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Transient error growth and local predictability: a study in the Lorenz system

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ABSTRACT

Lorenz's three-variable convective model is used as a prototypical chaotic system in order to develop concepts related to finite time local predictability. *Local* predictability measures can be represented by *global* measures only if the instability properties of the attractor are homogeneous in phase space. More precisely, there are two sources of variability of predictability in chaotic attractors. The first depends on the direction of the initial error vector, and its dependence is limited to an initial transient period. If the attractor has homogeneous predictability properties, this is the only source of variability of error growth rate and, after the transient has elapsed, all initial perturbations grow at the same rate, given by the first (*global*) Lyapunov exponent. The second is related to the local instability properties in phase space. If the predictability properties of the attractor are not homogeneous, this additional source of variability affects both the transient and post-transient phases of error growth. After the transient phase all initial perturbations of a particular initial condition grow at the same rate, given in this case by the first *local* Lyapunov exponent. We consider various currently used indexes to quantify finite time local predictability. The probability distributions of the different indexes are examined during and after the transient phase. By comparing their statistics it is possible to discriminate the relative importance of the two sources of variability of predictability and to determine the most appropriate measure of predictability for a given forecast time. It is found that a necessary premise for choosing a relevant local predictability index for a specific system is the study of the characteristics of its transient. The consequences for the problem of forecasting forecast skill in operational models are discussed.

1. Introduction

The discipline of operational NWP is experiencing a rapid evolution; on one hand more and more computer power has become available and, on the other, a more profound understanding of the mechanisms underlying weather and climate evolution has been attained. More demands have been set upon the scientific community involved in the field, among these, a precise a priori evaluation of the skill of a prediction. After the pioneering work of Lorenz (1965), predictability theory, developed for low-order chaotic systems, has provided the conceptual framework for assessing the limits of predictability in NWP and defining appropriate measures to quantify and predict forecast skill.

The problem of producing a quality evaluation of a prediction, along with the prediction itself, has been addressed in various ways at operational level. Current literature usually refers to it as forecasting forecast skill (Kalnay and Dalcher, 1987; Palmer and Tibaldi, 1988).

Several different predictors of skill have been adopted in studies on the subject. One of the most effective is the forecast spread among an ensemble of forecasts. A truly Monte Carlo simulation, however, is impracticable for NWP models for at least two reasons: the huge number of degrees of

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freedom and the fact that random perturbations to a chosen initial state could be in meteorologically irrelevant directions, and be removed by initialization procedures. Thus, several techniques have been proposed to select the ensemble of initial conditions (see for a review ECMWF, 1992), e.g., forecasts from different models, archived analysis fields, lagged forecasts, etc. Recently, Molteni and Palmer (1993) used the fastest growing perturbations obtained from a quasigeostrophic model linearized about the initial condition of the operational model. This method can be used to estimate an upper bound for the error of a particular forecast. Toth and Kalnay (1993) used perturbations obtained by rescaling the difference between a perturbed and an unperturbed previous forecast; their method, in a perfect model environment, would be equivalent to evaluating the first Lyapunov exponent.

Parallel to these studies on operational forecast models, much work has been done on low order minimal models of atmospheric circulations (Lorenz, 1982). The lack of realism of these models is compensated by advantages both practical and of interpretation: more and longer simulations are easily performed to give a statistically robust picture of the behavior of the system, and the development of the theory of low-order chaotic systems can be used as a guideline in the study of more complex models. Among the results emerged from the work on low-order systems, we wish to outline the framework and basic ideas that led us to the present study.

We are interested in measures of predictability of the first kind (Lorenz, 1975): we consider infinitesimally small errors, limiting our analysis to linear predictability. Evidence that this hypothesis is acceptable for short range forecast errors associated with the synoptic scales is given by Veyre (1992) and Vukićević (1993).

A classical measure of predictability is given by the first Lyapunov exponent (see, e.g., Lichtenberg and Lieberman, 1983). Lyapunov exponents are defined as a global average property of the attractor of a chaotic system, but one central concept addressed in the present study is that predictability is a local property in phase space. Operational forecasting experience indicates that the skill associated with individual forecasts may vary greatly. We are very far from being able, however, to assess with any statistical confidence to what extent the variability of forecast skill is related to the particular weather regime or to the particular structure of the initial error. One of the goals of our study is to delineate a strategy to answer this question in principle and we shall regard the Lorenz model as a benchmark.

Nicolis (1992) observes, in her study on the Lorenz system, a bimodal distribution of errors that can frustrate every attempt of defining a measure of predictability with the simple global mean error and its variance.

Nese (1989), in his study on the Lorenz system, considers the phase space dependent, asymptotic, local divergence rate of adjacent trajectories as a local predictability measure. The variation of this index along a trajectory is considerable. He shows that the local divergence rate is organized in phase space in regions of different predictability regimes, and classifies the regimes according to the average local divergence rate and its variance, estimating the predictability of predictability itself.

Trevisan (1993), hereafter referred to as T93, uses initial perturbations uniformly distributed on a sphere of small radius centered at decorrelated points on a trajectory spanning the attractor of the Lorenz system and then studies the time dependence of the average growth rate and its convergence to the first Lyapunov exponent.

Mukougawa et al. (1991) use the Lorenz index (Lorenz, 1965) as a measure of the local predictability (see, eq. 8 in their paper), given by the finite time linear RMS error amplification. It is a smooth function of the variables and of the time interval τ, for which the error matrix $A(t+\tau, t)$ is evaluated.

An important point is the existence, in a chaotic system, of a period of transient growth that can exceed Lyapunov exponential growth rate (T93, Krishnamurthy, 1993); the global average growth rate of errors initially in a random direction attains the asymptotic value given by the first Lyapunov exponent after a finite period of time. This transient can affect the short term prediction skill. The concept of transient enhanced exponential growth was introduced by Farrell (1985) and Lacarra and Talagrand (1988). A commonly used application of this concept to NWP consists in finding the initial perturbation that grows fastest in a given period of time (optimal perturbation, see, e.g., ECMWF, 1992).

Whatever measure of predictability we define, the growth of initial errors depends on several ingredients: location in phase space of the initial condition that we perturb, direction of the initial perturbation, time length of the forecast.

The goal of our paper is to compare the most commonly used finite time predictability indexes, with the aim of: inspecting the role of transiency on global (and local) predictability measures, and on the rate of convergence to the global (and local) first Lyapunov exponent; devise a strategy to evaluate the relative importance of phase space variability versus variability with initial error structure at different forecast times.

The outline of the paper is as follows: in Section 2 we shall give definitions and discuss general properties related to local predictability; in Section 3 we shall present the results obtained in the Lorenz system; in Section 4 we shall draw conclusions and discuss the implications for operational problems.

2. Definitions and general properties of predictability indexes

A classical measure of predictability of a chaotic system is given by the first Lyapunov exponent, which estimates the average over the attractor (global average) of the growth rate of infinitesimal perturbations to a central orbit. The first Lyapunov exponent is given by:

$$\sigma_1 = \lim_{t \to \infty} \lim_{d(0) \to 0} \frac{1}{t} \log \frac{d(t)}{d(0)} \tag{1}$$

where $d(t)$ is the distance between the perturbed and unperturbed trajectories.

