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The equatorial bulge, angular momentum and planetary wave motion

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ABSTRACT

The impact of the equatorial bulge on the equatorial angular momentum of planetary flows is investigated by considering both the situation on earth and on a spherical planet. First, a rigorous proof of the suggestion of Bell is given that both the standard nonlinear and linear primitive equations capture the impact of the bulge almost correctly, despite the fact that a spherical earth is assumed in these equations. Linear eigenmodes of shallow water flow with equivalent depth H are discussed both for the spheroid with a corresponding torque and a spherical planet without torque. Surprisingly, it is found that Rossby–Haurwitz waves as rotational modes for $H \to \infty$ are not affected by the bulge. For finite H, however, there are both gravitational and rotational modes with equatorial angular momentum which are affected by the bulge. In contrast, there is only one mode with equatorial angular momentum on the spherical planet. This mode is stationary.

1. Introduction

The equatorial bulge strongly affects atmospheric motion. The shape of the earth is adapted to the rotation such that the sum of gravity and centrifugal acceleration is directed normal to the earth's surface. This allows one to discard the centrifugal acceleration in the standard equations of motion and to assume that the earth is a sphere (Gill, 1982; Stommel and Moore, 1989). In other words, due to the bulge, the dynamics of the atmosphere are akin to those of a fluid on a sphere but without centrifugal acceleration. Moreover, the bulge exerts a substantial torque (Bell et al., 1991; Volland, 1996) which affects the equatorial components of global atmospheric angular momentum and excites polar motions. These torques are much larger than those acting on the axial component of global angular momentum. They are reflected in the dynamics of atmospheric wave motion. As demonstrated by Bell (1994), wavemodes with nonvanishing components of global angular momentum can propagate only because of the existence of the bulge. The partitioning of angular momentum between mass and wind terms is closely related to the phase speed of a wave.

There are still some open questions with respect to the dynamical impact of the bulge. For example, Rossby–Haurwitz waves are solutions to the barotropic vorticity equation on the sphere. The bulge is not represented in this equation and it is therefore difficult to see how the bulge can affect the motion of these waves. On the other hand, some Rossby–Haurwitz modes possess equatorial angular momentum. How, then, is this momentum altered during propagation? What happens, if there is no bulge as is the case for a truly spherical planet. Which modes possess equatorial angular momentum in that case? How do they propagate? In short, we want to extend Bell (1994), and to have

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a further look at the relationship of atmospheric motion and the bulge in the light of angular momentum conservation.

2. Basic equations

In this section, we will first give the prognostic equations for equatorial angular momentum (2.1). These equations are well-known, but are needed in what follows. Next these equations are specialized for the case of shallow water flow (2.2). Finally, the situation with a rigid lid is discussed (2.3).

2.1. Angular momentum equations

It is convenient to introduce an inertial Cartesian (x, y, z)-system and an inertial spherical (λ, ϕ, r)-system (Fig. 1). The z-axis coincides with the earth's rotation axis. The specific angular momentum

$$\boldsymbol{m} = \boldsymbol{r} \times \boldsymbol{v}_{\mathrm{a}} \tag{2.1}$$

($\boldsymbol{r}$ position; $\boldsymbol{v}_{\mathrm{a}}$ absolute velocity) is

$$\boldsymbol{m} = m_x \boldsymbol{i}_x + m_y \boldsymbol{i}_y + m_z \boldsymbol{i}_z \tag{2.2}$$

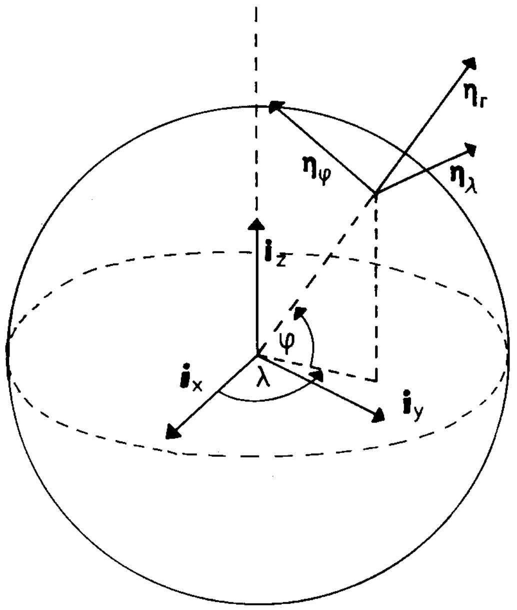

Fig. 1. Inertial Cartesian (x, y, z)-system with basic unit vectors $\boldsymbol{i}_x, \boldsymbol{i}_y, \boldsymbol{i}_z$ and the inertial spherical (λ, ϕ, r)-system with basic vectors $\boldsymbol{\eta}_\lambda, \boldsymbol{\eta}_\phi, \boldsymbol{\eta}_r$

