adjoint methods of finding the sensitivity of a chaotic model with respect to parameter changes. Specifically, we ask if results converge to the true "climate" sensitivity (which is independent of initial conditions) as the integration time increases. The kind of application we are interested in is illustrated by Hall (1986) who used the adjoint method to study the sensitivity of a simplified atmospheric General Circulation Model (GCM) to changes in various parameters including the level of CO_2. He based his analysis on 10-day integrations and raised the possibility of using much longer trajectories to analyse the sensitivity of a coupled GCM. We suggest here that a longer integration would not give useful results because, for chaotic systems, the adjoint sensitivity does not converge smoothly to a consistent climate sensitivity value with increasing integration time.

This question of finding a sensitivity which is not strongly dependent on initial conditions interested Corti and Palmer (1997) who used an adjoint model to estimate the sensitivity of atmospheric flow patterns to forcings that may occur in climate change scenarios. This problem also arises in 4-dimensional variational assimilation, in which adjoint sensitivities are used to minimise the model/data misfit over a specified assimilation window, and gives rise to an upper limit in the assimilation window length (Cong et al., 1998; Schröter et al., 1993).

To explore these issues requires many model integrations so using a realistic geophysical model such as an atmospheric or oceanic GCM is not feasible. Instead, the Lorenz (1963) system is used; this is much simpler than a GCM while retaining the same essential chaotic behaviour. In Section 2 the Lorenz (1963) system is reviewed. In Sections 3 and 4 numerical experiments are performed to compare the direct and adjoint-based estimates of the sensitivity. In Section 5, possible ways around the limitations found in the adjoint method are investigated. Section 6 contains the concluding discussion.

2. Lorenz equations

The model studied here is the Lorenz (1963) system, often used as an analogue for atmospheric behaviour because it mimics some of the properties of the large-scale atmospheric circulation; notably the occurrence of regime behaviour, distinct time scales and variations in local predictability (Palmer, 1993). The Lorenz equations are three non-linear ordinary differential equations derived from a simplified version of the Rayleigh–Bénard problem, which concerns fluid convection between two horizontal plates (Emanuel, 1994). That is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\sigma x + \sigma y, \tag{1a}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -xz + rx - y, \tag{1b}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = xy - bz. \tag{1c}$$

In this paper, we focus on z, which is related to the heat flow between the plates, and r, which is proportional to the temperature difference between the plates. In numerical examples, we use a 4th order Runge–Kutta method to integrate the Lorenz equations (1a–c) with a time-step of 0.005, where $\sigma = 10$ and $b = \frac{8}{3}$.

In a climate study, attention is focused on time-averaged properties rather than one particular trajectory. So in our analogue we consider the average value of z over a time-interval of length τ,

$$\bar{z}(r, \tau, x_0, y_0, z_0) = \frac{1}{\tau}\int_0^{\tau} z(t)\,\mathrm{d}t, \tag{2}$$

where x_0, y_0, z_0 are the initial conditions at $t = 0$.

We define the "climatological" value of z, $\bar{z}_\infty$, as the value of $\bar{z}$ as $\tau \to \infty$ (in this stationary, ergodic system, this is equivalent to the expectation value of z). For a given value of r, $\bar{z}$ consistently converges to a single value of $\bar{z}_\infty$, that is independent of initial conditions, for all but a small number of "pathological" initial conditions (such as setting x_0, y_0, z_0 to a fixed point). We shall therefore treat $\bar{z}_\infty$ as a single-valued function of r, while noting the possibility of these nongeneric trajectories and ensuring, of course, that we avoid them in our numerical experiments.

Fig. 1 shows that an increase in r raises the value of $\bar{z}$. We intuitively expect this behaviour because, in the Rayleigh–Bénard system, an increase in the temperature difference between the horizontal plates (r) should produce an increase in the average flux of heat ($\bar{z}$). The analogue in a climate GCM might be the dependence of the equator to pole heat flux on meridional temperature gradient, for instance. For the Lorenz system, the increase in $\bar{z}$ with r can be anticipated by considering the governing equations directly and determining the positions of the fixed points around which the trajectory orbits. In this case, the fixed points are given by $z = r - 1$, so $\bar{z}$ grows as the attractor surrounding the fixed points grows with r. In general, however, finding the positions of the fixed points may not be helpful in predicting the sensitivity of a quantity to a change in a model input. For example, changes in inputs may perturb the probability density function of the system attractor without significantly changing the locations of the fixed points (Palmer, 1999). Furthermore, in the case of the Lorenz system, there is a third unstable fixed point ($z = 0$): in a practical problem knowing how the fixed points affect the behaviour of the attractor may not be possible.