It is commonly suggested in the current specialized literature that Lyapunov exponents are of limited use in atmospheric predictability studies because they do not describe local properties and because they are asymptotic quantities (see, e.g., Abarbanel et al., 1991; Molteni and Palmer, 1993). These two objections might be viewed as two sides of the same coin: in fact according to (1), the first Lyapunov exponent can be evaluated as an asymptotic value obtained by following a single trajectory which spans over the whole attractor's phase space. However, it has been shown (T93) that the first Lyapunov exponent σ_1 can be obtained after a relatively short transient time, by ensemble averaging over a large number of initial conditions which cover the attractor.

Concerning the problem of evaluating the skill of individual forecasts, the first (global) Lyapunov exponent σ_1 may not be particularly useful, unless the attractor has homogeneous predictability properties: a large phase space variability could render the global average inadequate to measure local predictability. Following T93, we will elaborate on the concept of local Lyapunov exponents and will be able to distinguish the effects of transiency from those related to phase space variability.

To illustrate these points, we summarize the procedure followed by T93. Consider perturbing a trajectory, belonging to the attractor, by adding random isotropic errors. In practice a set of errors uniformly distributed over the surface of a hypersphere in the n-dimensional phase space, of infinitesimal radius $d(0)$, has been used (see (7) below). Different perturbations grow at different rates but after an initial transient, all error vectors align themselves along the same direction and, from then on, the variability of error growth rate will depend only upon the location in phase space. This behaviour was first demonstrated by Lorenz (1965). The interpretation of the mechanisms underlying the different phases of transient growth of random errors is found in T93: a first phase, which is characterized by small or negative average growth rates is dominated by the process of convergence towards the attractor of those perturbations which are along stable directions of the system. A second phase is characterized by enhanced (super-exponential) growth; this behaviour can be interpreted in terms of the process studied in systems linearized around a time-independent state where it can be related to the non-orthogonality of the normal modes (Lacarra and Talagrand, 1988; Farrell, 1990; Molteni and Palmer, 1993). Further support to the interpretation of the two phases of transient growth is given in T93 by comparing the behaviour of analogs (Lorenz, 1969) with that of random isotropic errors.

The time necessary for a single trajectory to span over the whole attractor may be much longer than the transient; as a consequence, to obtain a stable estimate of σ_1 from (1) it may be necessary to consider very long time intervals. If, instead, we consider an ensemble average growth rate obtained by repeating the computation for a sufficiently large number of initial conditions, the globally averaged time dependent growth rate will converge to the first Lyapunov exponent after a relatively short transient, T:

$$\langle \sigma(\boldsymbol{x}(t), \tau) \rangle = \left\langle \lim_{d(0) \to 0} \frac{1}{\tau} \log \left(\frac{d(t+\tau)}{d(t)} \right) \right\rangle$$
$$\approx \sigma_1, \qquad t > T, \qquad \tau > 0, \tag{2}$$

where the operator $\langle * \rangle$ indicates a phase space average over a large ensemble of initial states $\boldsymbol{x}_0 = \boldsymbol{x}(0)$. The point $\boldsymbol{x}(t)$ belongs to the trajectory originating at $\boldsymbol{x}(0)$. Notice that the points $\boldsymbol{x}(0)$ are chosen at regular time intervals; this ensemble average is therefore an average weighted with the natural density of the attractor. The transient T has been evaluated to be of order one in the Lorenz system, or approximately equal to σ_1^{-1}.

So far we have defined a finite time, globally averaged, first Lyapunov exponent. We turn now to the problem of estimating local predictability.

First local Lyapunov exponent: σ_l. We give the following definition of local first Lyapunov exponent:

$$\sigma_l(\boldsymbol{x}(t)) = \frac{\mathrm{d}}{\mathrm{d}t} \log(d(t))$$
$$= \lim_{\tau \to 0} \sigma(\boldsymbol{x}(t), \tau), \qquad t > T, \tag{3}$$

which is consistent with (2). This instantaneous local growth rate is univocally defined as a function of $\boldsymbol{x}$ only, provided t is large enough. In fact, at any time following the initial transient, $t > T$, the instantaneous growth rate of an arbitrary initial perturbation of an individual trajectory is a function only of the position in phase space, being associated with the particular direction naturally chosen by the system, i.e., the direction of the first local Lyapunov vector. This definition of local Lyapunov exponent was adopted by T93 and coincides with the measure of local divergence rate used by Nese, 1989. We notice that the are other definitions of local Lyapunov exponents used in the current literature (Abarbanel et al., 1991; Yoden and Nomura, 1993), which however do not share with (3) the property of giving, after ensemble averaging, the first (global) Lyapunov exponent.

It is common practice to evaluate growth rates over a finite time interval. For comparison with other local, finite time predictability indexes, we will also use the average, over τ of the local Lyapunov exponent, which coincides with the quantity appearing within $\langle\ \rangle$ in (2):

$$\overline{\sigma_l}^{\tau}(\boldsymbol{x}(t), \tau) = \frac{1}{\tau} \int_t^{t+\tau} \sigma_l(\boldsymbol{x}(t'))\, \mathrm{d}t'$$
$$= \lim_{d(0) \to 0} \frac{1}{\tau} \log \left(\frac{d(t+\tau)}{d(t)} \right), \qquad t > T, \tag{4}$$

where $\bar{*}^{\tau}$ indicates averaging over τ. Notice that the local Lyapunov exponent (3) does not depend on the time interval τ, whereas (4) does, in view of the time averaging procedure.

In the following, we shall use definition (4) substituting the symbol τ with the symbol t which is used in the definitions of the other indexes. This is to indicate that we evaluate the Lyapunov growth rate over the same forecast time as the other indexes, and relative to the same initial point $x_0 = x(0)$. Assuming that at time $t = 0$, the direction of the first Lyapunov vector is known, one can choose this particular perturbation as representative of the growth rate of the actual error.

Several other choices are possible: among them, weighting equally all possible directions of the initial errors can be called the null-hypothesis. We recall, for subsequent use, the definitions of the finite time predictability indexes adopted in the current literature on the subject. These depend in general on the location in phase space, the direction of the initial error vector, and on forecast time.

Growth rate of the optimal perturbation: σ_{O}. A possible choice is the growth rate associated with the error vector which grows fastest over a given time interval, usually referred to as optimal perturbation. This choice is useful if one wishes to assign an upper bound to the error of an individual forecast. The growth rate of the optimal perturbation is given by:

$$\overline{\sigma_{\mathrm{O}}}^{t}(\boldsymbol{x}_0, t) = \frac{1}{t} \log(\max a_i(\boldsymbol{x}_0, t)), \qquad i = 1, n, \tag{5}$$

where the a_i are the semiaxis of the ellipsoid that a hypersphere of unitary radius in the n-dimensional phase space, centered at $\boldsymbol{x}$ evolves into, in a time interval t, under tangent linear flow; the overbar $\bar{*}^t$ reminds us that we deal with an average over t. The squared values of the a_i are the eigenvalues of A^+A, A being the error matrix (see, e.g., Lorenz, 1965) and A^+ its adjoint.

