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Retrieval of balanced fields: an optimal control method

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ABSTRACT

The nonlinear balance equation links the stream function and the geopotential in a steady-state approximation of the atmosphere. It has been widely used to retrieve coherent wind and mass fields in limited areas under the assumptions that one of these fields was known without error within the domain and the other partially known on the boundary. The method which is proposed in this paper is based on optimal control techniques; it permits one to alleviate the above assumptions. In a first part of the paper, we describe an iterative algorithm, the geopotential and mass term of the balance equation are adjusted, and the boundary conditions are controlled in such a way that the discrepancy between the observations and a solution of the balance equation is minimized. The convergence of the method is discussed and a numerical example is carried out. The nonlinear balance equation uses the wind fields through the stream function. Of course, only the wind fields are observed, and the second part is devoted to an algorithm which permits one to compute the stream function and the potential velocity from wind observations in a limited area. The method is also based on the control of the boundary conditions.

1. Introduction

From the mathematical point of view, a model for numerical weather prediction can be seen as a differential equation:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = F(X),$$

where the vector X represents the state of the atmosphere, and F is the nonlinear partial differential operator governing the evolution of the atmosphere. The forecast from time T_0 to time T_1 is obtained by the numerical integration of (1), starting from an initial condition U which can be deduced from an observation X_{obs}, which is a given function of space and time, the term observation being here considered in a broad sense including physical observations of the state of the atmosphere as well as the results of numerical models. The initial condition has to be choosen in such a way that it satifies both the following constraints.

(i) It must be as close as possible to the observation. One difficulty follows from the fact that the observation and the variable X do not belong in the same space. If, for instance, the model as been discretized according to a finite difference scheme, then X stands for the values of the meteorological variables at the grid-points, whereas the observations represented by X_{obs} may be located at some other points of the physical space, or are not the meteorological variables used in the numerical model; for example, the quantity measured is radiance if derived from satellite observations. In order to compare the numerical variables and the physical observations, an operator C has to be defined in such a way that $C \cdot X$ belongs to the space of observations. A 2nd difficulty arises in the choice of the function, which should be a norm, measuring the discrepancy between $C \cdot X$ and X_{obs}.

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(ii) After the numerical integration of (1), the resulting solution must be in agreement with the properties of the atmosphere. For example, it is important to ensure that the numerical solution is almost free of gravity motion during the first steps of the integration.

A way to reduce the noise, due to gravity waves, at the beginning of a numerical prediction using primitive equations, is to impose a balance constraint between wind and mass fields. Traditional methods assume that one field (usually the mass field in mid-latitudes and the wind field in tropical latitudes), is measured without error. Then, the balance equation is used to deduce the other field; therefore, at the end of the analysis, we have at our disposal two balanced fields. However the price to be paid is a strong and unrealistic assumption with respect to the first field, not taking into account the observations of the other field. Haltiner et al. (1975) have derived a variational method using both wind and geopotential observations. Nevertheless some difficulties remain: (a) the balance equation has a unique solution if some ellipticity condition (Courant and Hilbert, 1962) is satisfied; this condition is written as an inequality constraint which must be enforced by the analyzed fields; (b) in limited domains, some boundary conditions have to be provided to solve the balance equation, and derive a stream function and velocity potential from the wind observations.

The method which is proposed is this paper is based on an optimal control approach (Le Dimet and Talagrand, 1986). In most of the applications, this method was carried out using a time-dependent model as a constraint, the initial condition being chosen in such a way that the resulting solution best fits the observations. In this paper, we will consider, as a constraint, a steady-state equation (the balance equation) for which the boundary conditions are unknown. These boundary conditions are used as control variables to achieve the best adjustment of the analyzed fields to the observations. The main advantage of this approach is to be independent of the hazardous assumptions, such that the fields, or part of them, are known without error on the boundary of the domain. These assumptions are implicitly used in most of the traditional methods which solve the balance equation. In the following, the derivation of the method is given, and then some numerical results are presented firstly with simulated data then with analyzed FGGE data.

The balance equation uses the stream function, which is not directly observable, but which has to be computed from the wind. In a limited area, there is also the problem of boundary conditions on the stream function. To solve this problem, a method, based on control of the boundary conditions, is proposed in the second part of this paper.

2. Balance equation and ellipticity condition

The balance equation is deduced from the horizontal divergence equation by neglecting the divergent part of the horizontal velocity and its temporal evolution. In a system of cartesian coordinates (x, y), where f denotes the Coriolis parameter, the balance equation is written as:

$$f\nabla^2\psi + \nabla f \nabla\psi + 2(\psi_{xx}\psi_{yy} - \psi_{xy}^2) - \nabla^2\Phi = 0. \quad (1)$$

In this expression, Φ is the geopotential and Ψ is the stream function. With respect to Ψ, eq. (1) is a nonlinear partial differential equation of the Monge-Ampère type. It is known (Courant and Hilbert, 1962) that with prescribed boundary conditions, it has at most two solutions, if the so-called ellipticity condition

$$2\nabla^2\Phi + f^2 - \nabla f \nabla\psi > 0. \quad (2)$$

is verified on the considered domain. If Φ is known and if adequate boundary conditions are given, then the stream function can be deduced from (1). On the other hand, if the stream function is known, then Φ can be deduced from (1) by solving a Poisson equation.

Many iterative methods have been designed to solve the balance equation; the convergence of these methods is linked to the enforcement of the ellipticity condition. In most of the algorithms, a crude change is performed on the data if the ellipticity condition is not verified. Observational evidence (Paegle and Paegle, 1974, 1976) shows the existence of non-elliptic regions in the actual atmosphere; these regions are often associated with intense convection, especially in the tropics. An analysis of the significance of non-elliptic regions is given in Kasahara (1982). Basically, it is shown that the approximation of the atmosphere by a balance equation is not valid in that case: an additional term must be added to the balance equation modifying the ellipticity condition.