3. Relationship between $\bar{z}$ and r

In most practical situations, we are interested in the response of a system to finite perturbations in parameters, which we refer to as the "macroscopic sensitivity" or slope. For a given set of initial conditions and integration time, the slope of the average value of z is

$$\frac{\Delta \bar{z}}{\Delta r} = \frac{\bar{z}(r + \Delta r) - \bar{z}(r)}{\Delta r}. \tag{3}$$

Fig. 2 suggests that, for the Lorenz (1963) system, the slope of $\bar{z}_\infty$ (the macroscopic climate sensitivity, $\Delta\bar{z}_\infty/\Delta r$) is a smoothly-varying, single-valued function of r over a wide range of values of r. That is, for a given r, $\Delta\bar{z}/\Delta r$ converges to a single value which is independent of x_0, y_0, z_0 and τ and, for small finite perturbations, also independent of Δr. Note that this does not necessarily follow from the assumption that $\bar{z}_\infty$ itself is a single-valued function of r. Lorenz (1964) draws attention to a system in which $\bar{z}_\infty$ is a highly discontinuous function of r, such that $\Delta\bar{z}_\infty/\Delta r$ would generally remain dependent on Δr no matter how small the perturbation.

The four panels in Fig. 2 show $\bar{z}(r)$ for 4 integration times and 2 initial conditions obtained in each case by performing a large number of integrations

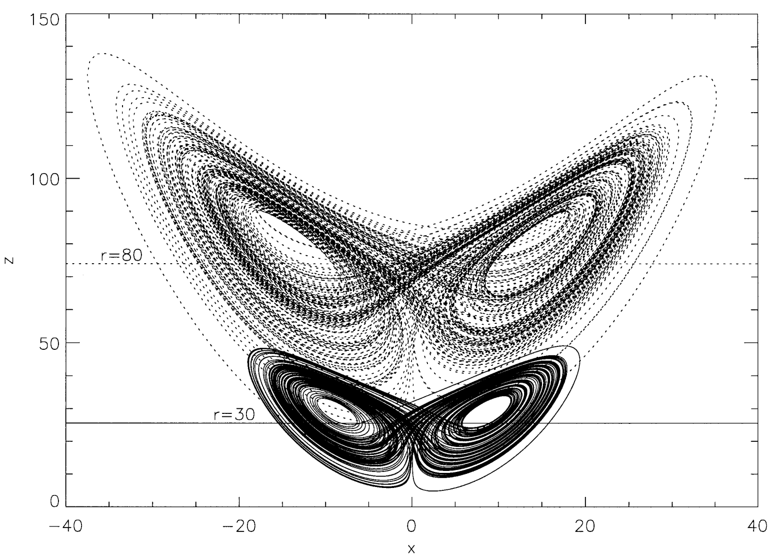

Fig. 1. Graphs of the Lorenz attractor for $r = 30$ and $r = 80$. The horizontal lines indicate the value of $\bar{z}$ in each case (for $\tau = 131.36$).