Growth rate associated with the Lorenz Index: σ_L. An alternative measure of transient local growth rate is based on the so called Lorenz index (Mukougawa et al., 1991) representing the RMS error averaged over the surface of the ellipsoid:

$$\alpha(\boldsymbol{x}_0, t) = \left(\frac{1}{n}\sum_{i=1}^{n} a_i^2(\boldsymbol{x}_0, t)\right)^{1/2},$$

and its associated growth rate:

$$\overline{\sigma_L}^t(\boldsymbol{x}_0, t) = \frac{1}{t}\log\alpha(\boldsymbol{x}_0, t). \tag{6}$$

Growth rate of random isotropic errors: σ_R. The final measure that we consider is the direction averaged growth rate (T93):

$$\sigma_R(\boldsymbol{x}_0, t) = \left[\frac{\mathrm{d}}{\mathrm{d}t}\log d(t)\right]_S, \tag{7}$$

where the operator $[*]_S$ indicates an average over errors which initially are on the surface of a hypersphere of infinitesimally small radius. This definition represents the mean growth rate of random isotropic errors; taking the ensemble average of this quantity is equivalent to taking the ensemble average of the growth rates of initial errors in a random direction. Its time average is given by:

$$\overline{\sigma_R}^t(\boldsymbol{x}_0, t) = \frac{1}{t}\left[\log\frac{\mathrm{d}(t)}{\mathrm{d}(0)}\right]_S. \tag{8}$$

The definitions of σ_L and σ_R are both based on the null-hypothesis. However we notice that (7) and (8) are obtained by averaging the growth rates, whereas (6) is obtained by taking the logarithm of the RMS error. In fact defining $\boldsymbol{y} = A(t)\,\boldsymbol{r}$, where $\boldsymbol{r}$ is a unitary vector, and A is the error matrix, we can evaluate the two indexes as follows:

$$\overline{\sigma_L}^t = \frac{1}{t}\log\sqrt{[y^+y]_S}$$

$$\overline{\sigma_R}^t = \frac{1}{t}[\log\sqrt{y^+y}]_S$$

See Appendix for further details.

Probability distribution functions (PDF) of the indexes defined above will be constructed and followed during their transient evolution. The distributions of the time averaged indexes defined by (4), (5), (6), and (8) will converge to a δ-function centered at σ_1, whereas the distributions of instantaneous indexes like (7) will converge to the distribution of the local Lyapunov exponent (3). For (5) and (6), which by definition are time averaged indexes, we shall compute instantaneous values by evaluating:

$$\sigma(t) = \lim_{\tau\to 0}\frac{1}{\tau}\left((t+\tau)\,\bar{\sigma}^{t+\tau} - t\bar{\sigma}^t\right). \tag{9}$$

3. Results and interpretation

The Lorenz (1963) three variable convective model is chosen here, as in many previous studies, as a prototype chaotic system for the purpose of investigating properties, and developing concepts, related to predictability. The model equations are:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -px + py,$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -xz + rx - y, \tag{10}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = xy - bz,$$

where $r = 28$, $b = 8/3$, $p = 10$.

Local predictability of this model is known to possess high variability in phase space (Nese, 1989; Mukougawa et al., 1991). Another aspect of the behavior of this model is the (enhanced exponential) transient evolution of global average error growth rate towards the first Lyapunov exponent (T93).

The present study deals specifically with the problem of variability of predictability and the systematic development of the appropriate tools for its interpretation. In particular, we shall address the problem of quantifying phase space variability of the indexes described in Section 2, by evaluating PDFs and their time evolution. We shall describe the effects of the transient on the variability of the indexes, and discriminate the relative importance of the variability due to location in phase space versus that due to the structure of the initial error.

As a first look at the behavior of system (10), let us consider a state whose position, $\boldsymbol{x}$, in phase

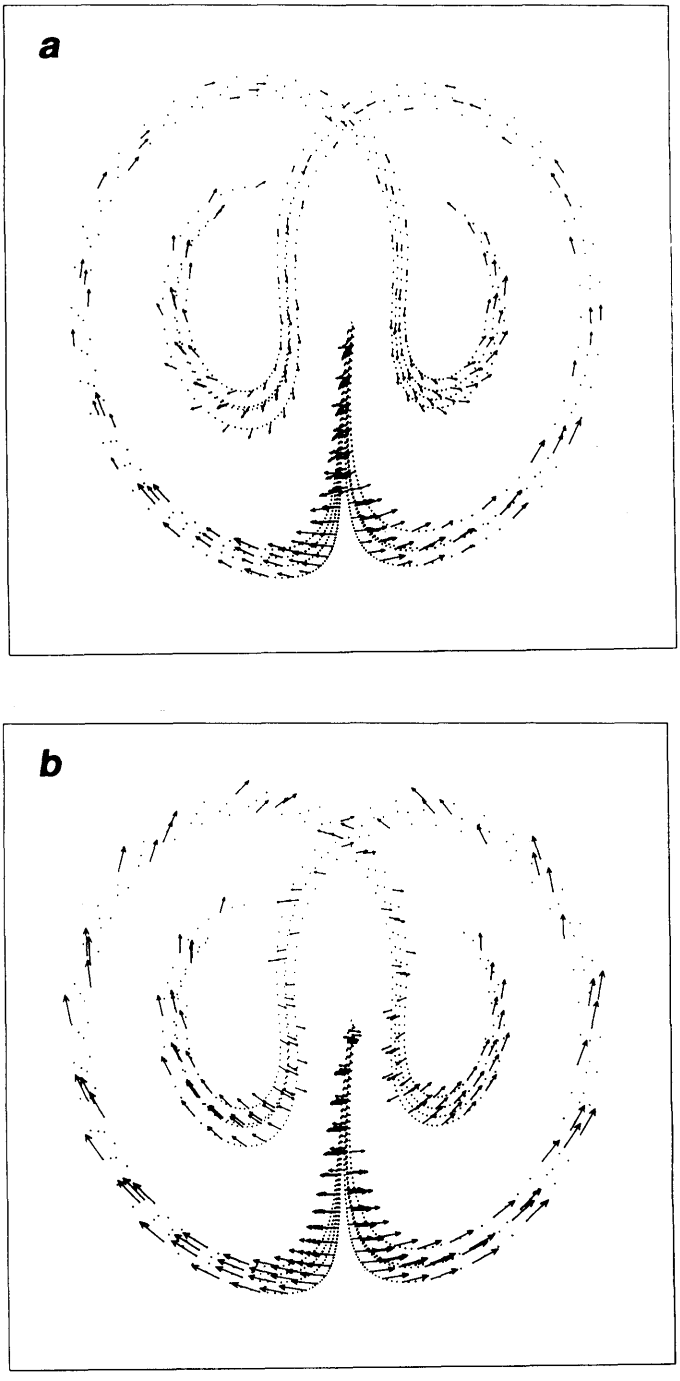

Fig. 1. Projections on the $y-z$ plane ($y \in [-30, 30]$, $z \in [0, 50]$) of: (a) optimal perturbations (semiaxis of ellipsoid) computed for $t=0.1$; (b) First Lyapunov vector computed for $\tau=0.1$, normalized according to amplification factor (see text). Dots plotted every time step ($dt=0.01$) represent basic trajectories. Vectors are plotted every 4 steps.

space is known within a given accuracy, along a trajectory belonging to the attractor, chosen as the initial condition of a particular forecast. In the limit of infinitesimally small errors, the growth rate of the initial error x' is a function of x and the direction of x'. In T93 it was shown that the global average linear error growth approaches the asymptotic value σ_1 in a typical time T of order one. We have argued that, after this time, which depends upon the properties of locally stable and unstable directions, all *individual* initial perturbations x', will align themselves with the first Lyapunov vector.