in the Cartesian system with unit vectors $\boldsymbol{i}_{x,y,z}$ and

$$\begin{aligned} m_x &= -u_{\mathrm{a}} a \sin\phi \cos\lambda + v_{\mathrm{a}} a \sin\lambda, \\ m_y &= -u_{\mathrm{a}} a \sin\phi \sin\lambda - v_{\mathrm{a}} a \cos\lambda, \end{aligned} \tag{2.3}$$

in the spherical system (Barnes et al., 1983), where u_{a}, v_{a} are the eastward, northward components of absolute velocity in the spherical system and where r is replaced by the earth's radius a. The conservation equation for m_x is

$$\begin{aligned} &\frac{\partial_{\mathrm{a}}}{\partial t}(\rho m_x) + \frac{1}{a\cos\phi}\frac{\partial}{\partial\lambda}(\rho u_{\mathrm{a}} m_x) \\ &\quad + \frac{1}{a\cos\phi}\frac{\partial}{\partial\phi}(\rho\cos\phi v_{\mathrm{a}} m_y) + \frac{\partial}{\partial r}(\rho w_{\mathrm{a}} m_x) \\ &= -\frac{1}{\cos\phi}\left[\frac{\partial}{\partial\phi}(p\cos\phi\sin\lambda) - \frac{\partial}{\partial\lambda}(p\sin\phi\cos\lambda)\right] \end{aligned} \tag{2.4}$$

(p pressure, ρ density (Egger and Hoinka, 1999)). Viscous effects are excluded in (2.4). The corresponding equation for m_y can be obtained by rotating (2.4) by $\pi/2$. The subscript a in the time derivative reminds us that we are in the absolute system.

The rotation of the earth is introduced by defining a relative zonal flow velocity

$$u = u_{\mathrm{a}} - \Omega a \cos\phi \tag{2.5}$$

and a deviation angular momentum m_x^* via

$$m_x = m_x^* + \bar{m}_x,$$

where

$$\bar{m}_x = -\Omega a^2 \cos\phi \sin\phi \cos\lambda, \tag{2.6}$$

is the angular momentum of the earth's rotation. With that (2.4) becomes

$$\begin{aligned} &\frac{\partial_{\mathrm{a}}}{\partial t}(\rho(m_x^* + \bar{m}_x)) \\ &\quad + \frac{1}{a\cos\phi}\frac{\partial}{\partial\lambda}(\rho(u + \Omega a\cos\phi)(m_x^* + \bar{m}_x)) \\ &\quad + \frac{1}{a\cos\phi}\frac{\partial}{\partial\phi}(\rho\cos\phi(v(m_x^* + \bar{m}_x))) \\ &\quad + \frac{\partial}{\partial r}(w\rho(m_x^* + \bar{m}_x)) = P_x, \end{aligned} \tag{2.7}$$

where $v = v_{\mathrm{a}}$, $w = w_{\mathrm{a}}$ and P_x represents the term on the right-hand side of (2.4). Note that almost no approximations enter in (2.7). Clearly, global angular momentum

$$M_x = \int_V \rho m_x \,\mathrm{d}v \tag{2.8}$$

(V volume of atmosphere; dv volume element) is conserved on a spherical earth where the integral over P_x vanishes. It is not conserved in presence of the bulge where the integral over P_x does not vanish. In what follows, mountains will not be considered.

2.2. *Shallow water flow*

The basic conservation equation (2.7) must be linearized with respect to a mean state of uniform rotation, if linear flow modes are to be investigated. It is convenient to discuss this problem for shallow water flow. A homogeneous fluid of density ρ_0 is assumed with a free surface. The mean thickness H of the fluid is assumed constant. The free surface is located at height

$$h = h_B + H + \eta, \tag{2.9}$$

where $a + h$ is the distance of the free surface to the center of the earth. The profile of the bulge is given by

$$h_B = \frac{\Omega^2 a^2}{2g}(\cos^2\phi - 1/2) \tag{2.10}$$

(Bell, 1994). The surface of the earth is located at $a + h_B$. The area average of the height deviation η vanishes.