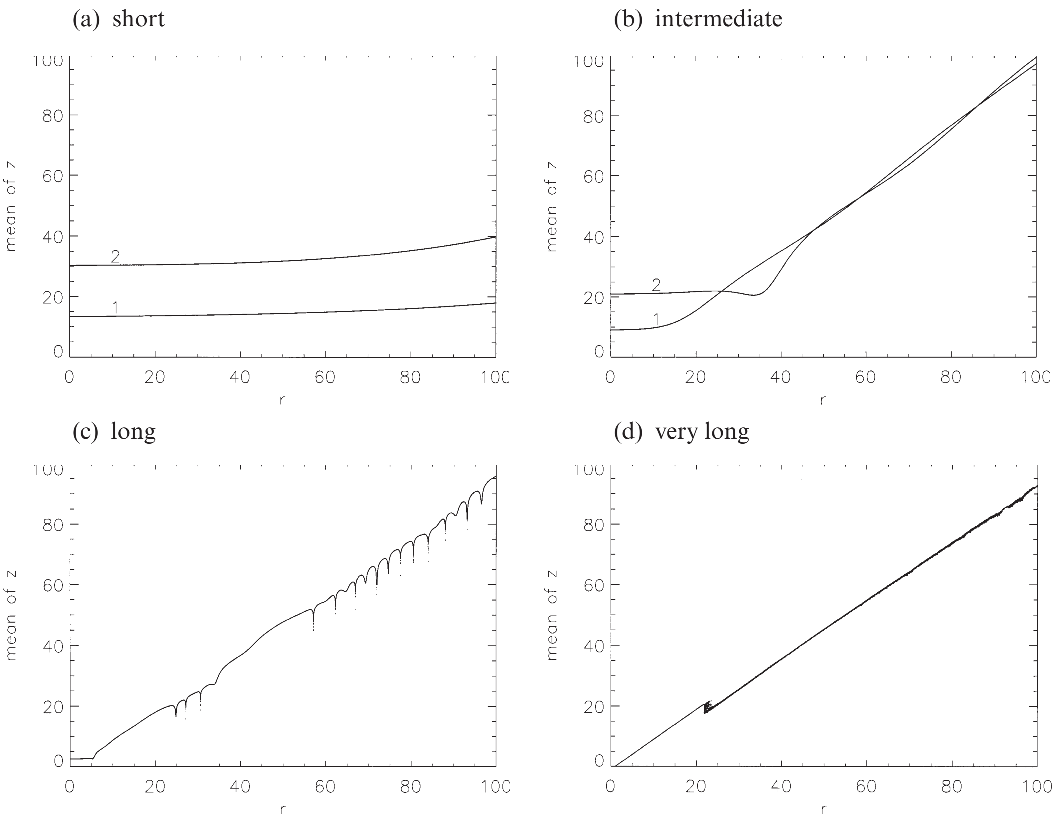

Fig. 2. Graphs of $\bar{z}$, the average of z over integration time (eq. (2)), against r. Panels show graphs for (a) a short integration ($\tau = 0.1$), (b) an intermediate integration ($\tau = 0.44$), (c) a long integration ($\tau = 2.26$), and (d) a very long integration ($\tau = 131.36$). In (a) and (b), line 1 is for initial conditions $x_0 = -2.40$, $y_0 = -3.70$, $z_0 = 14.98$ and line 2 is for initial conditions $x_0 = 8.00$, $y_0 = -2.00$, $z_0 = 36.05$. Graphs (c) and (d) show the results for the first set of initial conditions only. Values of $\bar{z}$ are calculated at intervals of $r = 0.005$.

scanning over a range of values of r at intervals of 0.005. The integration times are referred to as short, intermediate, long and very long, and are $\tau = 0.1$, $\tau = 0.44$, $\tau = 2.26$ and $\tau = 131.36$, respectively. The significance of these integration times is as follows: the short integrations only traverse part of an orbit around the Lorenz attractor; the intermediate integrations complete a full orbit; the long integrations complete several such orbits; and the very long integrations travel around the attractor O(100) times. The very long integrations are analogous to a climate integration of a GCM or a multi-year integration in an eddy-resolving ocean model where many life cycles of the chaotic geostrophic eddies are completed. In these cases, the effects of individual eddies average out, revealing the underlying climate sensitivity as r varies. This averaging or smoothing appears to have occurred for the very long integrations of the Lorenz system (Fig. 2d), giving a nearly linear dependence of $\bar{z}$ on r with a slope of around 0.96 (estimated by a linear fit to Fig. 2d), independent of initial conditions (at least, in these two cases and in several others we have tried, with the exception of fixed points noted above). This means that, in this case, the macroscopic climate sensitivity $\Delta\bar{z}_\infty/\Delta r$ is nearly independent of x_0, y_0, z_0 and Δr.

The assumption that $\bar{z}_\infty$ is a single-valued and continuous function of r appears only to be valid over a limited range of values of r. Discontinuities are apparent for $r > 90$, where the system switches in and out of limit cycles (Frøyland and Alfsen, 1984) and around $r = 24.74$ where the system switches from transient chaos to continuous chaos. In the neighbourhood of these bifurcations, $\bar{z}_\infty$ is (presumably) discontinuous and the gradient of the climatological value of z, $\partial\bar{z}/\partial r$, is not well-defined. Fortunately, for many practical applications this kind of discontinuous behaviour appears to be unusual. For example, recent laboratory experiments show a smooth dependence of heat flux on Rayleigh number in hard Rayleigh–Bénard turbulence (Glazier et al., 1999).