Fig. 1 illustrates how this process operates *locally*, and gives indications on its time scale. Some selected portions of trajectories, obtained from long integrations of (10), all passing through a chosen region in phase space, close to the z axis are chosen: these basic trajectories are represented in the figure by dots (plotted at every time step, $dt = 0.01$). We investigate the error growth for these trajectories by computing the optimal perturbation, whose growth rate is given by (2.4) for the points spanned by each trajectory. The vectors in Fig. 1a represent the projection on the $y-z$ plane of the vectors pointing in the direction of the major semiaxis of the ellipsoid ("optimal" error vector), computed for a time interval equal to 0.1. The length of the error vectors is proportional to $e^{0.1\overline{\sigma}_O}$. Fig. 1b shows the projection on the $y-z$ plane of the first Lyapunov vector computed over the same time interval, with norm $e^{0.1\overline{\sigma}_l}$, and plotted in correspondence to the same points. The time interval 0.1 is small enough that the Lyapunov exponents and the Lyapunov vectors can be considered local (see also discussion of Figs. 3, 4). The time interval used in constructing Fig. 1a is much smaller than the scale of the "transient", estimated on *average* to be order one, and the individual optimal perturbations are not yet aligned along the direction of the first local Lyapunov vectors. For time intervals $t = 1.0$, the direction of the *individual* optimal perturbations (not shown), coincides with that of the first Lyapunov vector, shown in Fig. 1b. Thus, after an initial period of time T into the forecast, the *local* Lyapunov exponent becomes the value of growth rate shared by all initial perturbations and for all subsequent times, provided of course that the linear tangent equations hold.

The problem remains of deciding which predictability index to choose to describe error growth during the transient phase, when it depends on the direction of the error vector. Since each of the indexes introduced in Section 2 represents a different choice of, or an average on, the direction of the error vector, they will exhibit a different time dependence during the initial stage of the forecast. Thus, we need to compare the behavior of the different indexes during the transient. First of all we shall focus on the time evolution of the first moment of the distributions of the various indexes, and subsequently we shall examine their PDFs. Clearly the first moment of the distribution of $\overline{\sigma}_l'$ does not depend upon t, and is equal to the estimated value $\sigma_1 = 0.9$. The other indexes, by definition, have an explicit dependence upon forecast time; as a consequence, also the first moment of their distribution is a function of t. Fig. 2 shows ensemble average growth rates as functions of time. Fig. 2a shows $\langle \bar{\sigma}_L \rangle$ and its corresponding instantaneous value $\langle \sigma_L \rangle$ computed using (9), together with $\langle \sigma_R \rangle$ (instantaneous by definition). These indexes exhibit an initial very short period of reduced exponential growth followed by a period of enhanced exponential growth. Fig. 2b shows $\langle \bar{\sigma}_O \rangle$ and the corresponding instantaneous value $\langle \sigma_0 \rangle$, together with $\langle \sigma_R \rangle$. This index shows enhanced exponential growth from the very beginning. All indexes in Fig. 2 converge by definition to σ_1, although the convergence of the time integrated indexes (solid curves) is obviously much slower.

These results point clearly to the fact that choosing any index at a particular time as a measure of predictability introduces a large degree of arbitrariness. In particular, extrapolating an estimate based on the growth over a finite time interval to future forecast times can be totally misleading. The use of Lyapunov exponents to estimate growth rates during the transient could also be misleading.

These considerations hold for systems which possess significant transient behavior regardless of phase space variability of divergence rates. In order to give a meaningful and reliable estimate of predictability it is therefore necessary to have an a priori knowledge of the properties of the transient for the particular system considered. Preliminary results indicate that transient growth is a significant feature also in more realistic models of the general circulation, with many more degrees of freedom (work in progress), whereas phase space

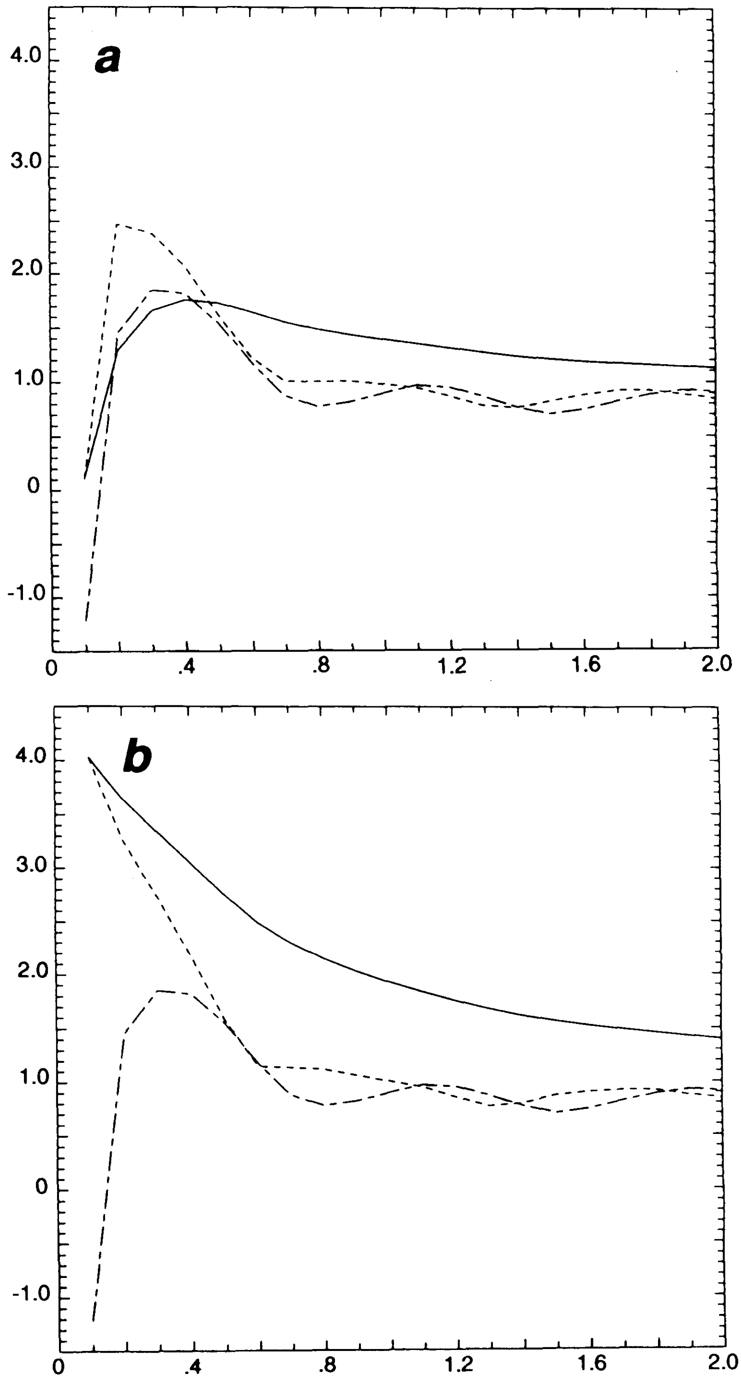

Fig. 2. Global average predictability indexes for the Lorenz system as functions of forecast time t: (a) $\langle \sigma_R \rangle$ dash-dotted line, $\langle \overline{\sigma_L} \rangle$ continuous line, $\langle \sigma_L \rangle$ dashed line; (b) $\langle \overline{\sigma_O} \rangle$ continuous line, $\langle \sigma_O \rangle$ dashed line, $\langle \sigma_R \rangle$ dash-dotted line.

variability is not as prominent as in the low order system at hand.