The equation of continuity is

$$\frac{\partial_a}{\partial t}\eta + \frac{1}{a\cos\phi}\frac{\partial}{\partial\lambda}((u + \Omega a\cos\phi)(H + \eta)) + \frac{1}{a\cos\phi}\frac{\partial}{\partial\phi}(\cos\phi\, v(H + \eta)) = 0. \tag{2.11}$$

Integration of (2.7) over the depth of the fluid yields

$$\begin{aligned}\frac{\partial_a}{\partial t}((m_x^* + \bar{m}_x)(H + \eta)) &+ \frac{1}{a\cos\phi}\frac{\partial}{\partial\lambda}((H + \eta)(u + \Omega a\cos\phi)(m_x^* + \bar{m}_x)) \\ &+ \frac{1}{a\cos\phi}\frac{\partial}{\partial\phi}(\cos\phi(H + \eta)v(m_x^* + \bar{m}_x)) \\ &= -\frac{1}{2\cos\phi}\frac{\partial}{\partial\phi}((H + \eta)^2 g\cos\phi\sin\lambda) \\ &+ \frac{1}{2\cos\phi}\frac{\partial}{\partial\lambda}((H + \eta)^2 g\sin\phi\cos\lambda) \\ &- (H + \eta)g\sin\lambda\frac{\partial h_B}{\partial\phi}.\end{aligned} \tag{2.12}$$

When deriving (2.12), the relations $w = \mathrm{d}h/\mathrm{d}t$ at $r = a + h$ and $w = \mathrm{d}h_B/\mathrm{d}t$ at $r = a + h_B$ have been used. It follows immediately

$$\frac{\mathrm{d}M_x}{\mathrm{d}t} = \Omega^2 a^4 \rho_0 \int_{-\pi}^{\pi}\int_{-\pi/2}^{\pi/2} \eta\cos^2\phi\sin\phi\sin\lambda\,\mathrm{d}\phi\,\mathrm{d}\lambda \tag{2.13}$$

where now

$$M_x = \int_{-\pi}^{\pi}\int_{-\pi/2}^{\pi/2} (H + \eta)\rho_0 m_x a^2\cos\phi\,\mathrm{d}\phi\,\mathrm{d}\lambda. \tag{2.14}$$

The bulge exerts a torque on the model atmosphere. However, M_x is conserved on a nonrotating earth.

Finally, (2.12) must be linearized with respect to the basic flow $\Omega a\cos\phi$, and $\bar{m}_x$ as specific angular momentum of the basic state. The resulting linear equation is:

$$\begin{aligned}&\frac{\partial_a}{\partial t}(m_x' H + n'\bar{m}_x) \\ &+ \frac{1}{a\cos\phi} \\ &\times\frac{\partial}{\partial\lambda}(H\Omega a\cos\phi\, m_x' + Hu'\bar{m}_x + \eta'\Omega a\cos\phi\,\bar{m}_a) \\ &+ \frac{1}{a\cos\phi}\frac{\partial}{\partial\phi}(Hv'\bar{m}_x\cos\phi) \\ &= -\frac{1}{\cos\phi}\frac{\partial}{\partial\phi}g(H\eta'\cos\phi)\sin\lambda \\ &+ \frac{1}{\cos\phi}\frac{\partial}{\partial\lambda}(H\eta'\sin\phi\cos\lambda) \\ &- \eta' g\sin\lambda\frac{\partial h_B}{\partial\phi}\end{aligned} \tag{2.15}$$

where

$$m_x' = -ua\sin\phi\cos\lambda + va\sin\lambda \tag{2.16}$$

and u, v, η are perturbations. Obviously, the linearization does not affect the torque by the bulge.

If, however, a spherical planet is assumed, the height h is still given by (2.9) but the surface of the earth is located at a. It is the free surface which forms a bulge. Thus, all factors $(H + \eta)$ in (2.11), (2.12) must be replaced by h and the last term on the right must be discarded. In (2.15), all factors H must be replaced by $H + h_B$ and the last term on the right disappears, of course. Thus there is no torque. The equation of continuity is

$$\frac{\partial \eta'}{\partial t} + i\Omega\eta' + \frac{(H+h_B)}{a\cos\phi}\frac{\partial u'}{\partial \lambda} + \frac{1}{a\cos\phi} \times \frac{\partial}{\partial \phi}((h+h_B)v'\cos\phi) = 0 \quad (2.17)$$

for a spherical planet.