As we are unable to perform integrations of infinite length to determine $\bar{z}_\infty(r)$, an important question is how long an integration is necessary to estimate the slope, $\Delta\bar{z}_\infty/\Delta r$, accurately? The short integrations (Fig. 2a) are not long enough because they have a smaller overall slope than either the intermediate, long or very long integrations. The reason for this is that a change in the value of r takes a non-vanishing time to affect the value of z. This can be seen by considering the sensitivity as $\tau \to 0$. For a sufficiently short integration, of length τ, we can use a Taylor expansion to yield

$$\frac{\partial \bar{z}}{\partial r} = \frac{1}{\tau}\int_0^\tau \frac{1}{2!}\, x_0^2 \Delta t^2 \, \mathrm{d}(\Delta t) + \mathrm{O}(\tau^3). \qquad (4)$$

Hence $\partial\bar{z}/\partial r \simeq (1/6)x_0^2\tau^2$. Therefore for very small τ (accurate for $\tau < 0.01$ in the current case) we expect a very small sensitivity, which depends on x_0 and τ but not on r. Unfortunately, there is no way we can infer the macroscopic climate sensitivity of ≈ 0.96 from eq. (4). For the short integrations ($\tau = 0.1$, Fig. 2a), eq. (4) is no longer accurate yet the integration is still too short to display the full macroscopic sensitivity.

This behaviour can also be seen in other systems. For example, Hall (1986) considers the sensitivity of near-surface air temperature to a change in the level of CO_2 in an atmospheric GCM. He finds a response of 0.12 K to a doubling of CO_2 using an adjoint sensitivity analysis of a 10-day integration. Although consistent with direct calculations with fixed sea surface temperatures (as Hall (1986) used), this sensitivity is much less than the range of coupled GCM estimates of the sensitivity to doubled-CO_2 of 1.5–4.5 K (Kattenberg et al., 1995). The coupled GCMs have a larger sensitivity because they react by changing the sea surface temperature in a response that takes longer than 10 days to fully evolve. Thus this discrepancy is directly analogous to the problem highlighted by Fig. 2a; whatever the size of the perturbation, the system still needs a finite time to display its response.

The intermediate, long and very long integrations have all completed one or more orbits around the attractor, and in contrast to the short integration all three cases display the same overall increase of $\bar{z}$ with r. Thus integrations of intermediate length or longer give a good estimate of $\bar{z}_\infty(r)$ and therefore of the macroscopic climate sensitivity. In detail, however, the intermediate, long and very long integrations do differ. Although $\bar{z}(r)$ in the intermediate integrations is smoothly varying, the long integration displays isolated extrema (for example at $r \approx 57$). The cause of these isolated extrema is chaotic error growth; a perturbation Δr produces a change in the model variables that grows exponentially leading, after sufficient time, to a completely different trajectory on the same basic attractor. This is seen in Fig. 3 which shows the trajectory of $z(t)$ for $r = 57$ and for $r = 57.175$. Intuitively, we expect an increase in r to produce an increase in $\bar{z}$, but in this case the small increase in $\bar{z}_\infty$ is completely overpowered by the exponential divergence of nearby trajectories. This gives a local, initial-condition-dependent *decrease* in $\bar{z}$, and an extremum in the graph of $\bar{z}$ against r.

As we increase the length of the integration, averaging over longer time scales, the size of these extrema are reduced. Fig. 4 shows that the

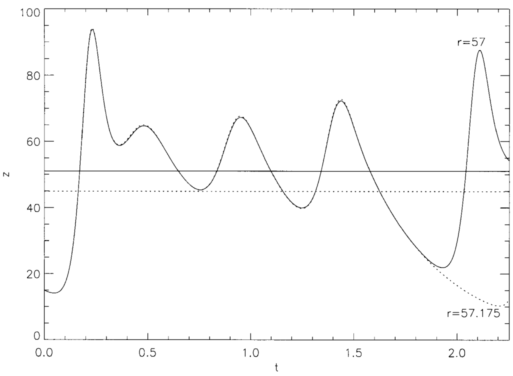

Fig. 3. The time evolution of $z(t)$ is plotted for $r = 57$ (solid curve) and $r = 57.175$ (dotted curve). Although r has increased, the time mean of z, $\bar{z}$ (horizontal lines), has decreased. This decrease in $\bar{z}$ produces the isolated extremum seen in Fig. 2c at $r = 57.175$.