So far, transient behavior has been examined in terms of phase space averaged error growth rate. The variability of predictability connected to the direction of the initial error, as opposed to phase space variability, is at the basis of the transient behavior; therefore it can be observed also in systems with homogeneous predictability properties. To illustrate this point, we examine transient behavior of the various indexes in the simple case of instability of a fixed point. The system we use is a generalization of a linear system introduced by Lacarra and Talagrand (1988). We introduce an explicit dependence on the angle α between the eigenvectors, which controls the intensity of enhanced exponential growth (see Appendix). Figs. 3, 4 show, in analogy with Figs. 2a, b, the time evolution of the predictability indexes obtained using different values of α. The similarity in the behavior of the same indexes in this and the chaotic model (10), is evident from comparison of Fig. 3 and Fig. 4 with Fig. 2a and Fig. 2b, in particular for the value of $\alpha = \pi/8$. The definition

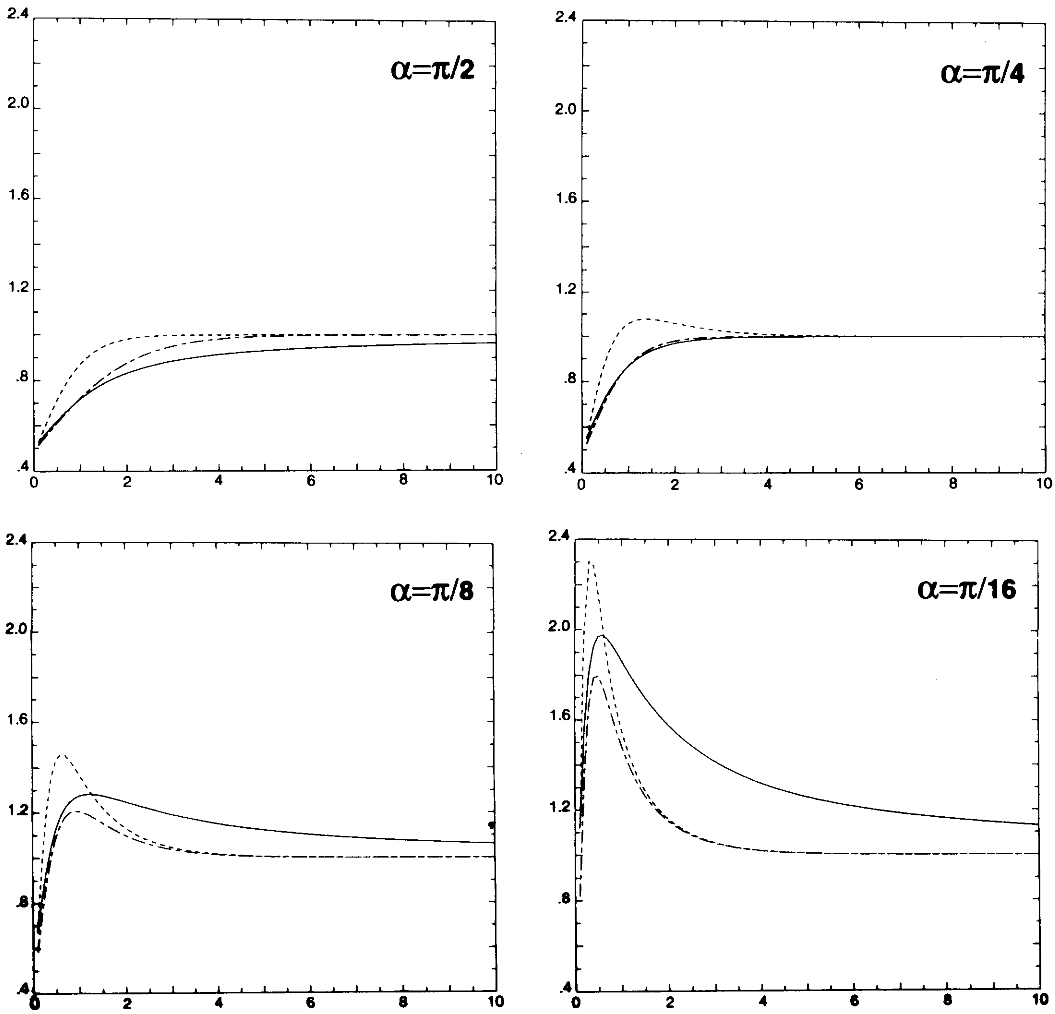

Fig. 3. Same as Fig. 2a, but for the linear system of the Appendix: $\langle\sigma_R\rangle$ dash-dotted line, $\langle\overline{\sigma_L}\rangle$ continuous line, $\langle\sigma_L\rangle$ dashed line. The value of α (see text) is indicated for each set of curves.

of a quantity corresponding to α in a chaotic system goes beyond the scope of this work. The problem, however, is interesting and well-worth investigating.

Having examined the time dependence of global average predictability indexes, we now turn to the study of the time evolution of their phase space variability and go back to the Lorenz system. Fig. 5 shows PDFs of the indexes evaluated during the experiment that produced the results presented in Fig. 2. Fig. 5a is relative to the first Lyapunov exponent $\overline{\sigma_l}$ averaged over $t = 0.1$, 0.4, 1.0. Fig. 5b, c are relative to the growth rates associated with the Lorenz index $\overline{\sigma_L}$ and the optimal perturbation $\overline{\sigma_O}$ averaged between $t = 0.0$ and $t = 0.1$, 0.4, 1.0. Fig. 5d refers to $\overline{\sigma_R}$.