2.3. *Rigid lid*

It is also of some interest to look at fluids constrained by a rigid lid at height h_T while the bottom is at distance $a + \varepsilon h_B$ to the center of the earth. The depth of the fluid is $d = h_T - \varepsilon h_B$ ($\varepsilon = 0$: spherical planet; $\varepsilon = 1$; earth). The angular momentum balance, is in analogy to (2.12):

$$\begin{aligned}
&\frac{\partial_a}{\partial t}((m_x^* + \bar{m}_x)d) \\
&\quad + \frac{1}{a\cos\phi}\frac{\partial}{\partial\lambda}((m_x^x + \bar{m}_x)(u + \Omega a\cos\phi)d) \\
&\quad + \frac{1}{a\cos\phi}\frac{\partial}{\partial\phi}((m_x^x + \bar{m}_x)v\cos\phi d) \\
&= -\frac{1}{2\cos\phi}\frac{\partial}{\partial\phi}(d^2 g\cos\phi\sin\lambda) \\
&\quad + \frac{1}{2\cos\phi}\frac{\partial}{\partial\phi}(d^2 g\sin\phi\cos\lambda) \\
&\quad - p_T\rho_0^{-1}\sin\lambda\frac{\partial h_T}{\partial\phi} \\
&\quad + (p_T\rho_0^{-1} + \mathrm{d}g)\sin\lambda\varepsilon\frac{\partial h_B}{\partial\phi}, \qquad (2.18)
\end{aligned}$$

where p_T is the pressure at the lid. Rossby–Haurwitz modes are eigenmodes of rigid lid flow with $\varepsilon = 1$, $d = H = \text{constant}$. It is seen that the angular momentum of Rossby–Haurwitz waves is not affected by torques because the torque exerted by the lid cancels that of the bulge. If, however, $\varepsilon = 0$ and $h_T = H + h_B$ as for a spherical planet, there is a torque at the lid.

3. Results

In this section, it is shown first on the basis of (2.7) that the standard equations of atmospheric dynamics include almost correctly the effect of the bulge (3.1). We proceed to discuss the angular momentum of the normal mode (3.2) for the nonrotating earth (3.3), the atmospheric case (3.4), and for the spherical planet (3.5).

3.1. *Torque of the bulge in nonlinear and linear models*

It has been suggested by Bell et al. (1991) that the torque by the bulge is only marginally affected by the traditional approximation, where the centrifugal acceleration is neglected and where a spherical earth is assumed (see also corresponding remarks by Barnes et al., 1983). In other words, there is a cancellation of errors. With (2.7), it is straightforward to provide a rigorous proof of this assertion. The impact of the centrifugal acceleration on angular momentum is described by the term:

$$\begin{aligned}
Z &= \frac{\rho}{a\cos\phi}\frac{\partial}{\partial\lambda}(\Omega a\cos\phi\bar{m}_x) \\
&= \Omega^2 a^2\rho\cos\phi\sin\phi\sin\lambda \qquad (3.1)
\end{aligned}$$

in (2.7). This term represents here the "advection" of the mean specific angular momentum $\bar{m}_x$ by the earth's rotation. We discard this term by adding a term Z on the right-hand side of (2.7). This modified equation must be integrated over the volume of the atmosphere under the assumption that the earth is spherical. In that case, the integral over P_x vanishes and one obtains

$$\frac{\mathrm{d}M_x}{\mathrm{d}t} = \int_V Z\,\mathrm{d}v. \qquad (3.2)$$

Assuming a hydrostatic atmosphere, one obtains

$$\frac{\mathrm{d}M_x}{\mathrm{d}t} = \int_{-\pi}^{\pi}\int_{-\pi/2}^{\pi/2} g^{-1}p_s\Omega^2 a^4 \times \cos^2\phi\sin\phi\sin\lambda\,\mathrm{d}\phi\,\mathrm{d}\lambda \qquad (3.3)$$

with surface pressure p_s. A direct integration of (2.4) yields the same result if the surface of the earth is at $a + h_B$ (see also Bell et al., 1991). This means that the torque by the bulge is incorporated almost correctly by global numerical weather prediction and climate models as has been suggested by Bell et al. (1991). The same is true for the standard linear flow equations on the sphere. In particular, a straightforward but tedious calculation shows that (2.15) and the related equation for m_y' can be converted to the linear standard equations