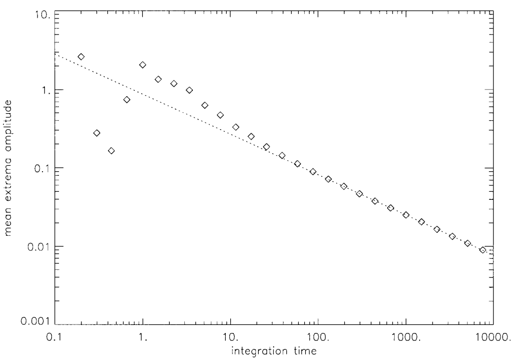

Fig. 4. Mean amplitude of the isolated extrema (Fig. 2) plotted against integration time τ. The mean amplitude of the extrema is found by considering $\bar{z}(r)$ for two different initial conditions, and subtracting them from one another to leave only the initial-condition-dependent extrema. The mean extrema amplitude is the standard deviation of this difference.

amplitude of the isolated extrema scales with $1/\sqrt{\tau}$, as if they were generated by a linear stochastic process. Thus, we expect the amplitude of these extrema to tend to zero as the integration time tends to infinity. In other words, the slope of the average value of z ($\Delta\bar{z}/\Delta r$) calculated on a finite time (for a fixed perturbation Δr) converges to the macroscopic climate sensitivity ($\Delta\bar{z}_\infty/\Delta r$) in the limit of a long integration time. We can therefore use finite-integration-time macroscopic sensitivities (the direct method) with reasonable confidence to estimate the underlying macroscopic climate sensitivity, provided that (a) we allow sufficient time for the system to display its true sensitivity, (b) Δr is large enough that the error due to local extrema is small, and, (c) Δr is small enough that the underlying curvature of the climate variable $\bar{z}_\infty$ is weak.

4. Relationship between $\partial\bar{z}/\partial r$ and r

In practice, for the case of sensitivity to many parameters, the direct method becomes prohibitively expensive. We now investigate whether the adjoint method, a promising alternative technique, can be used to find the climate sensitivity.

Fig. 5 shows $\partial\bar{z}/\partial r$ as a function of r for the same integration times and initial conditions used in Fig. 2. For each value of r, an exact value of $\partial\bar{z}/\partial r$ for this set of initial conditions, integration time and finite-difference formulation is computed using the model adjoint linearised about the original model trajectory (Errico and Vukicevic, 1992; Errico, 1997; Talagrand, 1997; Giering and Kaminski, 1998). This will, of course, be different from the value of $\partial\bar{z}/\partial r$ for the underlying continuous system, but the difference is irrelevant to the points we address here. The gradients for the short, intermediate and long integration times are as would be expected from differentiating the functions in Fig. 2. The long integrations, for example, show the effect of the isolated extrema leading to gradients that peak near $\pm 10^4$, compared to ≈ 0.96 for the macroscopic climate sensitivity. These values are highly unrepresentative of the macroscopic trend of $\bar{z}$ with r and also strongly dependent on the initial conditions. Furthermore, this behaviour deteriorates for the very long integrations where the gradients are $O(10^{100})$. These pathological values would not have been anticipated from visual inspection of Fig. 2 and indicate the presence of many small-amplitude but tightly packed extrema.

Fig. 6 shows how $|\partial\bar{z}/\partial r|$ varies with integration time τ for various ranges of r. It suggests that the infinitesimal or microscopic sensitivity can be written as

$$|\partial\bar{z}/\partial r| \propto e^{\lambda\tau}. \tag{5}$$

The values of the exponent λ are consistent with the first global Lyapunov exponents found by Frøyland and Alfsen (1984) (see also Trevisan and Legnani, 1995). This result supports the idea that the exponential growth in microscopic sensitivity is due to chaotic error growth produced by a change in r. Since the perturbation is infinitesimal this exponential growth will continue indefinitely without saturating, so there is no reason in principle why Fig. 6 cannot be extrapolated indefinitely to an arbitrarily long integration. Hence, we expect that as the integration length increases then $\partial\bar{z}/\partial r$ grows without bound, giving an infinite microscopic climate sensitivity. Thus, z_∞ appears to behave like a Weierstrass function (McShane, 1989), at least in the range $25 \leqslant r \leqslant 50$; that is, it is apparently continuous but its local gradients are undefined.