3. A variational solution for the balance equation

3.1. An optimal control algorithm

To retrieve balanced field, we assume to dispose over "observation" of the geopotential and of the stream function ψ_{obs} and Φ_{obs}. The stream function and geopotential are not directly observable; therefore we will assume (in a first step) that the estimation of these variables has been deduced from the measurement of meteorological variables. The balance equation can be written:

$$f\nabla^2\psi + \nabla f \nabla\psi + 2G(\nabla\psi) = \nabla^2\Phi. \tag{3}$$

In this expression, G is the determinant of the Hessian matrix:

$$G(\nabla\psi) = \frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial y^2} - \left[\frac{\partial^2\psi}{\partial x\,\partial y}\right]^2. \tag{4}$$

This nonlinear equation is solved by an iterative method of successive approximations linearizing eq. (3). The method can be described as follows: starting from an initial guess, $\Psi^{(0)}$, then at each iteration, $\Psi^{(n+1)}$ and $\Phi^{(n+1)}$ are determined by solving the equation:

$$f\nabla^2\psi^{(n+1)} + \nabla f \nabla\psi^{(n+1)} + 2G(\nabla\psi^{(n)}) = \nabla^2\Phi^{(n+1)}. \tag{5}$$

Eq. (5) has two unknowns, $\Psi^{(n+1)}$ and $\Phi^{(n+1)}$; furthermore, no boundary conditions are imposed, and it therefore has an infinity of solutions. Among all the solutions of eq. (5), we will choose the closest to the observations, which is defined by the following variational principle. Let Γ be the boundary of the considered domain Ω, u being a function defined on Ω, and with v and w two functions defined on the boundary Γ. We consider the following Poisson's problems:

Problem (1). Solve the equation:

$$f\nabla^2\psi^{(n+1)} + \nabla f \nabla\psi^{(n+1)} + 2G(\nabla\psi^{(n)}) = u \quad \text{in } \Omega, \tag{6}$$

with the boundary condition:

$$\psi_\Gamma^{(n+1)} = v. \tag{7}$$

u and v being given, *Problem (1)* has a unique solution, which can be numerically computed using standard methods for solving elliptic equations.

Problem (2). Solve the equation:

$$\nabla^2\Phi^{(n+1)} = u \tag{8}$$

with the boundary condition:

$$\Phi_\Gamma^{(n+1)} = w. \tag{9}$$

In the same way, Problem (2) has a unique solution when u and w are given. For each value of u, v and w let us define a *cost-function* J by:

$$J(u, v, w) = \tfrac{1}{2}\int_\Omega \alpha(C\psi^{(n+1)} - \psi_{\text{obs}})^2 + \beta(C\Phi^{(n+1)} - \Phi_{\text{obs}})^2\,\mathrm{d}\Omega, \tag{10}$$

where α and β are two given weight functions depending on the space variables. According to classical terminology used in optimal control theory, the variables u, v and w are *control variables*, and J is implicitly dependent on these variables through the stream function and the geopential computed as a solution of problems 1 and 2. The optimal choice of u, v and w minimizing J, allows the adjustment of the computed geopotential, the stream function and the boundary conditions to the observations.

The definition of the cost-function supposes that observations of the stream function and of the geopotential are available. Of course, this assumption has no physical sense; in the second part of the paper, we will see how to deduce these values from the observations. At the (outer) iteration $(n+1)$ of the algorithm, the optimal values u^*, v^*, w^* are chosen in such a way that they minimize $J(u, v, w)$; then $\Psi^{(n+1)}$ and $\Phi^{(n+1)}$ are deduced by solving problems (1) and (2). These optimal values are solutions of a problem of optimization and are computed as the limits of sequences built by an iterative method (inner iteration) of the descent type. To carry out a method of optimization, the gradient of the cost-function J, with respect to the control variables u, v and w has to be computed;

this can be done by using the adjoint of the linearized balance equation.

3.2. Optimality condition

In eqs. (6)–(9), for convenience, we will drop the iteration index. If we consider a perturbation $h = (h_u, h_v, h_w)$ on the control variables u, v, w, then $\hat{\Phi}$ and $\hat{\psi}$, the directional derivatives of Φ and ψ, in the direction h are solutions of:

$$f\nabla^2\hat{\psi} + \nabla f \nabla\hat{\psi} = h_u \qquad \text{in } \Omega, \tag{11}$$

$$\hat{\psi}_\Gamma = h_v, \tag{12}$$

$$\nabla^2\hat{\Phi} = h_u, \tag{13}$$

$$\hat{\Phi}_\Gamma = h_w. \tag{14}$$

Whereas the directional derivative of J is given by:

$$\hat{J}(u, v, w, h) = \int_\Omega \alpha(C\psi - \psi_{\text{obs}}, C\hat{\psi}) + \beta(C\Phi - \Phi_{\text{obs}}, C\hat{\Phi})\,\mathrm{d}\Omega, \tag{15}$$

by definition of the gradient, we have also:

$$\hat{J}(u, v, w, h) = \int_\Omega (\nabla J, h)\,\mathrm{d}\Omega.$$

To compute the gradient, we have to exhibit the linear dependence of the directional derivative of J with respect to the perturbation. Let us introduce P and Q, the *adjoint variables*; they will be precisely defined later. The inner products of (11) with P and of (13) with Q are taken; then integrated over Ω, we have:

$$\int_\Omega [(f\nabla^2\hat{\psi} + \nabla f\nabla\hat{\psi}, P) + (\nabla^2\hat{\Phi}, Q)]\,\mathrm{d}\Omega = \int_\Omega (h_u, P+Q)\,\mathrm{d}\Omega. \tag{16}$$

After using the Green formula in (16), it is easily seen that the adjoint variables are defined as:
P is the solution of:

$$\nabla^2(fP) - \nabla(P\nabla f) = C^t(C\psi - \psi_{\text{obs}}), \tag{17}$$

with the boundary condition:

$$P_\Gamma = 0. \tag{18}$$

Q is the solution of:

$$\nabla^2 Q = C^t(C\Phi - \Phi_{\text{obs}}) \tag{19}$$

with the boundary condition:

$$Q_\Gamma = 0. \tag{20}$$

Then the gradient of J is given by:

$$\nabla J(u, v, w) = \begin{pmatrix} \nabla J_u \\ \nabla J_v \\ \nabla J_w \end{pmatrix} = \begin{pmatrix} P + Q \\ \left(\dfrac{\partial(fP)}{\partial n}\right)_\Gamma \\ \left(\dfrac{\partial Q}{\partial n}\right)_\Gamma \end{pmatrix}. \tag{21}$$

Therefore the gradient is obtained after solving the adjoint system (17)–(20). We summarize the method as follows:

(i) The algorithm is started with a first guess of:

$\Psi^{(1)}$ for outer iterations;

$u_0^{(0)}$, $v_0^{(0)}$, $w_0^{(0)}$ are initial values for the control variables.