$$\left(\frac{\partial_a}{\partial t}+\Omega\frac{\partial}{\partial\lambda}\right)u'-2\Omega\sin\phi v'=-\frac{g}{a\cos\phi}\frac{\partial\eta'}{\partial\lambda}, \tag{3.5}$$

$$\left(\frac{\partial_a}{\partial t}+\Omega\frac{\partial}{\partial\lambda}\right)v'+2\Omega\sin\phi u'=-\frac{g}{a}\frac{\partial\eta'}{\partial\lambda}, \tag{3.6}$$

$$\left(\frac{\partial_a}{\partial t}+\Omega\frac{\partial}{\partial\lambda}\right)\eta'+\frac{H}{a\cos\phi}\frac{\partial u'}{\partial\lambda}+\frac{1}{a\cos\phi}H\frac{\partial}{\partial\phi}(v'\cos\phi)=0, \tag{3.7}$$

of shallow water flow on the sphere, where we can switch to the standard form for the rotating system via

$$\frac{\partial_a}{\partial t}+\Omega\frac{\partial}{\partial\lambda}=\frac{\partial}{\partial t}. \tag{3.8}$$

The mean angular momentum balance of the linear model is, according to (2.16):

$$\frac{\partial}{\partial t}\int_{-\pi}^{\pi}\int_{-\pi/2}^{\pi/2}\rho_0(m_x'H+\eta'\bar{m}_x)a^2\cos\phi\,\mathrm{d}\phi\,\mathrm{d}\lambda=\int_{-\pi}^{\pi}\int_{-\pi/2}^{\pi/2}\rho_0\eta'\Omega^2a^4\cos^2\phi\sin\phi\sin\lambda\,\mathrm{d}\phi\,\mathrm{d}\lambda \tag{3.9}$$

in analogy to (2.15). This means in turn that the torque by the equatorial bulge is included correctly in the linear shallow water equations. There is no torque for a spherical planet. We follow standard practice by calling the first term on the left wind term and the other one mass term.

3.2. *Normal modes*

The angular momentum of atmospheric normal modes is discussed extensively in Bell (1994) with particular emphasis on tides (see also Volland (1996)). These normal modes are eigenmodes of the divergence operator on the sphere, where the equations of motions are linearized with respect to state of rest in the rotating system. In particular, the eigenmodes of the shallow water equations (3.5)–(3.7) for the height field are Hough functions with equivalent depth H. The eigenmodes for a spherical rotating planet appear not to be known. The eigensolutions are of the form

$$\begin{pmatrix}\hat{u}_n(\phi)\\ \hat{v}_n(\phi)\\ \hat{\eta}_n(\phi)\end{pmatrix}\exp(i\lambda-i\omega_nt), \tag{3.10}$$

with index n. Inserting (3.10) in (2.15) or the related equation for the sphere yields:

$$\omega_n(M_{nx}'^w+M_{nx}'^m)=\varepsilon\Omega M_{nx}'^m, \tag{3.11}$$

where $\varepsilon=0(1)$ for the sphere (earth). The wind term in (3.11) is

$$M_{nx}'^w=\frac{1}{2}\int_{-\pi/2}^{\pi/2}(-\hat{u}_n\sin\phi+i\hat{v}_n)\times(H+(1-\varepsilon)h_\mathrm{B})\rho_0\cos\phi a^3\,\mathrm{d}\phi. \tag{3.12}$$

The mass term is

$$M_{nx}'^m=\frac{1}{2}\int_{-\pi/2}^{\pi/2}-\hat{\eta}\Omega^2a^4\cos^2\phi\sin\phi\,\mathrm{d}\phi. \tag{3.13}$$

It follows that

$$M_{nx}'^w=\left(\frac{\Omega\varepsilon}{\omega_n}-1\right)M_{nx}'^m. \tag{3.14}$$

This relation is a slight generalization of (20) of Bell (1994) in that the truly spherical case is included here. It is seen immediately that either

$$\omega_n=0 \tag{3.15}$$

or

$$M_{nx}'^w+M_{nx}'^m=0 \tag{3.16}$$

for the sphere. If there is no bulge, normal modes can propagate only if their angular momentum vanishes.