In summary, the microscopic sensitivity values found using the model adjoint are not useful in estimating the macroscopic climate sensitivities if an integration longer than intermediate length is used, indeed $\partial\bar{z}_\infty/\partial r$ is expected to be infinite. Unfortunately, too short an integration period produces a stable (i.e., only weakly dependent on initial

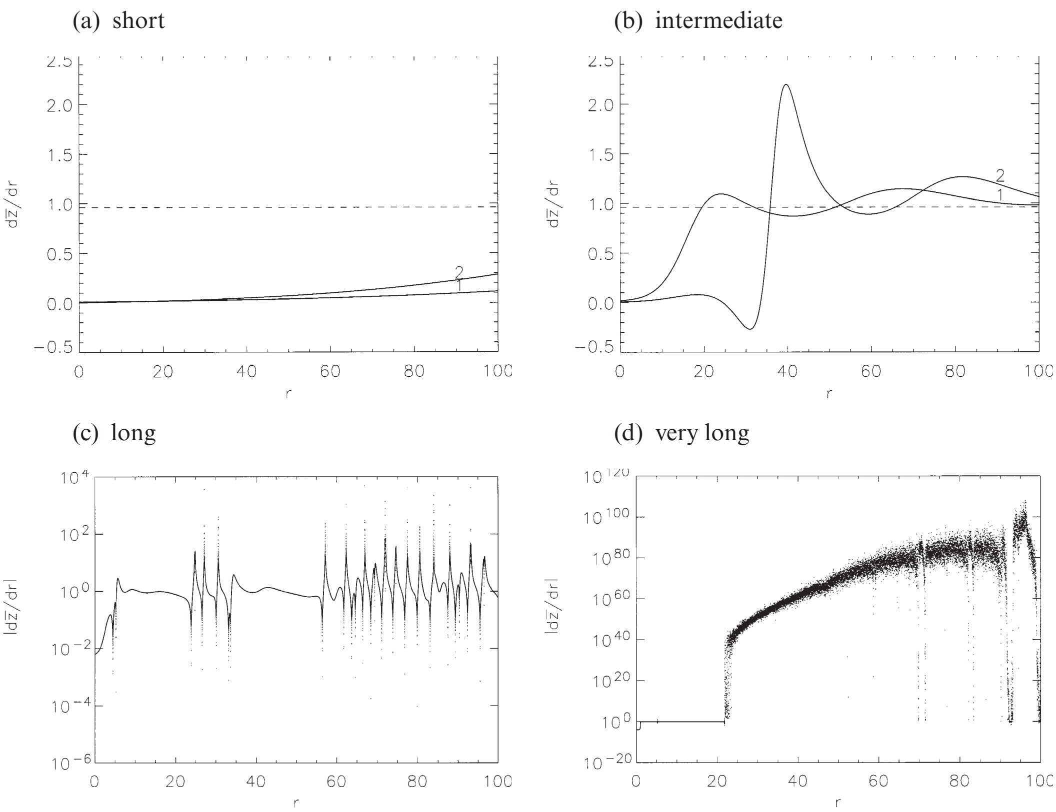

Fig. 5. Graphs of $\partial\bar{z}/\partial r$ against r for, (a) a short integration ($\tau = 0.10$), (b) an intermediate integration ($\tau = 0.44$), (c) a long integration ($\tau = 2.26$), and (d) a very long integration ($\tau = 131.36$). The dashed line, in (a) and (b), shows an estimate of the macroscopic climate sensitivity (0.9600 ± 0.0002). Note the change in ordinate scales for (c) and (d). In (a) and (b) line 1 is for initial conditions $x = -2.40$, $y = -3.70$, $z = 14.98$ and line 2 is for initial conditions $x = 8.00$, $y = -2.00$, $z = 36.05$. Graphs (c) and (d) show the results for the first set of initial conditions only. Values of $\partial\bar{z}/\partial r$ are calculated at intervals of $r = 0.005$.

conditions) but incorrect estimate of the sensitivity, exactly as in the direct method. For an intermediate or long trajectory, the estimate may be correct over much of the range, but wildly inaccurate for certain values of r in the vicinity of the isolated extrema. Simple microscopic sensitivities from adjoints are therefore essentially useless for analysing the macroscopic climate sensitivity of a chaotic system.