The smallest value of t for which we show results is equal to 0.1; further decreasing it does not affect the probability distribution function of Fig. 5a, as it is evident from comparison with Fig. 3a in Nese (1989), obtained with $t = 0.02$, and using a different time integration scheme. We may thus conclude that $t = 0.1$ is small enough to resolve the fine phase space structure of predictability on the

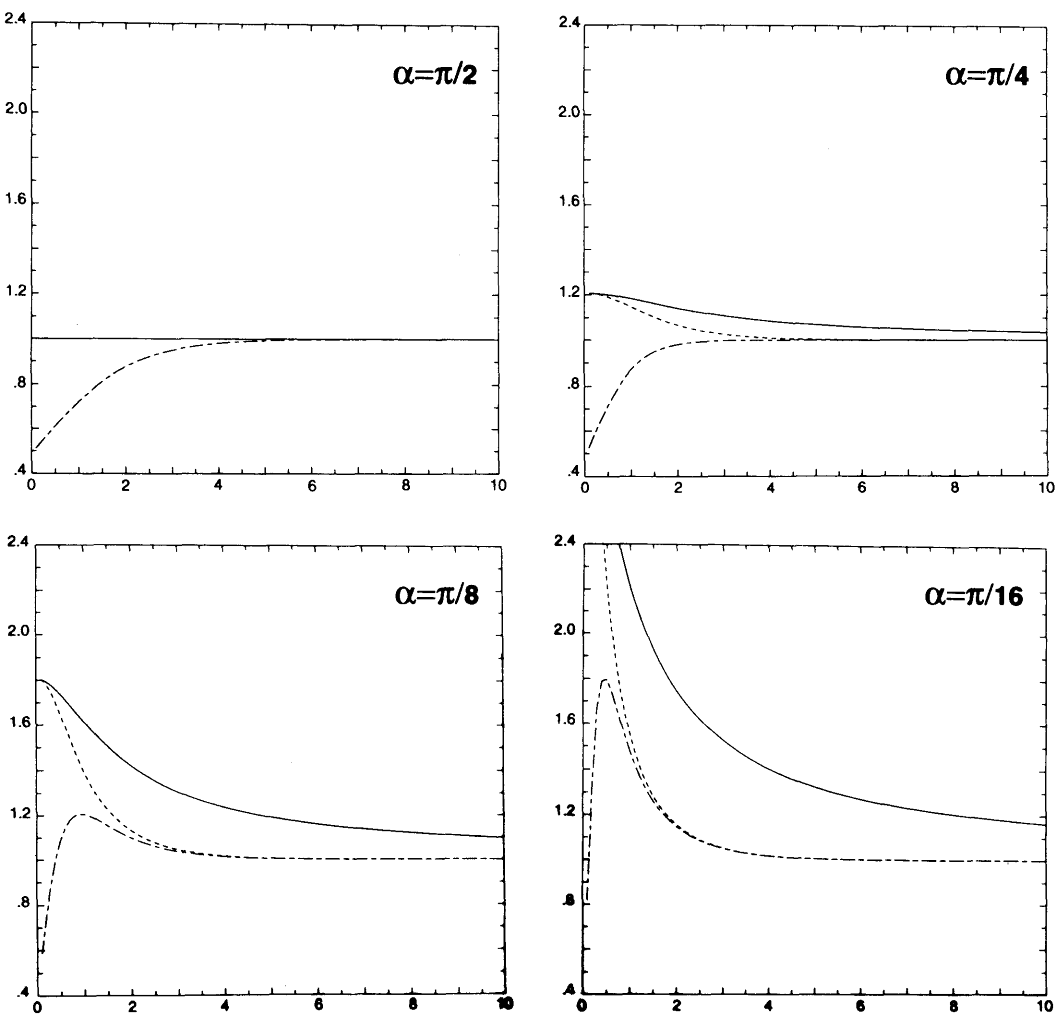

Fig. 4. Same as Fig. 2b, but for the linear system of the Appendix: $\langle \overline{\sigma_O} \rangle$ continuous line, $\langle \sigma_O \rangle$ dashed line, $\langle \sigma_R \rangle$ dash-dotted line. The value of α is indicated for each set of curves.

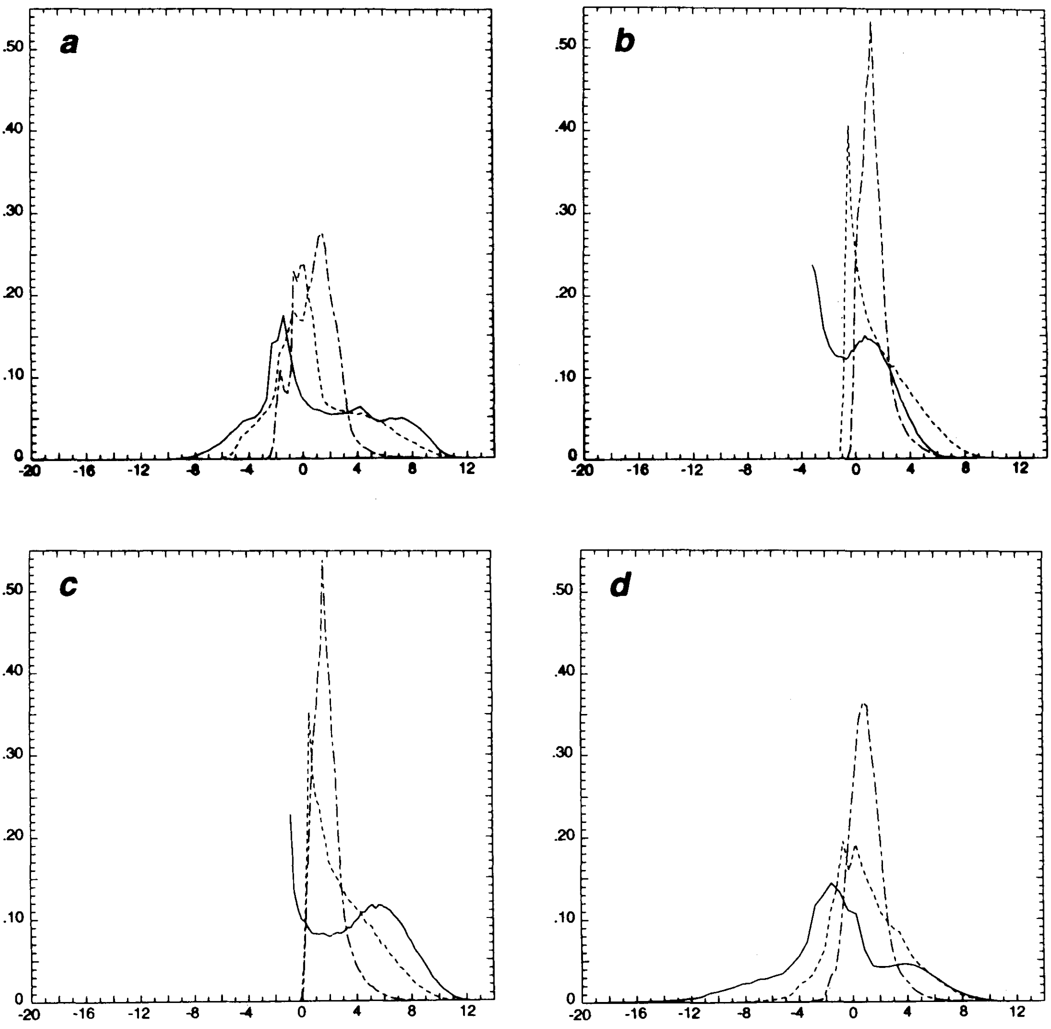

Fig. 5. PDFs of predictability indexes in the Lorenz system averaged over time intervals equal to 0.1 (continuous lines), 0.4 (dashed lines), 1.0 (dash-dotted lines): (a) $\overline{\sigma_l}$; (b) $\overline{\sigma_L}$; (c) $\overline{\sigma_O}$; (d) $\overline{\sigma_R}$.

attractor. The value $t = 0.4$ corresponds to a PDF well into the transient and $t = 1.0$ to a PDF belonging to the post-transient phase.