3.3. *Nonrotating earth*

The normal modes are easily found for a non-rotating earth. It follows from (3.5)–(3.7) that

$$\frac{\partial^2\eta'}{\partial t^2}+gH\nabla^2\eta'=0, \tag{3.17}$$

so that

$$\omega_n=\pm(gHn(n+1))^{1/2}a^{-1}, \tag{3.18}$$

and $\hat{\eta}_n=P_n^1(\phi)$, where $P_n^m(\phi)$ are associated Légendre polynomials. These are irrotational gravity modes. Since

$$\hat{u}_n = g\hat{\eta}_n/(a \cos \phi\omega_n),$$
$$\hat{v}_n = ig \frac{\partial \hat{\eta}_n}{\partial \phi} \Big/ (a\omega_n), \tag{3.19}$$

it follows that $M'^{w}_{nx} = 0$ from (3.13) after partial integration. This is, of course, generally true: irrotational winds do not contribute to the wind term (Bell, 1994). The mass term vanishes because of $\Omega = 0$. Therefore, the equatorial angular momentum of these gravity waves vanishes.

There is also a degenerate class of rotational modes with

$$u_n = -\frac{1}{a} \frac{\partial \hat{\psi}_n}{\partial \phi},$$
$$v_n = \frac{i}{a \cos \phi} \hat{\psi}_n, \tag{3.20}$$

where $\hat{\psi}_n$ is a streamfunction. One may choose $\hat{\psi}_n = P_n^1$. These modes are stationary: $\omega_{\rm n} = 0$. The angular momentum of this rotational flow is represented by the first mode with $\hat{\psi}_1 \sim \cos \phi$. All other $\hat{\psi}_n$ are orthogonal to $\hat{\psi}_1$ and are, therefore, without angular momentum. Of course, the normal modes given here for the nonrotating earth are equal to those derived by Longuet-Higgins (1967) for a rotating planet in the limit of a vanishing Lamb parameter $L = 4\Omega^2 a^2/gH \to 0$.

3.4. Atmospheric case

The eigenmodes for the rotating spherical planet are not known and must be evaluated numerically. The eigenmodes for the earth are well-known (Longuet-Higgins, 1967). For the sake of consistency, however, we will use the same numerical scheme to determine the eigenmodes both for the standard shallow water equations ($\varepsilon = 1$) and for the spherical case with $\varepsilon = 0$. The basic equations (3.5)–(3.7), (2.17) are discretized in a staggered grid assuming a structure (3.10). The meridional velocity $\hat{v}$ is defined at $N + 2$ grid points reaching from pole to pole, while $\hat{u}$, $\hat{\eta}$ are given at $N + 1$ intermediate points. Since $\hat{v}$ is not predicted at the poles, $3N + 2$ equations need to be solved. Care has been taken to ensure conservation of energy. The finite difference equations are given in Section 6.

As is well-known, the separation of gravity modes and slow modes as found in the nonrotating case extends to the rotating situation although less clearly. There is a class of fast gravity waves and another one of relatively slow rotational modes. It is customary to interpret the rotational modes in terms of Rossby–Haurwitz waves $(1, n)$ which are eigensolutions for the case $H = \infty$ ($L = 0$). They are nondivergent and $\hat{\psi}_n = P_n^1$. The frequency of a Rossby–Haurwitz wave is:

$$\omega_n = \Omega - 2\Omega/(n(n+1)). \tag{3.21}$$

The height fields of both the Rossby–Haurwitz modes (1, 1) and (1, 3) project onto P_2^1 (Volland, 1996), so that both modes appear to exert a torque and to contribute to the mass term. The wind term of mode (1, 3) vanishes. However, the torque exerted by Rossby–Haurwitz waves does not have an effect on their angular momentum. This conclusion follows directly from (2.15) which reduces to

$$\frac{\partial}{\partial t} m'_x + \frac{1}{a \cos \phi} \frac{\partial}{\partial \lambda} (\Omega a \cos \phi m'_x + \eta' \overline{m_x}) + \frac{1}{a \cos \phi} \frac{\partial}{\partial \phi} (v' \overline{m_x} \cos \phi) = -\frac{1}{\cos \phi} \frac{\partial}{\partial \phi} (g\eta' \cos \phi) \sin \lambda + \frac{1}{\cos \phi} \frac{\partial}{\partial \lambda} (\eta' \sin \phi \cos \lambda) \tag{3.22}$$