5. Can the limitations of adjoint sensitivity analysis be avoided?

Fig. 5b shows the adjoint method applied to an intermediate length integration provides a reasonable estimate of the macroscopic climate sensitivity but with errors of O(100%). A possible way to reduce these errors is to average results from an ensemble of intermediate length integrations, as used by Corti and Palmer (1997). To test this idea we take the very long integration, for each value of r, and split it into many intermediate length segments. We then perform an adjoint calculation on each segment, and average the result (Fig. 7a). This produces a reasonable estimate of the macroscopic climate sensitivity, particularly for $40 < r < 65$. The procedure requires only the same computing resources as a single adjoint calculation for the very long integration, which gave useless ($O(10^{100})$) gradients. Some scatter about the macroscopic climate sensitivity (O(10%)) still exists for an ensemble adjoint estimate made in this way.

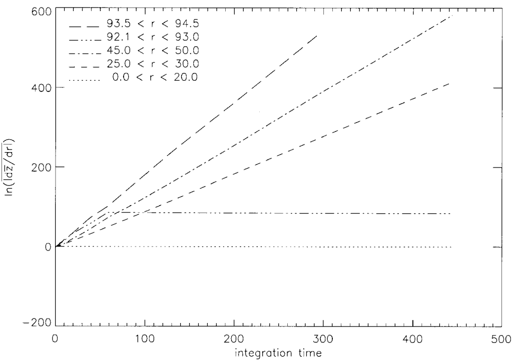

Fig. 6. Graph of the natural logarithm of the modulus of the gradient, $|\partial \bar{z}/\partial r|$, against integration time τ averaged over various ranges of r. The sensitivity grows exponentially with increasing integration time and the rate of growth depends on the value of r. Performing least squares exponential fits of the form $e^{\lambda\tau}$ gives for $0 \leqslant r < 20$, $\lambda = 0.002 \pm 0.001$; for $25 \leqslant r < 30$, $\lambda = 0.933 \pm 0.004$; for $35 \leqslant r < 40$, $\lambda = 1.121 \pm 0.004$; for $45 \leqslant r < 50$, $\lambda = 1.313 \pm 0.006$; and for $93.5 \leqslant r < 94.5$, $\lambda = 1.812 \pm 0.004$. These values of λ are consistent with the first Lyapunov exponents of Frøyland and Alfsen (1984). For $92.1 \leqslant r < 93$ a limit cycle is present. In this case, the trajectory is initially chaotic, and then enters the non-chaotic limit cycle; thus, the sensitivity initially grows exponentially with τ, but then ceases to grow once the limit cycle is achieved.

This is because the underlying trajectories about which the linearisation is performed are completely different even for two nearby values of r.

For $r < 40$, the ensemble adjoint estimates are also biased. This is because the individual integration segments are not long enough to display the full climate sensitivity. To reduce the bias, we repeat the multiple segment integration using a larger segment length of $\tau = 0.66$ (see Fig. 7b). As expected, the bias is reduced, although the scatter is increased (particularly for large r) for a given total computation time. This is because the effect of the isolated extrema (see Fig. 6 and eq. (5)) on each individual integration segment is larger, and also because fewer integrations are averaged. Fig. 7c shows that if the integration segment is too short, the sensitivity estimate is systematically low for all r, despite a greater precision.

Clearly the length of the integration segment must be chosen carefully for this ensemble-adjoint approach to work. In any practical application, several different segment lengths would have to be tested in a search for a stable estimate.