For the control variable, the upper index stands for the outer iteration, while the lower one is for the inner iteration.

(ii) $\Psi^{(1)}$ and $\Phi^{(1)}$ are computed by solving (6)–(9).

(iii) The adjoint system (17)–(20) is solved using $\Psi^{(1)}$ and $\Phi^{(1)}$

(iv) The gradient of J is computed according to (21). Using the gradient, a direction of descent D_k is computed (for instance it may be the direction corresponding to a conjugate gradient method, see Lemaréchal (1980)).

(v) The control variables are updated, in the inner iteration, according to:

$$u_{k+1}^{(1)} = u_k^{(1)} + r\,D_k,$$

$$v_{k+1}^{(1)} = v_k^{(1)} + r\,D_k,$$

$$w_{k+1}^{(1)} = w_k^{(1)} + r\,D_k,$$

where r, the stepsize, is choosen such it minimizes J along the direction of descent D_k. The stepsize and the descent direction are choosen according to a method of unconstrained optimization, e.g., the conjugate gradient or quasi-Newton methods (Le Dimet and Talagrand, 1986).

(f) If some stopping criterion is verified, then the nonlinear term in (6) is updated and we go back to (b) with $n = n + 1$.

3.3. Remark: practical determination of the adjoint model

There is no commutativity between the discretization and the operation of taking the adjoint; this is because these operations follow from the application of linear operators which do not commute, except if they have the same set of eigenvectors. Therefore, the derivation of the method has also to be carried out on the discretized model. To derive the adjoint model of the computer code, the direct code has to considered. Each statement of the direct model must be identified in terms of linear operators. Then, the adjoint model is obtained by starting from the end of the direct model and by transposing all the elementary statements. To carry out the direct model input variables, which are the control variables, provided by the code, each variable of the direct model has an associated adjoint variable. It can be seen (Le Dimet, 1988) that the component of the gradient of the cost function associated with an input variable is equal to the output value of its adjoint variable.

3.4. Numerical test

The method has been applied on a square domain $[0, 2\pi]$. The "true" stream function was defined by:

$$\psi(x, y) = \frac{100}{2\pi} \sin(x) \sin(y).$$

Then the geopotential was computed according to:

$$\nabla^2 \Phi = f \nabla^2 \psi + \nabla f \nabla \psi + 2G(\nabla \psi).$$

Pseudo-observations were simulated by adding a random noise to the stream function and to the geopotential:

$$\psi_{\text{obs}}(x, y) = \psi(x, y) + b(x, y),$$

$$\Phi_{\text{obs}}(x, y) = \Phi(x, y) + c(x, y).$$

The functions b and c are the values of random variables with a uniform distribution on the domain. Fig. 1 shows the exact value of the stream function and of the geopotential. Figs. 2a, b are the "observations" of the same fields. The model has been discretized with a centered difference scheme

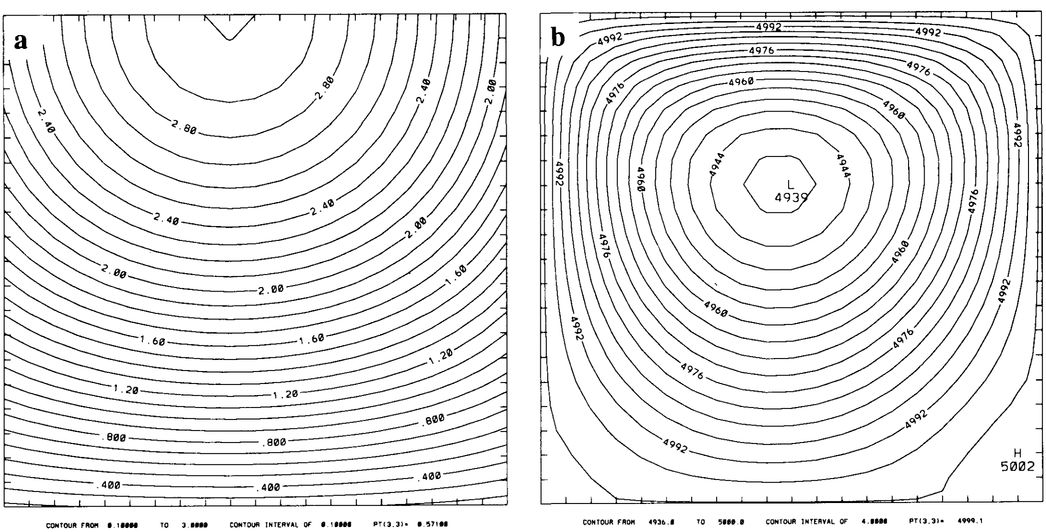

Fig. 1. Exact solution of the problem: (a) stream function, (b) geopotential height.