for $H \to \infty$. There is only a wind term left which is obviously conserved. On the other hand, both the waves (1, 1) and (1, 3) exert a torque. This "paradox" is resolved easily. As for (1, 1), the wind term becomes very large for $H \to \infty$. The torque is finite and does not affect the infinitely large wind term. Therefore $\omega_1 = 0$ (see eq. (3.21)) despite the fact that there is a torque. The wind term of mode (1, 3) vanishes so that (3.22) reduces to a diagnostic equation when applied to this mode. Again, the bulge does not affect this mode. Note, however, that (3.14) predicts $\omega_{\rm n} = \Omega$ for a vanishing wind term. Obviously, (3.14) is not satisfied by this Rossby–Haurwitz mode. These results are in accordance with Subsection 2.2, where it has been shown that the motion of Rossby–Haurwitz waves as solutions to the rigid lid problem is not affected by the bulge. In particular, (2.18) explains why (3.14) is not satisfied by Rossby–Haurwitz modes. The torque due to the lid is not accounted for by (3.14).

As for the gravity modes, (3.19) is valid for large

H except that Ω must be added to the frequency (Andrews et al., 1987). The gravity mode with $n = 2$ has $\hat{\eta} \sim P_2^1$ and therefore exerts a torque.

The situation is slightly more complicated for smaller values of H. In that case, more modes may project onto P_2^1. Here, we present the results for $H = 10^4$ m ($L = 0.08$). The numerically evaluated eigenmodes all have real frequencies, as must be expected for an energy conserving scheme. The projection $C(\hat{\eta}^1)$ of the height perturbation $\hat{\eta}$ on $\partial h_B/\partial \phi$ is evaluated and normalized such that $C = \pm 1$ if $\hat{\eta} \sim \sin \phi \cos \phi$. Vorticity and divergence are evaluated for each mode as well to see if the mode is dominantly rotational or not. The projections are found to be quite small for almost all gravity modes, so that these modes do not exert a torque and M'_{nx} vanishes. There are two exceptions with $C \sim 1$ and $n = 1$: there is an eastward propagating gravity mode with $\omega = 2.1 \times 10^{-4} \sim 3\Omega$, and a retrograde mode with $\omega = 9.8 \times 10^{-5}\ \mathrm{s}^{-1}$. In addition, there are two rotational modes. Both are prograde with $\omega = 1.1 \times 10^{-5}\ \mathrm{s}^{-1}$ and $\omega = 6.4 \times 10^{-5}\ \mathrm{s}^{-1}$. The first one corresponds to the Rossby–Haurwitz wave (1, 1), in the sense that this mode reduces to (1, 1) for $H \to \infty$. However, while the torques have no effect for $H \to \infty$, the Hough mode for $H = 10^4$ m exerts a torque and one finds from the model $M'^{w}_{1x}/M'^{m}_{1x} = 5.5$, in good agreement with (3.15). The $\hat{v}$ profile of the other rotational mode has two nodes. This is the Hough mode which reduces to the Rossby–Haurwitz wave (1, 3) for $H \to \infty$ with frequency $6.06 \times 10^{-5}\ \mathrm{s}^{-1}$ (see 3.22). The meridional profiles of $\hat{\eta}, \hat{u}, \hat{v}$ are close to those presented by Longuet-Higgins (1967) and, therefore, not shown.

The modes discussed so far have $C \sim 1$. There are, however, additional modes with $C \leqslant 0.1$. These correspond to modes with larger values of n.

3.5. *Spherical planet*

The mean height of the fluid is $h_B + H$ for the spherical planet. Correspondingly, the mean potential vorticity $f/(H + h_B)$ has a larger meridional gradient than for $\varepsilon = 1$. The waves feel an enhanced "β-effect". As before, we select those modes where $C(\hat{\eta}) \sim 1$. There are 3 eigenmodes which satisfy the criterion $C \sim 1$. Two of those satisfy the criterion $M_x \approx 0$ where the numerical error is less than 3% of the mass term. The frequencies of these modes are $-2.2 \times 10^{-4}\ \mathrm{s}^{-1}$, $9.7 \times 10^{-5}\ \mathrm{s}^{-1}$, $-4.5 \times 10^{-7}\ \mathrm{s}^{-1}$. Divergence and vorticity are both important. The height field of all three modes has just one node. The 3rd mode has nonvanishing equatorial momentum. This mode strongly resembles the classical Rossby–Haurwitz wave (1, 1). The frequency of this mode is $-4.6 \times 10^{-7}\ \mathrm{s}^{-1}$ instead of $\omega_n \sim 0$. The period is 64 days. The frequency of this mode depends on the numerical resolution. One obtains $\omega_n = 1.9 \times 10^{-6}\ \mathrm{s}^{-1}$ for $N = 21$, $\omega_n = 1.1 \times 10^{-6}\ \mathrm{s}^{-1}$ for $N = 41$, $\omega_n = 7.6 \times 10^{-7}\ \mathrm{s}^{-1}$ for $N = 61$. Thus, the frequency of this mode decreases with increasing number of gridpoints. Given the finite difference approximation (A.1)–(A.3) of (3.5)–(3.7), one can derive the numerical prognostic equation for M_x. Not surprisingly, it turns out that M_x is not conserved exactly in the numerical scheme. In other words, there are spurious sources and sinks of angular momentum which can be reduced only by increasing the resolution. One may think of designing a numerical scheme which conserves M_x, M_y exactly. This has been done by basing a linear model directly on (2.15), (3.7) and the equation for m_y. Given m_x, m_y and η at a moment in time, one can derive u and v via (2.3). Such schemes turned out to be unstable and had to be given up.