Comparing the PDFs, we notice large discrepancies among the indexes at $t = 0.1$ and $t = 0.4$ (see also Table 1). In fact, some of the possible directions of error growth are contracting, and other are expanding, while each index gives them different weights. In particular, σ_O is relative to the perturbation which grows fastest in the given time interval; σ_R is the average growth rate of a Monte

Table 1. *Mean and standard deviation of the PDFs of the indexes $\overline{\sigma_l}$, $\overline{\sigma_L}$, $\overline{\sigma_O}$, $\overline{\sigma_R}$, at time $t = 0.1$, 0.4, 1.0*

	$\overline{\sigma_l}$		$\overline{\sigma_L}$		$\overline{\sigma_O}$		$\overline{\sigma_R}$	
t	mean	SD	mean	SD	mean	SD	mean	SD
0.1	0.90	4.13	0.08	2.23	3.99	3.08	−1.21	4.30
0.4	0.90	3.08	1.72	2.14	3.00	2.21	0.95	2.61
1.0	0.90	1.48	1.37	1.05	1.90	1.07	1.02	1.27

Carlo ensemble; σ_L is the growth rate of the RMS error of the Monte Carlo ensemble. It is interesting to notice that on the time scale $t = 0.1$, the PDF of all indexes, including σ_O extends to negative values of growth rate, indicating that there are regions of the attractor's phase space where, locally and over a short time interval, all perturbations decay. Mapping of the Lorenz index by Mukougawa et al. (1991), computed for $t = 0.1$, shows in fact amplification factors that are less than one for large values of z.

The distributions of Fig. 5, for $t = 1.0$, provide estimates of growth rates, computed according to the definitions of the various indexes, averaged throughout the transient phase. Although reduced, there are still considerable differences among the various distributions, evident also by comparing their first and second moments (see Table 1). These differences give an evaluation of the integral effect of the transient on the different indexes. We notice that the standard deviation of the distributions is still large and comparable with the average growth rate (see Table 1). This effect is due to the extremely large phase space variability of this system. For subsequent time intervals of the forecast we can rely upon the local Lyapunov exponent as the relevant predictability index. This statement is supported by the evidence that the PDFs of instantaneous growth rate computed for any time $t > T$ and for any of the considered indexes, to an extremely good approximation, coincide with one another and with the PDF computed for the local Lyapunov exponent (not shown).

It is evident from inspection of Fig. 5 and Table 1 that large variability of local predictability measures in different regions of phase space is a prominent feature of the Lorenz system; other systems may not exhibit such strong variability. In such cases the phase space averaged quantities would be much more representative of local predictability. For each particular system the impact on predictability of phase space dependence versus the dependence upon the initial perturbation must be assessed specifically. We have seen in the present system that after the transient has died out, and all initial perturbations of an individual trajectory grow at the same rate, the standard deviation of growth rate of an ensemble of initial conditions covering the attractor's phase space is about as large as during the transient phase.

The transient time for the Lorenz system is comparable to an orbital period; this implies that large portions of the attractor are spanned when averaging measures of predictability over periods comparable with the transient. In fact, in this system the transient time is an order of magnitude larger than the "advective" time, defined as the time scale over which, following a trajectory, predictability properties change appreciably; we evaluated it to be of the order of 0.1, noticing that the PDFs do not change for shorter time intervals.

4. Conclusions

Finite time error growth is subject to transient behaviour due to the fact that not all initial perturbations grow at the same rate. The fact that for some directions of the initial error vector, errors decay over a finite time interval, while for others amplify, causes finite time predictability indexes to be crucially dependent upon the way the different possible error directions are weighed, according to the definition of the various indexes, and upon the length of the forecast time. Whereas the growth rate of the optimal perturbation $\overline{\sigma_O}$, which gives an upper bound for error growth, can be very large for short time intervals, the direction-average growth rate $\overline{\sigma_R}$ may be negative over the same time interval.

We showed that the transient behaviour of the indexes in a system with a fixed point (i.e., with homogeneous instability properties), is completely analogous to that of the Lorenz system, provided we consider the ensemble mean of indexes in the latter, thus eliminating the effect of phase space variability. The Lorenz system has very large phase space variability; there are even regions of phase space where all initial errors decay over a short time interval ($t = 0.1$). However, the ensemble mean of any index converges to the first Lyapunov exponent $\sigma_1 = 0.9$ in a finite time interval $T \approx 1$ (transient time).

In summary, at a sufficiently long forecast time: in an attractor with inhomogeneous predictability properties, all initial perturbations of the same initial condition grow at the rate given by the first local Lyapunov exponent; in an attractor with homogeneous predictability properties, all perturbations of any initial condition grow at the same rate given by the first global Lyapunov exponent, that in this case coincides with the local one. The length of the transient can be defined as the necessary time interval for all vectors to align along the direction of the first Lyapunov vector. After this time all local predictability indexes converge to the first *local* Lyapunov exponent, whereas the ensemble mean of the indexes converges to the first *global* Lyapunov exponent. This is consistent with the fact that, according to our definition, the first *global* Lyapunov exponent is the phase space average of the first *local* Lyapunov exponent. The rate of convergence is clearly more rapid for instantaneous indexes, and $\langle \sigma_R \rangle$ converges more rapidly than the other indexes.

The inhomogeneity of the indexes on the attractor must be studied in order to estimate the impact of phase space dependence on predictability. We examined the PDFs of different indexes averaged over time intervals belonging to both the transient and asymptotic regimes. As it is clear from inspection of PDFs of Lyapunov exponents, phase space variability is a prominent feature of the Lorenz system.

The same kind of analysis developed in the present work can be applied to any other system in order to obtain a complete picture of the variability of predictability. We are currently extending this analysis to an intermediate model of the general circulation of the atmosphere. Preliminary results show transient behavior also in this model, but the standard deviation of the PDF of the Lyapunov exponent is much smaller than the mean error growth rate, indicating that phase space variability of predictability is much less prominent than in the Lorenz system.

The implications of the present results for predictability in NWP models are discussed in the following.

We have shown that if the transient is comparable with the forecast time estimates of predictability are highly dependent on the choice of the predictability index and on the forecast time. Without a precise knowledge of the duration and intensity of the transient, it is not possible to assess the implications of a particular choice with respect to another. No estimate has ever been made of the intensity of the transient in NWP models; however the problems connected to the existence of the transient are important for short range forecast skill and its quantification.

At long range, i.e., for forecast times longer than the transient, the dependence of linear growth rate on the structure of the initial error tends to disappear, leaving only the dependence on phase space location. This fact has the following two consequences:

(a) At long-range, linear predictability measures become insensitive to the choice of index (or of the initial perturbation). However, we should be aware of the fact that nonlinear effects may become important before this happens; the procedure developed in the present study can be used to answer this question.

(b) The problem of quantifying phase space variability of predictability becomes feasible if use is made of the first local Lyapunov exponent.