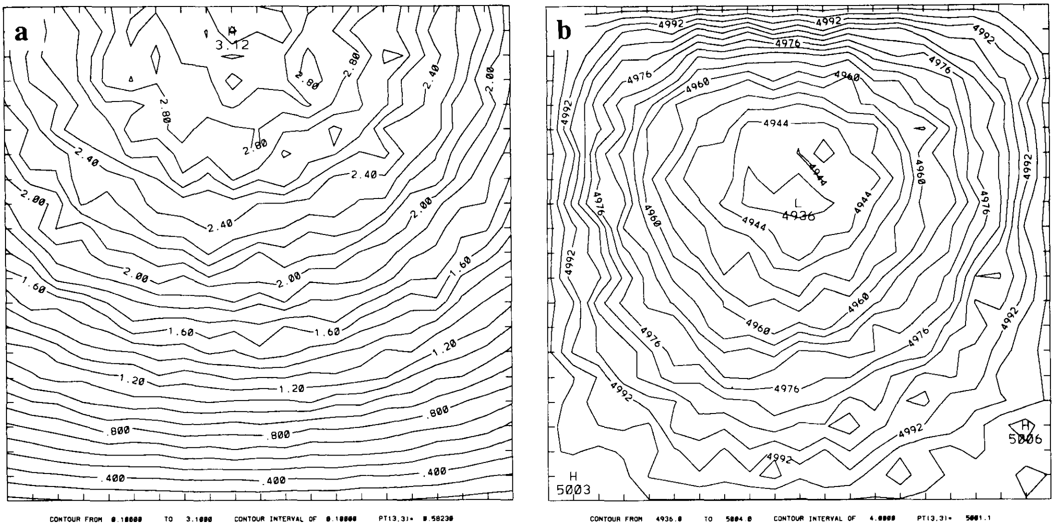

Fig. 2. "Observations": (a) stream function, (b) geopotential height.

with a grid size of $(2\pi/20)$. We assumed that an observation of the stream function and of the geopotential were available at each grid point.

The procedure of unconstrained optimization used in this test is a method of conjugate gradient, written in the subroutine M1GC3 of the Modulopt Library. Figs. 3, 4 display the retrieval of the stream function and geopotential after 5 outer iterations. Fig. 5 shows the value of the cost function versus the number of iterations; this value decreased by 4 orders of magnitude in 5 iterations. A comparison of Fig. 1 and Figs. 3–4 shows that the fields were retrieved with good accuracy.

From the numerical point of view, the results obtained are good, but at the price of a rather high computational cost. Nevertheless, the computational cost could be reduced by using a more efficient solver for the Poisson problem. For the

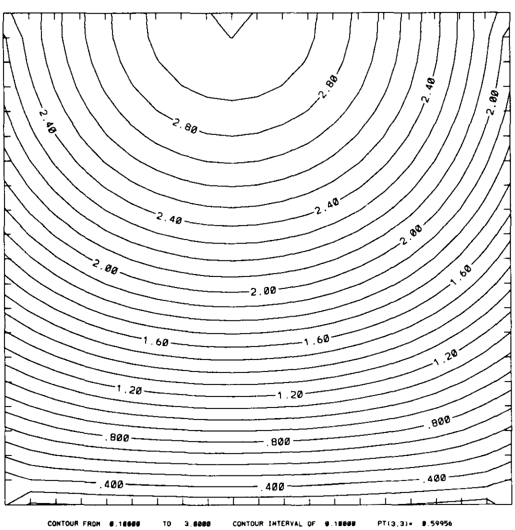

Fig. 3. Stream function at iteration 6.

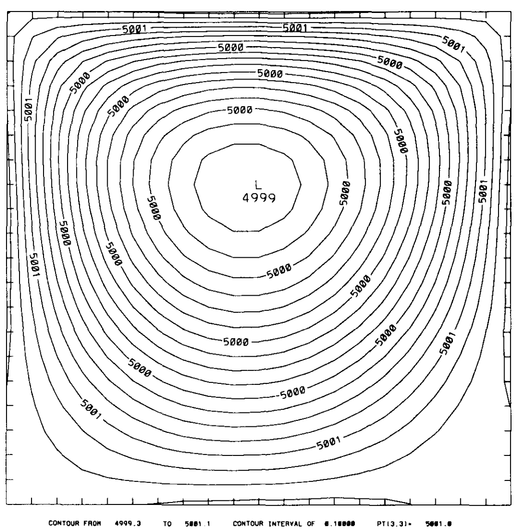

Fig. 4. Geopotential height at iteration 6.

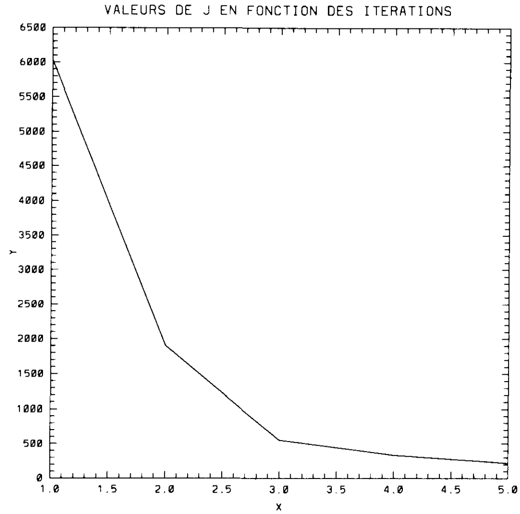

Fig. 5. Evolution of the cost function versus the number of iterations.

outer iteration, a classical successive approximation algorithm is carried out; therefore its convergence is limited by a verification of the ellipticity condition. It is worth noting that the ellipticity condition can be added as a constraint in the definition of the variational formulation, leading to a problem of constrained optimization, for which standard procedure can be followed.