There is a mode with $C \sim 0.75$ and $\omega = -5.4 \times 10^{-5}\ \mathrm{s}^{-1}$. This mode is rotational with two zeros of $\hat{v}$. The meridional profiles of $\hat{\eta}, \hat{u}, \hat{v}$ of this mode differ, of course, somewhat from those of the corresponding atmsopheric mode. We have not been able to find a simple explanation of these differences. There is another rotational mode with $C \sim 0.28$ and $\omega \sim 6.3 \times 10^{-5}\ \mathrm{s}^{-1}$ (four nodes). There are many more modes with even smaller values of C.

4. Conclusion

The results of this paper are based on the angular momentum equation (2.7). In particular, the corresponding new equation (2.12) for shallow water shows rather clearly the impact of the equatorial bulge on global angular momentum. Given these equations, it was straightforward to provide a rigorous proof of the suggestions of Bell et al. (1991) and Bell (1994), that the primitive equations describe almost correctly the impact of the bulge on angular momentum both for nonlinear and linear flow.

The normal modes for shallow water flow on the earth and on a spherical planet have been evaluated. As for the atmospheric case, it is found that the classical Rossby–Haurwitz modes are not affected by torques. The relation (3.14) is not valid for these waves. For finite values of H, there are, of course, gravity as well as rotational modes with nonvanishing angular momentum which exert a torque. There is only one mode with angular momentum on the spherical planet. This mode is stationary because torques are absent. The equatorial angular momentum of all other modes vanishes. However, a torque is exerted if a rigid lid is assumed on top of the fluid.

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6. Appendix

Numerical model

The numerical scheme is based on $N+2$ grid points covering all latitudes from pole to pole. The meridional velocity $\hat{v}_j$ is defined at these points with N forecast equations while $\hat{u}_{j+1/2}$, $\hat{\eta}_{j+1/2}$ are defined at the intermediate points, so that there are altogether $3N-2$ equations. The mean depth of the fluid is denoted by d. The equations are

$$\frac{\partial_a \hat{v}_j}{\partial t} + i\Omega\hat{v}_j + \frac{f_j}{2}(\hat{u}_{j+1/2} + \hat{u}_{j-1/2}) = -g(\hat{\eta}_{j+1/2} - \hat{\eta}_{j-1/2})/(aD\phi), \quad (j = 2, \ldots, N-1), \quad \text{(A.1)}$$

$$\frac{\partial_a \hat{u}_{j+1/2}}{\partial t} + i\Omega\hat{u}_{j+1/2} - (f_j v_j\, d_j \cos\phi_j + f_{j-1} v^{j-1} d_{j-1} \cos\phi_{j-1})/(2d_{j+1/2}\cos\phi_{j+1/2}) = -\,i\hat{\eta}_{j+1/2}/(a\cos\phi_{j+1/2}), \quad (j = 1, N+1), \quad \text{(A.2)}$$

$$\frac{\partial_a \hat{\eta}_{j+1/2}}{\partial t} + i\Omega\hat{\eta}_{j+1/2} + id_{j+1/2}\hat{u}_{j+1/2}/(a\cos\phi_{j+1/2}) - (\hat{v}_{j+1}\cos\phi_{j+1} d_{j+1} - \hat{v}_j \cos\phi_j d_j)/(a\cos\phi_{j+1/2} d_{j+1/2} D\phi), \quad (j = 1, N+1), \quad \text{(A.3)}$$

where $D\phi = \pi/(N+1)$. Eigenvalues and eigenvectors are obtained from a standard routine.

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