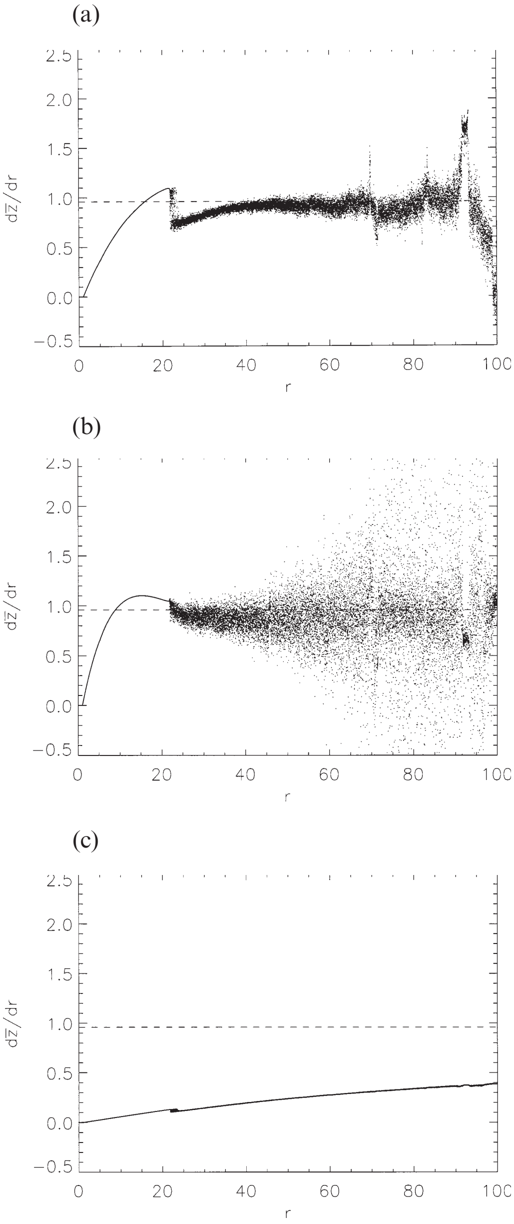

Fig. 7. Estimates of the macroscopic climate sensitivity from averaging an ensemble of microscopic sensitivities from adjoint calculations: (a) average of the gradient of 299 intermediate integrations ($\tau = 0.44$); (b) average of the gradient of 199 longer integrations ($\tau = 0.66$); (c) average of the gradient of 1314 short integrations ($\tau = 0.1$). All of these ensembles span the same integration time as one very long integration ($\tau = 131.36$). The dashed line, in each figure, shows an estimate of the macroscopic climate sensitivity (0.9600 ± 0.0002) from fitting a straight line to Fig. 2d.

Furthermore, although our results for the Lorenz equations are encouraging, other chaotic systems may well behave differently. Nevertheless, in view of the prohibitive expense of the direct method, any potentially useful adjoint technique should be investigated. We defer detailed discussion of these issues to another article.

6. Discussion

The question we have addressed is as follows: do the direct and adjoint methods of estimating sensitivity converge to the macroscopic sensitivity as the integration time tends to infinity? This work shows that the direct method does converge for the Lorenz (1963) system. The adjoint method does not tend to a useful climate sensitivity value, however; rather it grows exponentially with increasing integration time. This suggests studying the climate response of a chaotic GCM using sensitivities from an adjoint model would not yield the macroscopic climate sensitivity required.

The reason for this problem with the adjoint method is that, for climate studies, the sensitivity of a single model realisation to an infinitesimal change in the model parameters is not the quantity of most interest. In this case, the microscopic sensitivity from the adjoint model is highly dependent on initial conditions because it is primarily dependent on chaotic changes in the solution trajectory. Of greater interest is the component of the response that is independent of the initial conditions. Neither the adjoint nor the direct sensitivity methods provide this unequivocally. Our results suggest that the direct method, used, for example, in model predictions of the effect of increased CO_2 on the Earth's climate (Kattenberg et al., 1995) can, subject to certain caveats, produce reliable estimates of this macroscopic climate sensitivity. The crucial practical disadvantage to this technique is that a separate integration is required to find the sensitivity to each individual parameter.

Another possible method, namely averaging microscopic sensitivities obtained from an ensemble of intermediate length adjoint calculations, has been investigated. Our interest in this approach arises from the apparently useful sensitivity information shown by the adjoint integration in Fig 5b. This method has the potential advantage of yielding the sensitivity values to many model inputs simultaneously at similar computational cost to a single direct method calculation. Intermediate length or longer integrations must be used in the ensemble as shorter integrations do not display the macroscopic climate sensitivity, at least for the Lorenz (1963) system. Averaging an ensemble of intermediate length microscopic sensitivities has not yet been attempted for more complex systems. For example, it is not clear whether an appropriate intermediate time scale exists in a complex climate GCM where there are multiple chaotic time scales. Finding a suitable intermediate time scale for the system of interest is vital because choosing too short an integration segment produces a biased estimate of the macroscopic climate sensitivity. Alternatively, if the integration segments are too long, the amplitude of the chaotic, initial-condition-dependent effects are too large for a reasonably-sized ensemble of segments to uncover the underlying macroscopic climate sensitivity.

Overall, the direct perturbation method is the most reliable for studying macroscopic climate sensitivity. The simple adjoint method, yielding microscopic sensitivities, is not useful for finding macroscopic climate sensitivity in chaotic systems. The ensemble-adjoint method may be of some use however, although difficult issues remain to be resolved before it is applied to a complex GCM. In any event, direct sensitivity calculations will probably still need to be used to confirm results for parameters of most interest.

7. Acknowledgments

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