4. Computing velocity potential and stream function

4.1. General relations and usual methods

The algorithm described above uses the stream function and its observation. Of course, the stream function is not directly observable: it has to be deduced from the wind. Wind, stream function Ψ and velocity potential are linked to the vorticity and to the divergence D of the wind through the relations:

$$\nabla^2\psi = \zeta, \tag{22}$$

$$\nabla^2\chi = D. \tag{23}$$

Vorticity and divergence are directly computed from the wind by:

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \tag{24}$$

$$D = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}. \tag{25}$$

From the observed wind, divergence and vorticity may be easily derived, if a correct discretization has been carried out; then the stream function and velocity potential are deduced by solving the Poisson equations (22) and (23), if boundary conditions have been provided for these equations. Let V_s and V_n be the tangent and normal components of the wind on the boundary; they are related to the stream function and the velocity potential by the relations:

$$V_s = \frac{\partial\psi}{\partial n} + \frac{\partial\chi}{\partial s}, \tag{26}$$

$$V_n = -\frac{\partial\psi}{\partial s} + \frac{\partial\chi}{\partial n}. \tag{27}$$

The method of Sangster (1960) assumes that the velocity potential χ vanishes on the boundary. With this assumption, the velocity potential can be computed by solving (23) with homogeneous Dirichlet boundary conditions. Then eq. (27) can be integrated around the boundary, using the normal component of the observed wind on the boundary (therefore making the assumption that there is no error in this component). In the last step, the stream function is computed by solving the Poisson equation with the Dirichlet boundary condition. There will be a solution if the following condition, deduced from Gauss' theorem, is satisfied:

$$\int_\Omega D\,\mathrm{d}\Omega = \int_\Gamma V_n\,\mathrm{d}\Gamma. \tag{28}$$

Various algorithms have been proposed to solve this problem. All the methods assumed that a part of the wind (either its tangential component or its normal one) can be observed without error. For practical applications, we can expect only a few observations on the boundary, and therefore this assumption may induce large error in the retrieved field.

4.2. A variational algorithm

The method that we propose is based on the control of the solution of the Poisson's equations (22) and (23), by their boundary values under Dirichlet conditions. With the boundary conditions:

$$\chi = \alpha, \tag{29}$$

$$\psi = \beta, \tag{30}$$

the eqs. (22)–(30) and (23)–(29) both have a unique solution.

An observation of the wind being given, a cost function G can be defined, depending implicitly on the boundary conditions, α and β, through the solutions of eqs. (22)–(30) and (23)–(29):

$$G(\alpha, \beta) = \frac{1}{2} \int_\Omega \left(\frac{\partial \chi}{\partial x} - \frac{\partial \psi}{\partial y} - u_{\text{obs}} \right)^2 + \left(\frac{\partial \chi}{\partial y} + \frac{\partial \psi}{\partial x} - v_{\text{obs}} \right)^2 \mathrm{d}\Omega. \tag{31}$$

The problem is to determine optimal values for α and β, i.e., such that they minimize the cost function G. The principle of the determination of these optimal values is the same as above: we need to exhibit the gradient of G with respect to α and β to get the optimal condition and to insert the gradient in a procedure of unconstrained optimization.

For a perturbation, $h = (h_\alpha, h_\beta)$ on the boundary conditions, the directional derivatives of χ, ψ and G verify:

$$\nabla^2 \hat{\psi} = 0, \tag{32}$$

$$\hat{\psi} = h_\alpha \quad \text{on } \Gamma, \tag{33}$$

$$\nabla^2 \hat{\chi} = 0, \tag{34}$$

$$\hat{\chi} = h_\beta \quad \text{on } \Gamma, \tag{35}$$

$$\hat{G}(\alpha, \beta, h) = \int_\Omega \left(\frac{\partial \chi}{\partial x} - \frac{\partial \psi}{\partial y} - u_{\text{obs}}, \frac{\partial \hat{\chi}}{\partial x} - \frac{\partial \hat{\psi}}{\partial y} \right) + \left(\frac{\partial \chi}{\partial y} + \frac{\partial \psi}{\partial x} - v_{\text{obs}}, \frac{\partial \hat{\chi}}{\partial y} + \frac{\partial \hat{\psi}}{\partial x} \right) \mathrm{d}\Omega. \tag{36}$$

Let P and Q be the adjoint variables of χ and ψ if they are defined as solutions of:

$$\nabla^2 P = \nabla^2 \chi - \nabla u_{\text{obs}}, \tag{37}$$

$$P = 0 \quad \text{on } \Gamma; \tag{38}$$

$$\nabla^2 Q = -\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} - \nabla v_{\text{obs}} \tag{39}$$

$$Q = 0 \quad \text{on } \Gamma. \tag{40}$$

Then, after taking the inner product of (32) and (34) with P and Q, using the Green formula, the definition of the gradient and eq. (36), it is derived as a factor of the perturbation h of the control variables:

$$\nabla G = \begin{pmatrix} \nabla G_\alpha \\ \nabla G_\beta \end{pmatrix} = \begin{pmatrix} \nabla \chi - \dfrac{\partial \psi}{\partial y} + \dfrac{\partial \psi}{\partial x} - u_{\text{obs}} - v_{\text{obs}} - \dfrac{\partial P}{\partial n} \\ -\dfrac{\partial \psi}{\partial x} + \dfrac{\partial \psi}{\partial y} - \dfrac{\partial \chi}{\partial y} - \dfrac{\partial \chi}{\partial x} - u_{\text{obs}} - v_{\text{obs}} - \dfrac{\partial Q}{\partial n} \end{pmatrix}. \tag{41}$$

The gradient is evaluated on the boundary of the domain. For practical uses, a similar derivation of the gradient must be carried out on the discretized system.

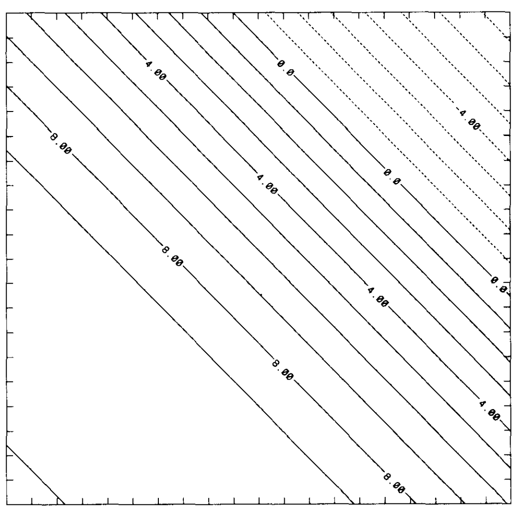

Fig. 6. Velocity potential as given by eq. (42).