Common practice for predicting forecast skill is based on ensemble forecasts; different methods are used to choose the members of the ensemble. One criterion is to construct the ensemble by selecting the perturbations which grow fastest in a given time interval (Molteni and Palmer,1993). Based on the results of the present work, this procedure provides an upper bound for error growth but could largely overestimate the expected growth rate, if the fastest growing modes are computed for forecast times shorter than or comparable to the transient time.

The breeding method, recently developed by Toth and Kalnay (1993), is based on a procedure which in principle* selects the first Lyapunov vector, and uses it as representative of the ensemble. This method is very efficient in the event that the first Lyapunov exponent is an adequate measure of predictability for the forecast time for which it is used. This can be ascertained after the characteristics of the transient have been evaluated. Furthermore, the authors argue that, due to initialization procedures which rely upon model forecasts, the initial error is not random but has a large projection on the breeding mode. In this circumstance, one would expect a very good performance of the method even during the transient.

In order to improve our ability of forecasting forecast skill further work is needed on the following lines: to evaluate the characteristics of the various phases of error growth in GCMs; to

* In a perfect model environment and if the initial errors are sufficiently small.

compare the duration of the transient with the range of validity of linear error dynamics for typical errors in the definition of the initial state; to study how the error in the analysis projects on different growing modes initially and during the various phases of transient growth; to compare the relative importance of phase space variability with variability associated with the structure of the initial error.

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6. Appendix

Predictability indexes for a system with a fixed point

Lacarra and Talagrand (1988) examined the behavior of a linear two dimensional system whose eigenvectors are not orthogonal (see their Appendix A), a condition which gives rise to enhanced exponential transient growth. We generalize the system they used to allow for an explicit dependence on the angle α between the eigenvectors. We compute, for this system, the growth rate according to the different indexes (see Section 2), and show how the transient behavior changes with the parameter α.

The system we consider is:

$$\dot{x}_1 = a x_2$$

$$\dot{x}_2 = x_2.$$

The eigenvalues of the resolvent matrix A associated with this system are $\lambda_1 = 1$ and $\lambda_2 = \mathrm{e}^t$, and the corresponding eigenvectors are $\boldsymbol{e}_1 = (1, 0)$ and $\boldsymbol{e}_2 = (1/\sqrt{1+a^2})(a, 1)$. The angle between the eigenvectors is $\alpha = \arctan(1/a)$.

The Lorenz index is usually written in terms of the eigenvalues of the matrix A^+A. It can also be written explicitly as a function of the eigenvalues of the matrix A and of the angle α. We shall use the latter formulation in order to study the dependence of the indexes on the angle α.

Consider a circle of unitary radius. Each vector $\boldsymbol{r}$ belonging to it will be transformed in a vector $\boldsymbol{y} = A(t)\,\boldsymbol{r}$ belonging to an ellipse whose semi-axis are a_1 and a_2, whose squares are the eigenvalues of the matrix A^+A given by:

$$a_{1,2}^2 = \tfrac{1}{2}(a^2(\mathrm{e}^t - 1) + \mathrm{e}^{2t} + 1) \pm \tfrac{1}{2}\sqrt{(a^2(\mathrm{e}^t-1)^2 + \mathrm{e}^{2t} + 1)^2 - 4\mathrm{e}^{2t}}$$

If we write $\boldsymbol{r}$ as a linear combination of the eigenvectors of A, i.e., $\boldsymbol{r} = c_1\boldsymbol{e}_1 + c_2\boldsymbol{e}_2$, and express the vectors in terms of a Cartesian system $\hat{\imath}$ and $\hat{\jmath}$, we have:

$$\boldsymbol{e}_1 = \hat{\imath}\cos\alpha_1 + \hat{\jmath}\sin\alpha_1,$$

$$\boldsymbol{e}_2 = \hat{\imath}\cos\alpha_2 + \hat{\jmath}\sin\alpha_2,$$

$$\boldsymbol{r} = \hat{\imath}\cos\theta + \hat{\jmath}\sin\theta,$$

where α_1, α_2, and θ are the angles between the corresponding vector and the axis $\hat{\imath}$. The mean square distance of the transformed vectors $\boldsymbol{y} = A(t)\,\boldsymbol{r} = \lambda_1 c_1 \boldsymbol{e}_1 + \lambda_2 c_2 \boldsymbol{e}_2$ is:

$$[|y^+ y|]_\theta = \lambda_1^2[c_1^2]_\theta + \lambda_2^2[c_2^2]_\theta + 2\lambda_1\lambda_2[c_1c_2]_\theta\cos\alpha = \frac{\lambda_1^2}{2\sin^2\alpha} + \frac{\lambda_2^2}{2\sin^2\alpha} - \lambda_1\lambda_2\frac{\cos^2\alpha}{\sin^2\alpha},$$

where the operator $[*]_\theta$ indicates averaging over θ, $\alpha = \alpha_1 - \alpha_2$, and we used

$$[c_1^2]_\theta = \left[\left(-\frac{\sin(\alpha_2-\theta)}{\sin(\alpha_1-\alpha_2)}\right)^2\right]_\theta = \frac{1}{2\sin^2(\alpha_1-\alpha_2)},$$

$$[c_2^2]_\theta = \left[\left(\frac{\sin(\alpha_1-\theta)}{\sin(\alpha_1-\alpha_2)}\right)^2\right]_\theta = \frac{1}{2\sin^2(\alpha_1-\alpha_2)},$$

$$[c_1c_2]_\theta = -\frac{\cos(\alpha_1-\alpha_2)}{2\sin^2(\alpha_1-\alpha_2)}.$$

Finally, we can write the growth rate associated with the Lorenz index:

$$\overline{\sigma_L}' = \frac{1}{t}\log\sqrt{[|y^T y|]_\theta} = \frac{1}{2t}\log\left(\frac{1}{\sin^2\alpha}\left(\frac{1+\mathrm{e}^{2t}}{2} - \mathrm{e}^t\cot^2\alpha\right)\right),$$

or, in terms of the semi-axis of the ellipse:

$$\overline{\sigma_L}' = \frac{1}{2t} \log \frac{a_1^2 + a_2^2}{2}.$$

The direction average index $\overline{\sigma_R}'$ involves taking the average of logarithms:

$$\overline{\sigma_R}' = \frac{1}{t} [\log \sqrt{y^+ y}]_0$$

$$= \frac{1}{t} [\log \sqrt{\lambda_1^2 c_1^2 + \lambda_2^2 c_2^2 + 2\lambda_1 \lambda_2 c_1 c_2 \cos \alpha}]_0.$$

We compute $\overline{\sigma_R}'$ numerically except for the case $\alpha = \pi/2$, for which we can write $\overline{\sigma_R}'$ in terms of the semi-axis of the ellipse

$$\overline{\sigma_R}' = \frac{1}{t} \log \left(\frac{a_1 + a_2}{2} \right),$$

where we have made use of the identity

$$\frac{1}{\pi} \int_0^\pi \log(\lambda_1^2 \sin^2 \theta + \lambda_2^2 \cos^2 \theta) \, d\theta$$

$$= 2 \log \left(\frac{\lambda_1 + \lambda_2}{2} \right).$$

The growth rate of the optimal perturbation is:

$$\overline{\sigma_O}' = \frac{1}{t} \log a_1$$

Note that the overline indicates a time average, and that by use of (9) we can derive instantaneous values for all these indexes.

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