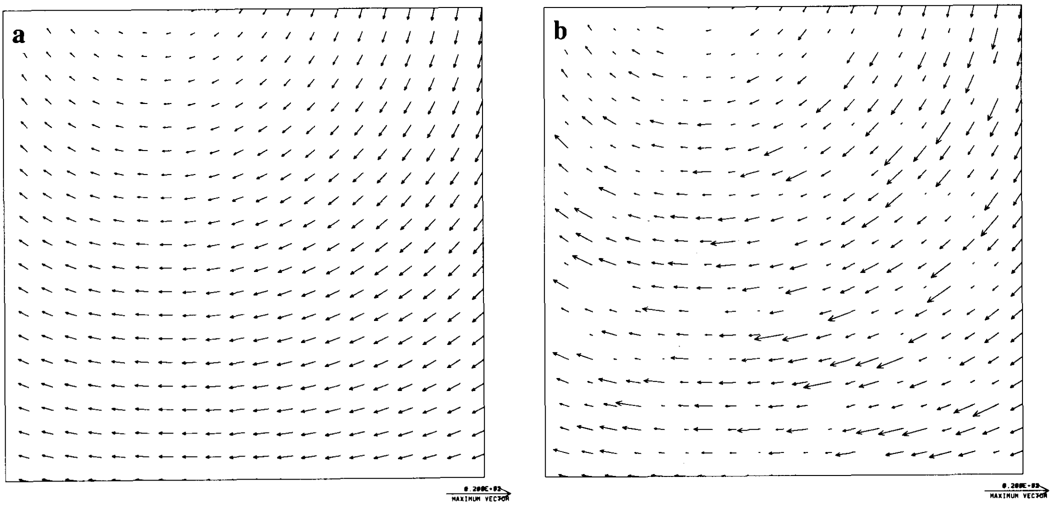

Fig. 7. (a) Wind deduced from the velocity potential, (b) "observed" wind.

4.3. Numerical example

A numerical application has been performed using analytic data; we have assumed that the velocity potential (Fig. 6) is given by:

$$\chi(x, y) = \frac{50}{2\pi} \sin(x + y). \tag{42}$$

from which an analytic wind (Fig. 7a) is deduced through:

$$u(x, y) = \frac{\partial \chi}{\partial x} - \frac{\partial \psi}{\partial y}, \tag{44}$$

$$v(x, y) = \frac{\partial \chi}{\partial y} + \frac{\partial \psi}{\partial x}. \tag{45}$$

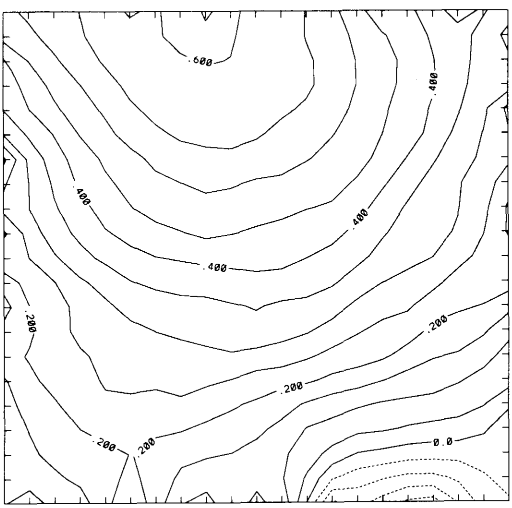

Fig. 8. "Observed" stream function.

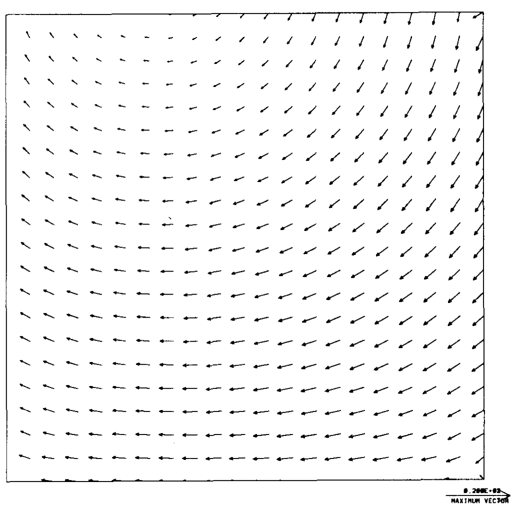

Fig. 9. Retrieved wind.

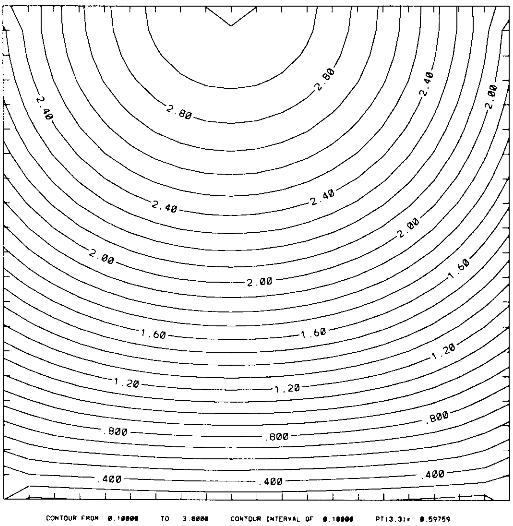

Fig. 10. Retrieved stream function.

The domain and the discretization are the same as in the numerical example developed in the Subsection 4.2. An observed wind (Fig. 7b) was deduced by adding to (44) and (45), a random noise.

Fig. 8 shows the "observed" stream function deduced from the wind. Fig. 9 shows the corresponding wind from the stream function retrieved after 5 iterations, displayed in Fig. 10.

4.4. *Application to real data*

The complete retrieval of a balanced field from observations was carried out using the FGGE data on height and wind fields for the 500 hPa level at 0:00 UTC, 26 May 1979 on a β-plane approximation centered at (32° N, 130° E). Data were extrapolated to the grid points of the model. Figs. 11a, b represent the analyzed wind and geopotential field, from which the stream function is deduced (Fig. 12). After 12 outer iterations of the method, the cost function remained stationary after being divided by a factor of 2. The retrieved stream function is displayed in Fig. 13, while wind and geopotential fields are shown in Figs. 14a, b. They satisfy the nonlinear balance equation to good accuracy, without any assumption on the exactness of one of the fields.

Nevertheless, an arbitrary choice is required by this method: the definitition of the cost function. In numerical experiments, the general form of J was taken as:

$$J=\frac{1}{2}\int_{\Omega}\frac{\alpha}{\sigma_{\Phi}^{2}}\,\|\phi-\phi_{\rm obs}\|^{2}$$

$$+\frac{\beta}{\sigma_{\Psi}^{2}}\,\|\Psi-\Psi_{\rm obs}\|^{2}\,d\Omega.$$

The first numerical experiment was carried out

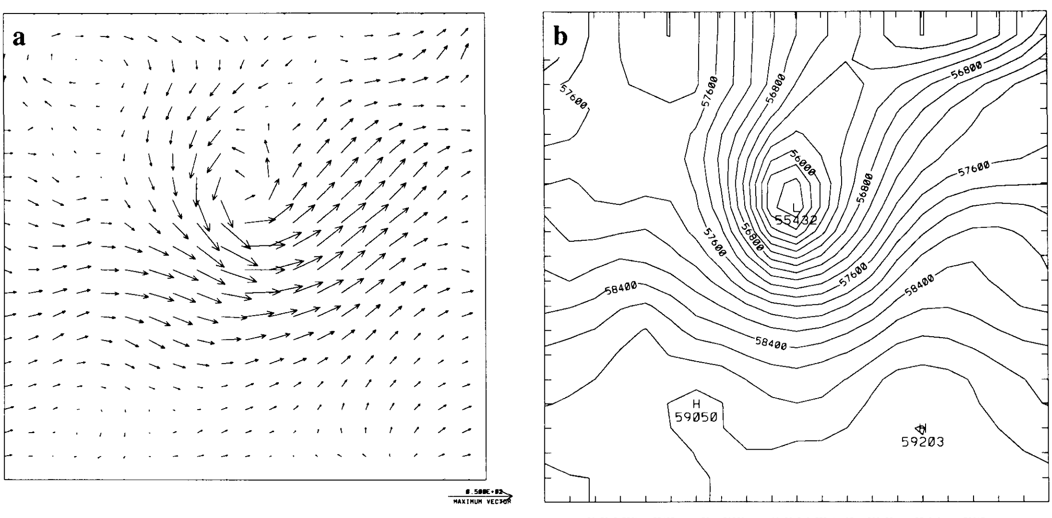

Fig. 11. Analyzed fields given by FGGE data: (a) wind field, (b) geopotential field.

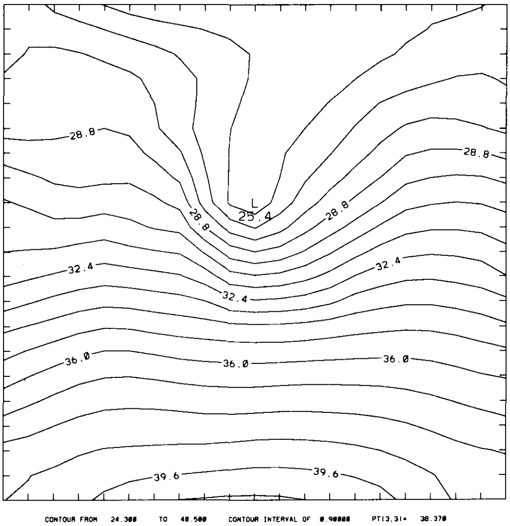

Fig. 12. Velocity deduced from analyzed fields given by FGGE data.

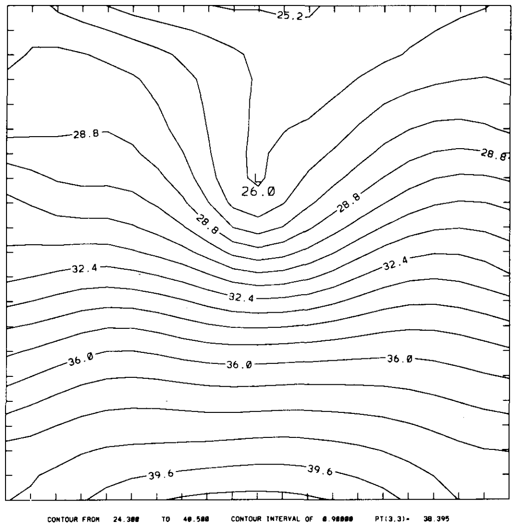

Fig. 13. Retrieved fields: stream function ($\alpha = 100$ and $\beta = 1$).

with $\alpha = 100$ and $\beta = 1$. A second numerical experiment was performed with the same observations and the same initial guess but different values for the weights, namely $\alpha = 1$ and $\beta = 100$. The convergence is obtained after 30 iterations. The retrieved values of the stream function, geopotential and wind field are displayed in Figs. 15, 16, 17. They are slighlty different from the former results. An open question remains: what is the best choice for the cost function?

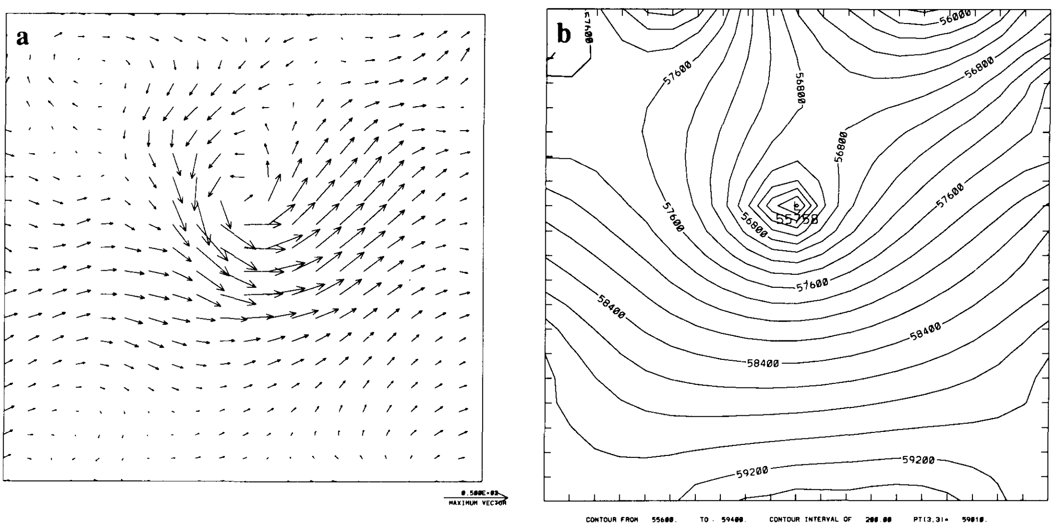

Fig. 14. Retrieved fields: (a) wind field, (b) stream function ($\alpha = 100$ and $\beta = 1$).

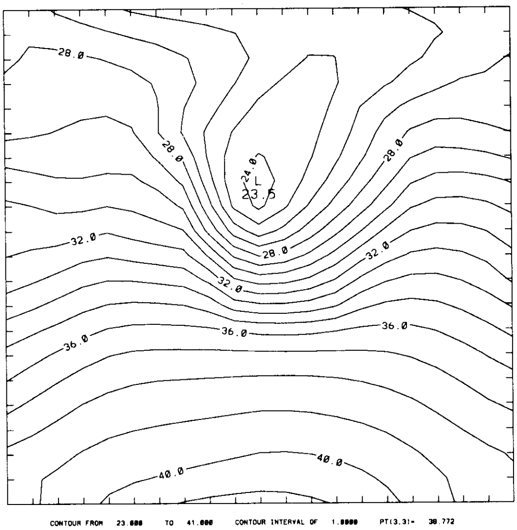

Fig. 15. Retrieved fields: stream function ($\alpha = 1$ and $\beta = 100$).

5. Conclusion

The method which has been proposed in this paper permits the retrieval of balanced fields from unbalanced observations without any assumption on the exactness of the observations on the boundary of the domain. The basic tools to implement the method are: (a) a Poisson equation solver which can be used for the direct system as well as for the adjoint one; (b) an optimization algorithm.

Both of these codes are available in most of the computer libraries; therefore there is no great difficulty to implement the algorithm.

In all the applications carried out in this paper, we have supposed, for sake of simplicity, the observed fields given on a regular grid. It would be worthwhile to consider only the real points of observation; this can be done either by considering an l-interpolation from the grid points to the points of observation, or by using a finite-element discretization with nodes located at the points of observation.

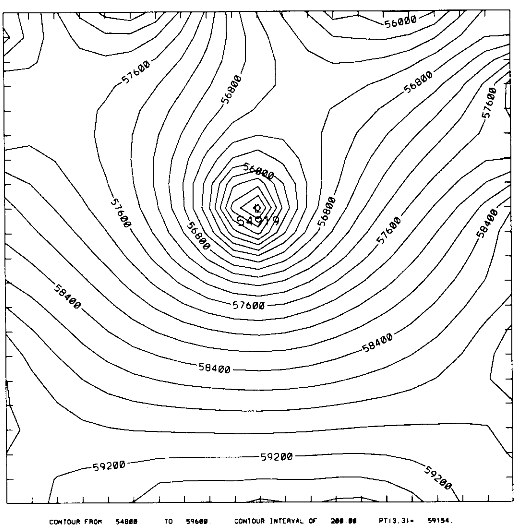

Fig. 16. Retrieved fields: geopotential height field ($\alpha = 1$ and $\beta = 100$).

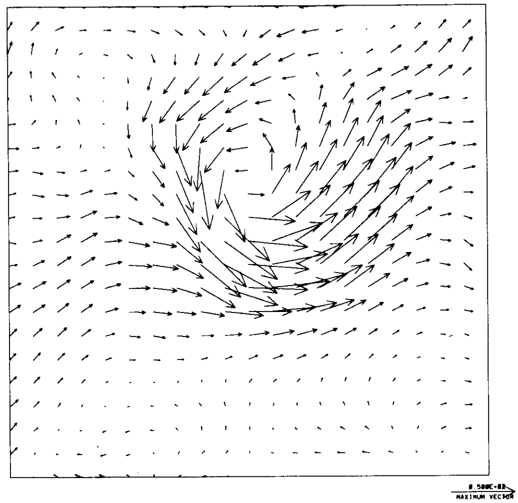

Fig. 17. Retrieved fields: wind field ($\alpha = 1$ and $\beta = 100$).

6. Acknowledgment

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REFERENCES

Courant, R. and Hilbert, D. 1962. *Methods of mathematical physics*, Vol. II, Interscience, 830 pp.

Haltiner, G. J., Sasaki, Y. K. and Barker, E. H. 1975. A variational procedure for obtaining global balanced winds. *Proceedings of the JOC Study Group Conference on four-dimensional assimilation.* Paris, 17–21 November, 11, 198–223. WMO, Geneva.

Kasahara, A. 1982. Significance of non-elliptic regions in balanced flows of the tropical atmosphere. *Mon. Wea. Rev.* **110**, 1956–1967.

Le Dimet, F.-X. and Talagrand, O. 1986. Variational algorithms for analysis and assimilation of meteorological observations. Theoretical aspects. *Tellus* **38A**, 97–110.

Le Dimet, F.-X. 1988. *Determination of the adjoint of a numerical weather prediction model.* SCRI Scientific Report, Florida State University, Tallahassee Florida. 26 pp.

Lemaréchal, C. 1980. *Non-differentiable optimization. Theory and algorithm.* Birkauser, Boston.

Paegle, J. and Paegle, J. N. 1974. An efficient and accurate approximation to the balance wind with application to non-elliptic data. *Mon. Wea. Rev.* **102**, 838–846.

Paegle, J. and Paegle, J. N. 1976. On geopotential data and ellipticity of the balance equation. A data study. *Mon. Wea. Rev.* **104**, 1279–1288.

Sangster, W. E. 1960. A method of representing the horizontal pressure force without reduction at sea level. *J. Meteor.* **17**, 166–176.